ON KRUSKAL'S CASCADES AND COUNTING CONTAINMENTS IN A SET OF SUBSETS

DAVID E. DAYKIN AND PETER FRANKL

Abstract. Let $\mathcal{F}$ be a set of m subsets of $X = \{1, 2, ..., n\}$. We study the maximum number λ of containments $Y \subset Z$ with $Y, Z \in \mathcal{F}$. THEOREM 9. $\lambda = (1+o(1))\binom{m}{2}$, *if, and only if,* $m^{1/n} \to 1$. When n is large and members of $\mathcal{F}$ have cardinality k or $k-1$ we determine λ. For this we bound $(\Delta N)/N$ where ΔN is the shadow of Kruskal's k-cascade for the integer N. Roughly, if $m \sim N + \Delta N$, then $\lambda \sim kN$ with infinitely many cases of equality. A by-product is Theorem 7 of LYM posets.

§1. *Introduction.* Let $\mathcal{F}$ be a set of m subsets of the set $X = \{1, 2, ..., n\}$. Kleitman determined the minimum number of pairs Y, Z in $\mathcal{F}$ with $Y \subset Z$ in [**9**]. We became interested in finding the maximum number. Obviously there can be at most $\binom{m}{2}$ such pairs. In Theorem 9 we show that there can be $(1+o(1))\binom{m}{2}$ pairs, if, and only if, $m^{1/n} \to 1$. In Section 5 we deal with the case in which $Y \in \mathcal{F}$ implies that the cardinality $|Y|$ is k or $k-1$. For this case we need to introduce Kruskal's cascades in Section 2, and bound their growth in Section 3. An application to LYM posets is Theorem 7.

§2. *Kruskal's cascades.* The facts in this section were discovered in 1963 by Kruskal [**11**] and in 1966 independently by Harper [**6**, **7**] and Katona [**8**]. A simple proof is in [**2**], and some years ago the author observed that virtually the same proof shows that the shadow of the shift of the family is in fact a subset of the shift of the shadow of the family. (See also [**3**].)

Let k be a fixed member of the set $\mathbb{Z}$ of positive integers. Then each $N \in \mathbb{Z}$ has one and only one representation

$$N = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \dots + \binom{a_t}{t}, \tag{1}$$

with

$$a_k > a_{k-1} > \dots > a_t \geqslant t \geqslant 1 . \tag{2}$$

This representation (1) is called the *k-cascade* for N. Let

$$\Delta N = \Delta_k N = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \dots + \binom{a_t}{t-1}, \tag{3}$$

and note that (3) is the $(k-1)$-cascade for ΔN, if, and only if, $t > 1$. When $t = 1$ the binomial coefficient identity

$$\binom{r}{s} = \binom{r-1}{s} + \binom{r-1}{s-1}, \tag{4}$$

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for $0 \leqslant s \leqslant r$ but not $0 = s = r$, can be used from the right on (3) a number of times to turn (3) into the $(k-1)$-cascade for ΔN.

For $h \in \mathbb{Z}$ let $\mathscr{I}(h)$ be the set of all subsets of cardinality h of $\mathbb{Z}$. If $\mathscr{G} \subset \mathscr{I}(k)$, let $\Delta\mathscr{G}$ be the set of all maximal proper subsets of members of $\mathscr{G}$ so that $\Delta\mathscr{G} \subset \mathscr{I}(k-1)$. The most important fact is that $|\Delta\mathscr{G}| \geqslant \Delta|\mathscr{G}|$. We order the members of each $\mathscr{I}(h)$ by putting $Y < Z$, if $\max\{Y\setminus Z\} < \max\{Z\setminus Y\}$. When $\mathscr{G}$ is the first N sets in this ordering of $\mathscr{I}(k)$, then $\Delta\mathscr{G}$ is precisely the first ΔN sets of $\mathscr{I}(k-1)$.

