COMMUNICATION

A NEW SHORT PROOF FOR THE KRUSKAL–KATONA THEOREM

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We give a very short proof for the Kruskal-Katona theorem and Lovász's version of it: given \( \binom{k}{\xi} \) \( k \)-element sets there are at least \( \binom{k-1}{\xi-1} \) \( k-1 \)-element sets which are contained in at least one of the \( k \)-sets.

1. Introduction

Suppose \( X = \{1, 2, \ldots, n\} \) and \( \mathcal{F} \) is a family of subsets of \( X \), i.e. \( \mathcal{F} \subseteq 2^X \). Define \( \Delta(\mathcal{F}) = \{E \subseteq X : \text{ for some } F \in \mathcal{F}, E \subseteq F, |F - E| = 1\} \). Given \( k, m \geq 1 \) such that \( \mathcal{F} \subseteq \binom{X}{\xi} \), i.e., \( |F| = k \) for all \( F \in \mathcal{F} \), and \( |\mathcal{F}| = m \), what can one say about \( |\Delta(\mathcal{F})| \)? In general one cannot improve on the trivial upper bound \( |\Delta(\mathcal{F})| \leq km \). Best possible lower bounds for every \( m \) were obtained independently by Kruskal [3] and Katona [2]. To state their result one writes \( m \) in \( k \)-cascade form:

\[
m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}, \quad a_k > a_{k-1} > \cdots > a_s \geq 1.
\]

Note that every positive integer has a unique \( k \)-cascade representation.

Theorem 1 (Kruskal–Katona). If \( \mathcal{F} \subseteq \binom{X}{\xi} \), \( |\mathcal{F}| = m = \binom{a_k}{k} + \cdots + \binom{a_s}{s} \), then

\[
|\Delta(\mathcal{F})| \geq \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_s}{s-1}.
\]  

A short proof of Theorem 1 was given by Daykin [1]. Because of the \( k \)-cascade representation, the Kruskal–Katona theorem is often very clumsy for applications. Lovász [4] proposed the following slightly weaker but much handier form.

Theorem 2 (Lovász). Suppose \( \mathcal{F} \subseteq \binom{X}{\xi} \), \( |\mathcal{F}| = m = \binom{a_k}{k} \). Then

\[
|\Delta(\mathcal{F})| \geq \binom{x}{k-1} \quad \text{where } x \geq k \text{ is real.}
\]
The aim of this short note is to give a simple, unified argument for both these theorems.

2. Shifting

For $1 < j \leq n$ and $\mathcal{H} \subseteq 2^X$, let us define:

$$S_j(H) = \begin{cases} (H \setminus \{j\}) \cup \{1\} & \text{if } j \in H, 1 \notin H, ((H \setminus \{j\}) \cup \{1\}) \notin \mathcal{H}, \\ H & \text{otherwise}, \end{cases}$$

$$S_j(\mathcal{H}) = \{S_j(H) : H \in \mathcal{H}\}.$$

**Proposition.** $\Delta(S_j(\mathcal{F})) \subseteq S_j(\Delta(\mathcal{F}))$ holds for $1 < j \leq n$.

**Proof.** We must show that for every $F \in \mathcal{F}$, $S_j(F) \subseteq S_j(\Delta(\mathcal{F}))$ holds. This follows easily if $F = S_j(F)$. Suppose $F \neq S_j(F)$, i.e., $j \in F, 1 \notin F, S_j(F) = (F \setminus \{j\}) \cup \{1\}$. If $E \subseteq S_j(F), 1 \notin E$, then $E \in S_j(\Delta(\mathcal{F}))$ follows easily from $E \subseteq F$. However, $1 \in E$ implies $E' = E \setminus \{j\} \subseteq F$ and $E \in S_j(\Delta(\mathcal{F}))$ follows from the definition of $S_j(\Delta(\mathcal{F}))$. \[\square\]

Iterating the operation $S_j$ for $2 \leq j \leq n$, the number of sets containing 1 is increasing. Thus after a finite number of steps we obtain a family $\mathcal{G}$ satisfying $|\mathcal{F}| = |\mathcal{G}|, |\Delta(\mathcal{G})| \geq |\Delta(\mathcal{F})|$ and $S_j(\mathcal{F}) = \mathcal{G}$ for $2 \leq j \leq n$.

