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On the Number of Sets in a Null t-Design

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In this paper we prove that the symmetric difference of any two distinct $S_\lambda(n, k, t)$ Steiner-systems contains at least 2^{t+1} different sets (Corollary 2). The proof also yields an extremal set theoretical result of Sauer (Theorem 2).

1. INTRODUCTION

Let X be an n-element set, and let k, t, λ be positive integers, $n \geq k \geq t$. A system $\mathcal{S}$ of k-element subsets of X is called an $S_\lambda(n, k, t)$ Steiner-system if every t-element subset of X is contained in exactly λ members of $\mathcal{S}$. (Notice that two members of $\mathcal{S}$ are not necessarily distinct.) We denote by $h_{\mathcal{S}}$ the characteristic function of $\mathcal{S}$ i.e. $h_{\mathcal{S}}$ is defined on all subsets of X, and for any $B \subseteq X$, $h(B)$ is the number of occurences of B in $\mathcal{S}$.

Further, let $V(X)$ denote the vector space of all real valued functions $f: 2^X \to R$. (Obviously, dim $V(X) = 2^n$.)

DEFINITION. For a fixed integer t, $0 \leq t \leq n$, we say that f is a null t-design if for every $A \subseteq X$, $|A| \leq t$, we have

$$\sum_{A \subseteq F \subseteq X} f(F) = 0. \tag{1}$$

Notice that for $t = n$ the only null t-design is the identically zero one. A null t-design is called k-uniform if $f(F) \neq 0$ implies $|F| = k$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two $S_\lambda(n, k, t)$ Steiner-systems, then $h_{\mathcal{S}_1} - h_{\mathcal{S}_2}$ is a k-uniform null t-design.

Of course, both null t-designs and k-uniform null t-designs form vector spaces.

These vector spaces were considered in several papers (Graver and Jurkat [4], Graham, Li and Li [3], Deza and Frankl [1]): the dimensions were determined and bases were exhibited.

For simplicity we identify a set $\{x_1, x_2, \ldots, x_l\}$ with the product $x_1x_2 \cdots x_l$, and a null t-design, f, with the polynomial

$$\sum_{F \subseteq x} f(F) \prod_{x \in F} x.$$

In this terminology the simplest null t-design is $(1 - x_1) \cdots (1 - x_{t+1})$ (the constant term is just $f(\varnothing)$, while the simplest k-uniform null t-design is $(x_1 - x_2)$ $(x_3 - x_4) \cdots (x_{2t+1} - x_{2t+2})x_{2t+3} \cdots x_{k+t+1}$.

In both these designs there are exactly 2^{t+1} terms. The aim of this note is to prove:

THEOREM 1. *Let f be a nonidentically zero null t-design. Then f has at least 2^{t+1} nonzero terms, i.e.,*

$$|\{F \subseteq X : f(F) \neq 0\}| \geq 2^{t+1}.$$

COROLLARY 1. *A nonidentically zero k-uniform null t-design has at least 2^{t+1} nonzero terms.*

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COROLLARY 2. *The symmetric difference of any two distinct $S_\lambda(n, k, t)$ Steiner-systems contains at least 2^{t+1} distinct sets.*

REMARK 1. The truth of Corollary 2 in the special case $k=3$, $t=2$ follows from a result of Lindner and Rosa [5]. The statement of Corollary 1 was conjectured by Singhi [7].

THEOREM 2. *Suppose*

$$\mathcal{F} \subseteq 2^X, \qquad |\mathcal{F}| > \sum_{0<i<t} \binom{n}{i}.$$

Then there exists an $H \subseteq X$, $|H| = t+1$, such that for every $H_0 \subseteq H$ one can find $F \in \mathcal{F}$ with $F \cap H = H_0$ (i.e. $\{F \cap H : F \in \mathcal{F}\} = 2^H$).

REMARK 2. Theorem 2 was originally proved by Sauer [6], and in [2] a simpler proof is given. However, we think the present proof gives more insight.

2. Proof of the Results

For a null t-design f and a subset S of X we define the trace $f_S: 2^S \to R$ by

$$f_S(G) = \sum_{F \subseteq X, F \cap S = G} f(F), \quad \text{for every } G \subseteq S.$$

PROPOSITION 1. *The function f_S is a null t-design on S.*

PROOF. In fact, let $A \subseteq S$ and $|A| \leq t$. Then we have

$$\sum_{A \subseteq G = X} f_S(G) = \sum_{A \subseteq G \subseteq S} \sum_{\substack{F \subseteq X, \\ F \cap S = G}} f(F) = \sum_{A \subseteq F \subseteq X} f(F) = 0.$$

We now give the proof of Theorem 1. Let f be a nonidentically zero null t-design. Let s be the maximal integer for which f is a null s-design (in most cases $t=s$ but in any case $t \leq s \leq n-1$). Then we can find a $(s+1)$-element subset S of X for which (1) is violated, i.e.

$$\sum_{S \subseteq F \subseteq X} f(F) = a \neq 0. \tag{2}$$

Let us consider the trace of f on S, i.e. f_S. In view of Proposition 1, f_S is a null s-design.

PROPOSITION 2. *For $G \subseteq S$ we have $f_S(G) = (-1)^{|S-G|}a$.*

PROOF. We apply induction on $|S-G|$. If $G=S$, then the statement is just (2). Suppose now we are given some G and we know the proposition holds for all its supersets H, $G \subseteq H \subseteq S$. Since f_S is a null s-design and $|G| \leq s$, we have

$$\begin{aligned} 0 &= \sum_{G \subseteq H \subseteq S} f_S(H) = f_S(G) + \sum_{G \subsetneq H \subseteq S} f(H) \\ &= f_S(G) + a \sum_{0<i \leq |S-G|} \binom{|S-G|}{i} (-1)^{|S-G|-i} \\ &= f_S(G) - (-1)^{|S-G|}a. \end{aligned}$$

Now Theorem 1 is immediate: $f_S(G) \neq 0$ implies that for some $F \subseteq X$, $F \cap S = G$ we have $f(F) \neq 0$. Since $|2^S| = 2^{s+1}$, this gives us $2^{s+1} \geqslant 2^{t+1}$ nonzero terms, as desired. As for Theorem 2, for every $F \in \mathscr{F}$, define $f^F \in V(X)$ by

$$f^F(G) = \begin{cases} 1 & \text{if } G \subseteq F, |G| \leqslant t \\ 0, & \text{otherwise.} \end{cases}$$

All these functions are in a $(\sum_{0 \leqslant i \leqslant t} \binom{n}{i})$-dimensional subspace $V_{\leqslant t} = \{f \in V(X): f(G) = 0,$ whenever $|G| > t\}$. Thus they cannot be linearly independent. Let $\sum_{F \in \mathscr{F}} \alpha(F) f^F = 0$ be a linear dependence among them. This means that the function g defined by $g(F) = \alpha(F)$ if $F \in \mathscr{F}$, and $g(F) = 0$ otherwise, is a null t-design. Now, in the terminology of the proof of Theorem 1, for an arbitrary $(t+1)$-element subset T of S we have

$$\{T \cap F: \alpha(F) \neq 0\} = 2^T,$$

and in particular

$$\{T \cap F; F \in \mathscr{F}\} = 2^T,$$

which proves Theorem 2.

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