§3. *The function* $f_k(N) = (\Delta_k N)/N$. The trivial identity

$$\binom{r}{s} = \binom{r}{s-1}\frac{r-s+1}{s} \quad \text{for } 0 < s \leqslant r,$$

will often be used. As always k is a fixed integer in $\mathbb{Z}$. For $N \in \mathbb{Z}$ we define

$$f(N) = f_k(N) = (\Delta N)/N = (\Delta_k N)/N.$$

LEMMA 1. *If* $1 \leqslant k < a$ *and* $\binom{a}{k} \leqslant N$, *then*

$$f(N) < \frac{k}{a-k} = f\left(\binom{a-1}{k}\right).$$

Proof. Consider the k-cascade (1) for N. We must have $a_k \geqslant a$. First we suppose that $a_k = a$. Then (2) implies that

$$a-k+i \geqslant a_i \geqslant i, \qquad \text{for } k-1 \geqslant i \geqslant t.$$

We must prove that $\Omega > 0$ where

$$\Omega = kN - (a-k)\Delta N.$$

Now two terms of Ω are

$$k\binom{a}{k} - (a-k)\binom{a}{k-1} = \binom{a}{k-1} = \binom{a-1}{k-1} + \binom{a-2}{k-2} + \ldots + \binom{a-k+t}{t} + I = \Psi$$

say, where

$$I = \binom{a-k+t-1}{t-1} + \binom{a-k+t-2}{t-2} + \ldots + \binom{a-k}{0} \geqslant \binom{a-k}{0} > 0.$$

If $k = t$, then $\Omega = \Psi = I > 0$, as required, so assume $k > t$.

For $k-1 \geqslant i \geqslant t$ and $p \geqslant k$, define $I_i(p)$ by

$$I_i(p) = \binom{p-k+i}{i} + k\binom{a_i}{i} - (p-k)\binom{a_i}{i-1}.$$

Then

$$\Omega = I_{k-1}(a)+I_{k-2}(a)+\ldots+I_t(a)+I\,,$$

and to prove that $\Omega > 0$ we shall show that

$$I_i(a) \geqslant I_i(a-1) \geqslant \ldots \geqslant I_i(a_i-i+k) > 0$$

for each i. If $p-k+i > a_i$, then (4) says that

$$\binom{p-k+i}{i} = \binom{p-1-k+i}{i} + \binom{p-1-k+i}{i-1}.$$

Since

$$\binom{p-1-k+i}{i-1} \geqslant \binom{a_i}{i-1},$$

we see that $I_i(p) \geqslant I_i(p-1)$. It follows that $I_i(a) \geqslant \ldots \geqslant I_i(a_i-i+k)$. Finally (5) gives

$$I_i(a_i-i+k) = \binom{a_i}{i-1}\{((k+1)(a_i-i+1)/i)-(a_i-i)\} > 0$$

and the case $a_k = a$ of the lemma is established.

The case $a_k > a$ is now easy. If $e = a_k$ then $\binom{e}{k} \leqslant N$, so by the case $a_k = a$ we have $f(N) < k/(e-k) < k/(a-k)$ and the lemma is proved.

Next we define three subsets of $\mathbb{Z}, \mathbb{Z}^2$ namely $\mathscr{L}$ for lower bound, $\mathscr{U}$ for upper bound, and $\mathscr{P}$ for pair of bounds.

$$\mathscr{L} = \{L : 1 \leqslant M \leqslant L \text{ implies } f(M) \geqslant f(L)\}\,.$$

$$\mathscr{U} = \{K : K < N \text{ implies } f(K) > f(N)\}\,.$$

$$\mathscr{P} = \{(P, Q) : 1 \leqslant M \leqslant P < Q \leqslant N \text{ implies } f(M) > f(N)\}\,.$$

That $P < Q$ does not imply $(P, Q) \in \mathscr{P}$ is shown by the case $k = 3, P = 10, Q = 11$.

LEMMA 2. *If* $1 \leqslant k < a$, *then* $\binom{a-1}{k} \in \mathscr{L}$.