Hence in proving Theorems 1 and 2—by eventually replacing $\mathcal{F}$ by $\mathcal{G}$—we may assume $S_j(\mathcal{F}) = \mathcal{F}$ holds for $2 \leq j \leq n$ or equivalently—with the notation $\mathcal{F}_0 = \{F \in \mathcal{F} : 1 \notin F\}$:

$$E \in \Delta(\mathcal{F}_0) \implies (E \cup \{1\}) \in \mathcal{F}. \quad (3)$$

3. The Proof of Theorems 1 and 2

Define $\mathcal{F}(1) = \{F \setminus \{1\} : 1 \in F \in \mathcal{F}\}$.

$$|\Delta(\mathcal{F})| \geq |\mathcal{F}(1)| + |\Delta(\mathcal{F}(1))|. \quad (4)$$

We apply double induction on $k$ and $m$. For $k = 1$ and $m$ arbitrary, both (1) and (2) hold trivially. We first prove (2). If $|\mathcal{F}(1)| \geq (\begin{pmatrix} k-1 \\ 2 \end{pmatrix})$ then by the induction hypothesis $\Delta(\mathcal{F}(1)) \geq (\begin{pmatrix} k-1 \\ 3 \end{pmatrix})$. Thus (4) yields

$$|\Delta(\mathcal{F})| \geq \begin{pmatrix} x - 1 \\ k - 1 \end{pmatrix} + \begin{pmatrix} x - 1 \\ k - 2 \end{pmatrix} = \begin{pmatrix} x \\ k - 1 \end{pmatrix},$$

as desired.

Suppose next $|\mathcal{F}(1)| < (\begin{pmatrix} k-1 \\ 2 \end{pmatrix})$. Then $|\mathcal{F}_0| = |\mathcal{F} - |\mathcal{F}(1)| > (\begin{pmatrix} k-1 \\ 2 \end{pmatrix})$, and so, by induction, $|\Delta(\mathcal{F}_0)| \geq (\begin{pmatrix} k-1 \\ 3 \end{pmatrix})$. But (3) implies $|\mathcal{F}(1)| \geq (\begin{pmatrix} k-1 \\ 2 \end{pmatrix})$, a contradiction.
Now we prove (1). If
\[ |\mathcal{F}(1)| \geq \binom{a_k - 1}{k - 1} + \cdots + \binom{a_s - 1}{s - 1}, \]
then by induction
\[ \Delta(\mathcal{F}(1)) \geq \binom{a_k - 1}{k - 2} + \cdots + \binom{a_s - 1}{s - 2}. \]
Note that \( a_1 = 0 \). By (4) we have
\[ \Delta(\mathcal{F}) \geq \left( \binom{a_k - 1}{k - 1} + \binom{a_k - 1}{k - 2} \right) + \cdots + \left( \binom{a_s - 1}{s - 1} + \binom{a_s - 1}{s - 2} \right) \]
\[ = \binom{a_k}{k - 1} + \cdots + \binom{a_s}{s - 1}, \]
as desired. Suppose
\[ |\mathcal{F}(1)| < \binom{a_k - 1}{k - 1} + \cdots + \binom{a_s - 1}{s - 1}. \]
Then from \(|\mathcal{F}_0| = |\mathcal{F}| - |\mathcal{F}(1)|\) we infer
\[ |\mathcal{F}_0| > \binom{a_k - 1}{k - 1} + \binom{a_k - 1}{k - 2} + \cdots + \binom{a_s}{s}. \]
By induction and (3)
\[ |\mathcal{F}(1)| \geq |\Delta(\mathcal{F}_0)| \geq \binom{a_k - 1}{k - 1} + \binom{a_k - 1}{k - 2} + \cdots + \binom{a_s - 1}{s - 1} \]
follows, which is a contradiction. \( \square \)

References