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A NEW SHORT PROOF FOR THE KRUSKAL–KATONA THEOREM

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We give a very short proof for the Kruskal-Katona theorem and Lovász's version of it: given $\binom{x}{k}$ k-element sets there are at least $\binom{x}{k-1}$ $(k-1)$-element sets which are contained in at least one of the k-sets.

1. Introduction

Suppose $X=\{1,2,\ldots,n\}$ and $\mathcal{F}$ is a family of subsets of X, i.e. $\mathcal{F}\subseteq 2^X$. Define $\Delta(\mathcal{F})=\{E\subseteq X:$ for some $F\in\mathcal{F}$, $E\subseteq F$, $|F-E|=1\}$. Given $k, m\geq 1$ such that $\mathcal{F}\subseteq\binom{X}{k}$, i.e., $|F|=k$ for all $F\in\mathcal{F}$, and $|\mathcal{F}|=m$, what can one say about $|\Delta(\mathcal{F})|$? In general one cannot improve on the trivial upper bound $|\Delta(\mathcal{F})|\leq km$. Best possible lower bounds for every m were obtained independently by Kruskal [3] and Katona [2]. To state their result one writes m in *k-cascade* form:

$$m=\binom{a_k}{k}+\binom{a_{k-1}}{k-1}+\cdots+\binom{a_s}{s},\quad a_k>a_{k-1}>\cdots>a_s\geq 1.$$

Note that every positive integer has a unique *k-cascade* representation.

Theorem 1 (Kruskal–Katona). *If* $\mathcal{F}\subseteq\binom{X}{k}$, $|\mathcal{F}|=m=\binom{a_k}{k}+\cdots+\binom{a_s}{s}$, *then*

$$|\Delta(\mathcal{F})|\geq\binom{a_k}{k-1}+\binom{a_{k-1}}{k-2}+\cdots+\binom{a_s}{s-1}. \tag{1}$$

A short proof of Theorem 1 was given by Daykin [1]. Because of the *k-cascade* representation, the Kruskal–Katona theorem is often very clumsy for applications. Lovász [4] proposed the following slightly weaker but much handier form.

Theorem 2 (Lovász). *Suppose* $\mathcal{F}\subseteq\binom{X}{k}$, $|\mathcal{F}|=m=\binom{x}{k}$. *Then*

$$|\Delta(\mathcal{F})|\geq\binom{x}{k-1}\quad \textit{where } x\geq k \textit{ is real.} \tag{2}$$

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The aim of this short note is to give a simple, unified argument for both these theorems.

2. Shifting

For $1<j\leqslant n$ and $\mathscr{H}\subseteq 2^X$, let us define:

$$S_j(H)=\begin{cases}(H-\{j\})\cup\{1\} & \text{if } j\in H,\ 1\notin H,\ ((H-\{j\}\cup\{1\})\notin\mathscr{H},\\ H & \text{otherwise},\end{cases}$$

$$S_j(\mathscr{H})=\{S_j(H)\colon H\in\mathscr{H}\}.$$

Proposition. *$\Delta(S_j(\mathscr{F}))\subseteq S_j(\Delta(\mathscr{F})$ holds for $1<j\leqslant n$.*

Proof. We must show that for every $F\in\mathscr{F}$, $\Delta(S_j(F))\subseteq S_j(\Delta(\mathscr{F}))$ holds. This follows easily if $F=S_j(F)$. Suppose $F\neq S_j(F)$, i.e., $j\in F$, $1\notin F$, $S_j(F)=(F-\{j\})\cup\{1\}$. If $E\subseteq S_j(F)$, $1\notin E$, then $E\in S_j(\Delta(\mathscr{F}))$ follows easily from $E\subseteq F$. However, $1\in E$ implies $E'=E-\{1\}\cup\{j\}\subseteq F$ and $E\in S_j(\Delta(\mathscr{F}))$ follows from the definition of $S_j(\Delta(\mathscr{F}))$. □

Iterating the operation S_j for $2\leqslant j\leqslant n$, the number of sets containing 1 is increasing. Thus after a finite number of steps we obtain a family $\mathscr{G}$ satisfying $|\mathscr{F}|=|\mathscr{G}|$, $|\Delta(\mathscr{F})|\geqslant|\Delta(\mathscr{G})|$ and $S_j(\mathscr{G})=\mathscr{G}$ for $2\leqslant j\leqslant n$.

Hence in proving Theorems 1 and 2 – by eventually replacing $\mathscr{F}$ by $\mathscr{G}$ – we may assume $S_j(\mathscr{F})=\mathscr{F}$ holds for $2\leqslant j\leqslant n$ or equivalently – with the notation $\mathscr{F}_0=\{F\in\mathscr{F}\colon 1\notin F\}$:

$$E\in\Delta(\mathscr{F}_0)\quad\text{implies}\quad(E\cup\{1\})\in\mathscr{F}. \tag{3}$$

3. The Proof of Theorems 1 and 2

Define $\mathscr{F}(1)=\{F-\{1\}\colon 1\in F\in\mathscr{F}\}$.

$$|\Delta(\mathscr{F})\geqslant|\mathscr{F}(1)|+|\Delta(\mathscr{F}(1))|. \tag{4}$$

We apply double induction on k and m. For $k=1$ and m arbitrary, both (1) and (2) hold trivially. We first prove (2). If $|\mathscr{F}(1)|\geqslant\binom{x-1}{k-1}$ then by the induction hypothesis $\Delta(\mathscr{F}(1))\geqslant\binom{x-1}{k-2}$. Thus (4) yields

$$|\Delta(\mathscr{F})|\geqslant\binom{x-1}{k-1}+\binom{x-1}{k-2}=\binom{x}{k-1},$$

as desired.

Suppose next $|\mathscr{F}(1)|<\binom{x-1}{k-1}$. Then $|\mathscr{F}_0|=|\mathscr{F}|-|\mathscr{F}(1)|>\binom{x-1}{k}$, and so, by induction, $|\Delta(\mathscr{F}_0)|\geqslant\binom{x-1}{k-1}$. But (3) implies $|\mathscr{F}(1)|\geqslant\binom{x-1}{k-1}$, a contradiction.

Now we prove (1). If

$$|\mathcal{F}(1)| \geq \binom{a_k - 1}{k-1} + \cdots + \binom{a_s - 1}{s-1},$$

then by induction

$$\Delta(\mathcal{F}(1)) \geq \binom{a_k - 1}{k-2} + \cdots + \binom{a_s - 1}{s-2}.$$

Note that $\binom{a}{-1} = 0$. By (4) we have

$$\Delta(\mathcal{F}) \geq \left(\binom{a_k - 1}{k-1} + \binom{a_k - 1}{k-2}\right) + \cdots + \left(\binom{a_s - 1}{s-1} + \binom{a_s - 1}{s-2}\right)$$
$$= \binom{a_k}{k-1} + \cdots + \binom{a_s}{s-1},$$

as desired. Suppose

$$|\mathcal{F}(1)| < \binom{a_k - 1}{k-1} + \cdots + \binom{a_s - 1}{s-1}.$$

Then from $|\mathcal{F}_0| = |\mathcal{F}| - |\mathcal{F}(1)|$ we infer

$$|\mathcal{F}_0| > \binom{a_k - 1}{k} + \binom{a_{k-1} - 1}{k-1} + \cdots + \binom{a_s - 1}{s}.$$

By induction and (3)

$$|\mathcal{F}(1)| \geq |\Delta(\mathcal{F}_0)| \geq \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \cdots + \binom{a_s - 1}{s-1}$$

follows, which is a contradiction. □

References

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