Proof. Let $1 \leqslant M \leqslant \binom{a-1}{k}$ and $\mathscr{G}$ be the first M sets of $\mathscr{I}(k)$ in the Kruskal ordering, so $\Delta\mathscr{G}$ is the first ΔM sets of $\mathscr{I}(k-1)$. Then the 1928 lemma of Sperner concerning the set of all subsets of $\{1, 2, \ldots, a-1\}$ says that $(\Delta M)/M \geqslant \binom{a-1}{k-1}/\binom{a-1}{k}$ and this is just what Lemma 2 says.

LEMMA 3. *If* $1 \leqslant k < a$ *then*

$$\left(\binom{a-1}{k}, \binom{a}{k}\right) \in \mathscr{P}.$$

This third lemma follows immediately from Lemmas 1 and 2.

Next we generalize Lemma 2.

LEMMA 4. *If the k-cascade* (1) *for N has the property*

$$(a_k+1)/k \leqslant (a_{k-1}+1)/(k-1) \leqslant \ldots \leqslant (a_t+1)/t\,, \tag{6}$$

then $N \in \mathscr{L}$.

Proof. Given M in $1 \leqslant M \leqslant N$ we must show that $f(M) \geqslant f(N)$. Let the k-cascade for M be

$$M = \binom{b_k}{k} + \binom{b_{k-1}}{k-1} + \ldots + \binom{b_u}{u},$$

$$b_k > b_{k-1} > \ldots > b_u \geqslant u \geqslant 1\,.$$

Case $k = t$. Here the result is Lemma 2.

Case $k = u$. By (5) we have

$$f\left(\binom{k}{k}\right) = k > f\left(\binom{k+1}{k}\right) = k/2 > \ldots > f\left(\binom{b_k}{k}\right).$$

Hence we shall assume that $b_k = a_k$ and $k > t$. Then the inequality $f(M) \geqslant f(N)$ is

$$\binom{a_k}{k-1}\left\{\binom{a_k}{k} + \ldots + \binom{a_t}{t}\right\} \geqslant \binom{a_k}{k}\left\{\binom{a_k}{k-1} + \ldots + \binom{a_t}{t-1}\right\}.$$

Using (5) we see that (6) says that

$$\binom{a_k}{k-1}\binom{a_i}{i} \geqslant \binom{a_k}{k}\binom{a_i}{i-1} \qquad \text{for } k \geqslant i \geqslant t\,,$$

and summing over i gives the result for this case $k = u$.

Now we use induction on k. The case $k = 1$ is trivial. So we assume that $k > 1$, and further that $k > t, u$. For convenience write

$$A = \Delta N, \qquad B = \binom{a_k}{k-1}, \qquad D = \binom{a_k}{k}, \qquad E = \Delta M\,.$$

Thus $D < N$ and applying the case $k = u$ to $D < N$ we find that $f(D) \geqslant f(N)$. Since $M \leqslant N$, we have $b_k \leqslant a_k$. If $b_k < a_k$ then $M < D$ and applying the case $k = t$ to $M < D$ shows that $f(M) \geqslant f(D) \geqslant f(N)$. Hence we shall now assume that $b_k = a_k$.

Next we use our induction hypothesis. We have $1 \leqslant M-D \leqslant N-D$ and the part of (6) from $k-1$ to t has not changed, so by induction $f(M-D) \geqslant f(N-D)$. In other words

$$\frac{E-B}{M-D} \geqslant \frac{A-B}{N-D}\,. \tag{7}$$

We define positive rational numbers α, β by $\alpha = BN/A = BM/\beta$. Now $f(D) \geqslant f(N)$ simply says that $\alpha \geqslant D$. Because $M \leqslant N$,

$$1 - \frac{M-\alpha}{M-D} = \frac{\alpha-D}{M-D} \geqslant \frac{\alpha-D}{N-D} = 1 - \frac{N-\alpha}{N-D}\,.$$

Notice that $M-D$ and $N-D$ are positive. We cancel the ones and multiply by A/N on the right and by $\beta/M = A/N$ on the left to obtain

$$\frac{A-B}{N-D} \geqslant \frac{\beta-B}{M-D}.$$

Hence (7) shows that $E \geqslant \beta$, or in other words that $f(M) \geqslant f(N)$, and the lemma follows.

LEMMA 5. *If the k-cascade* (1) *for N has property* (6), *then so too has the* $(k-1)$*-cascade for* ΔN.

Proof. If $k = t$, then property (6) holds vacuously for both N and ΔN, so we assume $k > t$. If $t > 1$ then (3) is the $(k-1)$-cascade for ΔN. Now $k/(k-1) < (k-1)/(k-2)$ and by (6) we have $(a_k+1)/(a_{k-1}+1) \leqslant k/(k-1)$ so $(a_k+1)/(k-1) \leqslant (a_{k-1}+1)/(k-2)$. It is now clear that ΔN has property (6). When $t = 1$ we must use (4) as described in Section 2, but we omit the detailed explanation.

LEMMA 6. *For* $2 \leqslant q \in \mathbb{Z}$ *put*

$$K = K(q) = \binom{qk-1}{k} + \binom{q(k-1)-1}{k-1} + \binom{q(k-2)-1}{k-2} + \dots + \binom{q-1}{1},$$

then $K(q) \in \mathscr{U}$.

Proof. If $K < N$ we must show that $f(K) > f(N)$. By using (5) for each binomial coefficient of K, we note that $f(K) = 1/(q-1)$. Again we use the cascade (1) for N. The proof is by induction on k and the case $k = 1$ is trivial.

Case $\binom{qk}{k} \leqslant N$. Here the result is Lemma 1.

Case $K < N < \binom{qk}{k}$. Here by the uniqueness of cascades we must have $a_k = qk-1$. For convenience put $F = \binom{qk-1}{k-1}$ and $G = \binom{qk-1}{k}$. Then we must show that

$$\frac{1}{q-1} > f(N) = \frac{F+\Delta_{k-1}(N-G)}{G+(N-G)},$$

but this is exactly the induction hypothesis that $1/(q-1) > f_{k-1}(N-G)$, so the proof is complete.

Actually, answering a question of Erdős, the following was proved by Katona [**8**]. If $k \geqslant 2$, then $K(2)$ is the largest number h such that for any $Y_1, \dots, Y_h$ in $\mathscr{I}(k)$ there are $Z_1, \dots, Z_h$ in $\mathscr{I}(k-1)$ such that $Z_i \subset Y_i$ for $1 \leqslant i \leqslant h$. The numbers $K(2)$ have $f = 1$ and were also used by Ahlswede and Katona in [**1**]. It would be interesting to know the sets $\mathscr{L}, \mathscr{U}$ precisely. The condition for $\mathscr{U}$ may be

$$\frac{a_k+1}{k} \sim \frac{a_{k-1}+1}{k-1} \sim \dots \sim \frac{a_1+1}{1}.$$

Lemmas 4 and 6 show that $\{K(q): 2 \leqslant q\} \subset \mathscr{L} \cap \mathscr{U}$ and we wonder if there is equality.

§4. *LYM Posets.* Suppose that the k-cascade (1) for N has property (6). Let T be the set consisting of the first N sets of $\mathscr{I}(k)$. Let Λ be the set of all subsets of members of T. We order members of Λ by inclusion to make it a poset. For $0 \leqslant h \leqslant k$ put $J(h) = \Lambda \cap \mathscr{I}(h)$. By repeated use of Lemma 5 the h-cascade for $|J(h)|$ has property (6) for $0 < h \leqslant k$. Choose arbitrarily $0 < h \leqslant k$ and $S \subset J(h)$. Then the result of Kruskal says that $\Delta_h(|S|) \leqslant |\Delta S|$. Further Lemma 4 says that $f_h(|S|) \geqslant f_h(|J(h)|)$. Combining these two results we find what is sometimes called the generalized matching property, namely that

$$\frac{|J(h-1)|}{|J(h)|} \leqslant \frac{|\Delta S|}{|S|} \quad \text{for} \quad 0 < h \leqslant k \text{ and } S \subset J(h).$$

In other words we have proved

THEOREM 7. *In the above notation Λ is a* LYM *posets.*

For literature on LYM posets see [**4**, **5**, **10**].

§5. *The number of edges in a bipartite graph.* Let $k, m \in \mathbb{Z}$ be fixed. Consider a bipartite graph Γ with a set V of m vertices. We suppose that $V = U \cup W$ where $U \subset \mathscr{I}(k)$ and $W \subset \mathscr{I}(k-1)$ and two vertices are joined by an edge, if, and only if, one is a proper subset of the other.

Example 1. $k = 3, m = 23$. Let U be the first $11 = \binom{5}{3}+\binom{2}{2}$ sets of $\mathscr{I}(3)$. Let $W = \Delta U$ so W is the first $12 = \Delta(11) = \binom{5}{2}+\binom{2}{1}$ sets of $\mathscr{I}(2)$. Here Γ has 33 edges.

Example 2. $k = 3, m = 23$. Let U, W be the first 12, 11 sets of $\mathscr{I}(3)$, $\mathscr{I}(2)$ respectively. Here Γ has 34 edges.

We study the maximum attainable number $\mu(k, m)$ of edges in Γ over all choices of V. The reader can easily evaluate $\mu(2, m)$. Example 1 shows that, if $m \geqslant N+\Delta N$, then $\mu(k, m) \geqslant kN$. That this does not always give the best bound is shown by Example 2. However numbers N with property (6), and which therefore lie in $\mathscr{L}$, are not too far apart. Hence the theorem which we next present shows that the bound kN for μ from Example 1 is of the correct asymptotic form.

THEOREM 8. *If*

$$\binom{k^2+k}{k} \leqslant N \in \mathscr{L} \quad \textit{and} \quad m = N+\Delta_k N,$$

then $\mu(k, m) = kN$.

Proof. Let the number of vertices in U of degree k be r, and the number of other vertices in U be s, so $|U| = r+s$. Let e be the number of edges on U so $e \leqslant kr+(k-1)s$. If $r+s < N$, then $e < kN$ so we assume $N \leqslant r+s$.

Case $N < r$. Clearly $\Delta N \leqslant \Delta r$ and by Kruskal's theorem $\Delta r \leqslant |W| = m-r-s$. Hence

$$N + \Delta N < r+\Delta N \leqslant r+\Delta r \leqslant |U|+\Delta r \leqslant |U|+|W| = m = N+\Delta N,$$

showing that this case never arises.

Case $r \leqslant N$. Now $N \in \mathscr{L}$ implies that $\delta \leqslant \Delta r$ where $\delta = rf(N)$. As before $\Delta r \leqslant m-r-s$, so $r+s \leqslant m-\delta$. Hence

$$e \leqslant kr+(k-1)s = k(r+s)-s \leqslant k(m-\delta)-s.$$

Using Lemma 1 with $a = k^2+k$ shows that $f(N) < 1/k$, and we assumed that $N \leqslant r+s$, and $0 \leqslant s$, so combining these gives

$$1 \leqslant (r+s)/N \leqslant (r/N)+(s/k\Delta N).$$

Multiplying this inequality by $k\Delta N$ gives

$$k\Delta N \leqslant (kr\Delta N/N)+s = \delta k+s.$$

Adding kN to both sides gives

$$k(N+\Delta N) = km \leqslant \delta k+s+kN.$$

Hence $e \leqslant k(m-\delta)-s \leqslant kN$, proving the theorem.

§6. *Counting containments in a set of subsets.* If $\mathscr{F}$ is a set of subsets of the set $X = \{1, 2, ..., n\}$, we let $v(\mathscr{F})$ be the number of containments in $\mathscr{F}$, so

$$v(\mathscr{F}) = |\{(Y, Z) : Y, Z \in \mathscr{F}, Y \subset Z, Y \neq Z\}|.$$

For $1 \leqslant m \leqslant 2^n$ we study the function $\lambda(m, n)$ defined as the maximum value of $v(\mathscr{F})$ over all $\mathscr{F}$ of cardinality $|\mathscr{F}| = m$. Trivially

$$\lambda(m, n) \leqslant \binom{m}{2},$$

and we can have equality for $1 \leqslant m \leqslant n+1$ with $\mathscr{F}$ a chain. Also it is not hard to see that

$$\lambda(2^n, n) = 3^n-2^n.$$

Usually we think of m as a function $m(n)$ of n.

Example 3. Let $n = pq$ and $X = X_1 \cup ... \cup X_p$ where $X_1 = \{1, 2, ..., q\}$, $X_2 = \{q+1, q+2, ..., 2q\}$ and so on. Next let $\mathscr{F}$ be $\mathscr{F}_1 \cup ... \cup \mathscr{F}_p$ where

$$\mathscr{F}_i = \{Y : X_1 \cup ... \cup X_{i-1} \subset Y \subset X_1 \cup ... \cup X_i\} \quad \text{for} \quad 1 \leqslant i \leqslant p.$$

Then $m = |\mathscr{F}| = p2^q - (p-1)$ and

$$v(\mathscr{F}) = p(3^q - 2^q) + \binom{p}{2} 2^{2q} - (p-1).$$

Hence as $p, q \to \infty$ we have $m^{1/n} \to 1$ and $v(\mathscr{F}) = (1+o(1))\binom{m}{2}$.

This example gives half of

THEOREM 9. *$\lambda(m, n) = (1+o(1))\binom{m}{2}$, if, and only if, $m^{1/n} \to 1$.*

Example 4. This comes from the case $p = 2$ of Example 3, so $n = 2q$. For $\mathscr{F}$ we take any $\lfloor m/2 \rfloor$ and $\lceil m/2 \rceil$ members of $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively, other than X_1. Then $|\mathscr{F}| = m \leqslant 2(2^q - 1)$ and $v(\mathscr{F}) \geqslant \lfloor m/2 \rfloor \lceil m/2 \rceil$. Thus we see that for $c \leqslant \sqrt{2}$ we can have $m \sim c^n$ and $v(\mathscr{F}) \sim m^2/4$. This leads us to make

CONJECTURE 1. *If $\sqrt{2} < c$ and $m(n) > c^n$, then $\lambda(m, n) = o(m^2)$.*

In this connection we mention

CONJECTURE 2. (Erdős). *If $n = 2r$ and $m = 2^{r+1}$ and*

$$\pi(\mathscr{F}) = |\{Y, Z) : Y, Z \in \mathscr{F}, \quad Y \cap Z = \varnothing\}|,$$

then $\pi(\mathscr{F})$ is asymptotically maximal when

$$\mathscr{F} = \{Y : Y \subset X, \ Y \cap W = \varnothing \text{ or } Y \cap (X \setminus W) = \varnothing\},$$

where $W = \{1, 2, ..., r\}$.

Since $Y \cap Z = \varnothing$, if, and only if, $Y \subset X \setminus Z$, the conjectures raise a

PROBLEM. *Given two sets $\mathscr{G}, \mathscr{H}$ of subsets of X put*

$$\pi(\mathscr{G}, \mathscr{H}) = |\{Y, Z) : Y \in \mathscr{G}, Z \in \mathscr{H}, Y \cap Z = \varnothing\}|.$$

For fixed $g, h \in \mathbb{Z}$ what is the maximum value of $\pi(\mathscr{G}, \mathscr{H})$ over all $\mathscr{G}, \mathscr{H}$ with $|\mathscr{G}| = g$ and $|\mathscr{H}| = h$?

Proof of Theorem 9. Our Example 3 showed that we can obtain the number of containments in the statement of the theorem, so now we show that we cannot obtain more.

The hypothesis $m^{1/n} \to 1$ means that given $c > 1$ there is an $N = N(c)$ such that $m = m(n) < c^n$ for $n > N$. Using Stirling's approximation for factorials one sees the following facts. For the given c there are integers L, M and a real number d in $1 < d < c$ such that $\binom{n}{\lceil n/L \rceil} < d^n$ for $n > M$.

Now let $n > M, N$ and assume that $\mathscr{F}$ is a set of $m = \lfloor c^n \rfloor$ subsets of $\{1, 2, ..., n\}$. For $1 \leqslant i \leqslant L$ put

$$\mathscr{F}_i = \{Y : Y \in \mathscr{F}, n(i-1)/L < |Y| < ni/L\}.$$

In case the empty set $\varnothing$ is in $\mathscr{F}$ we ignore it. We proceed to get the crude bound $o(c^{2n})$ for the number of containments in any one $\mathscr{F}_i$. Suppose that $Z \in \mathscr{F}_i$. If $n(i-1)/L < j < |Z|$, the number of subsets Y of X with $Y \subset Z$ and $|Y| = j$ is

$$\binom{|Y|}{j} = \binom{|Y|}{|Y|-j} < \binom{n}{|Y|-j} \leqslant \binom{n}{\lceil n/L \rceil} < d^n.$$

So the number of containments in $\mathscr{F}_i$ is bounded by d^n times the number of choices for Z times the number of choices for j. The former number is at most c^n and the latter number is at most $\lceil n/L \rceil$. Hence we have the bound $o(c^{2n})$. Clearly the number of containments $Y \subset Z$ where Y, Z can both lie in any of $\mathscr{F}_1, \ldots, \mathscr{F}_L$ is $Lo(c^{2n}) = o(c^{2n})$.

The above bound $o(c^{2n})$ is much smaller than the crude bound $(1-(1/L))\binom{m}{2}$ for the number of containments $Y \subset Z$ in $\mathscr{F}$ where $Y \in \mathscr{F}_i$ and $Z \in \mathscr{F}_j$ but $i < j$. The number of such containments will certainly not exceed the number we should have under the following two assumptions, which produce the bound. We first assume that $Y \in \mathscr{F}_i$ and $Z \in \mathscr{F}_j$ and $i < j$ always imply that $Y \subset Z$. Second we assume that all $\mathscr{F}_i$ have $\lfloor m/L \rfloor$ or $\lceil m/L \rceil$ members. This ends the proof.

References

1. R. Ahlswede and G. O. H. Katona. Graphs with maximal number of adjacent pairs of edges. *Acta Math. Acad. Sci. Hungar.*, 32 (1978), 97–120.
2. D. E. Daykin. A simple proof of the Kruskal–Katona theorem. *J. Combinatorial Theory*, 17 (1974), 252–253.
3. D. E. Daykin. An algorithm for cascades giving Katona-type inequalities. *Nanta Math.*, 8 (1975), 78–83.
4. D. E. Daykin and P. Frankl. Inequalities for subsets of a set and KLYM posets. *SIAM J. on Algebraic and Discrete Methods*, 4 (1983), 67–69.
5. R. L. Graham and L. H. Harper. Some results on matching in bipartite graphs. *SIAM J. Appl. Math.*, 17 (1969), 1017–1022.
6. L. H. Harper. Optimal assignments of numbers to vertices. *J. Soc. Indust. Appl. Math.*, 12 (1964), 131–135.
7. L. H. Harper. Optimal numberings and isoperimetric problems on graphs. *J. Combinatorial Theory*, 1 (1966), 385–393.
8. G. Katona. A theorem for finite sets. *Theory of Graphs, Proc. of Colloquium, Tihany, Hungary* (1966), Eds. P. Erdős and G. Katona, 187–207.
9. D. J. Kleitman. A conjecture of Erdős–Katona on commensurable pairs among subsets of an n-set. *Theory of Graphs, Proc. of Colloquium, Tihany, Hungary* (1966), Eds. P. Erdős and G. Katona, 215–218.
10. D. J. Kleitman. An extremal property of antichains in partial orders: The LYM property and some of its implications and applications. *Combinatorics*, Eds. M. Hall and J. H. van Lint, *Math. Centre Tracts*, Vol. 55 (Math. Centre, Amsterdam, 1974) 77–90.
11. J. B. Kruskal. The number of simplices in a complex. *Math. Optimization Techniques*, Ed. R. Bellman (University of California Press, 1963), 251–278.

Dr. D. E. Daykin,
Dept. of Mathematics,
The University of Reading,
Whiteknights,
Reading RG6 2AX

Dr. P. Frankl,
C.N.R.S.,
Paris,
France.

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