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Linear Dependencies among Subsets of a Finite Set

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A 0–design for given integers k, t; $k > t \geq 0$ is a family of k-subsets $\{F_1 \cdots F_m\}$ along with rational numbers $\alpha_1 \cdots \alpha_m$ such that for any t-element set G, $\sum_{G \subseteq F_i} \alpha_i = 0$ holds. We prove that if $\binom{t+1}{1}, \ldots, \binom{t+1}{k-t-1}$ have a common divisor d which does not divide α_i than $\exists i'$ such that $|F_i \cap F_{i'}| = k-t-1$ (Theorem 1).

We deduce some extremal set theoretic consequences (Theorems 2 and 3), relating to problems of Erdös.

1. Introduction

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of n elements. For an integer k, $0 \leq k \leq n$ we denote by $\binom{X}{k}$ the k-subsets of X. Further we denote by $V(n, k)$ the free vector space generated by the elements of $\binom{X}{k}$ over the rationals, i.e. $V(n, k)$ is consisting of all the formal rational linear combinations of k-subsets of X. We will also think of an element $\{x_{i_1}, x_{i_2}, \ldots x_{i_k}\}$ of $\binom{X}{k}$ as square—free monomial $x_{i_1}x_{i_2} \cdots x_{i_k}$. Let t be a fixed integer, $0 \leq t \leq k$.

Definition. An element

$$\nu = \sum_{F \subseteq \binom{X}{k}} \alpha_F F$$

of $V(n, k)$ is a 0-design if for every $G \in \binom{X}{t}$ the following holds

$$\sum_{G \subseteq F} \alpha_F = 0.$$

Obviously the 0-designs form a subspace of dimension $\binom{n}{k} - \binom{n}{t}$ of $V(n, k)$. We say that the 0-design $\nu = \sum \alpha_F F$ is *primitive* if all the α_Fs are integers and their g.c.d. is 1. Notice that for every 0-design there exists a unique rational number c such that $c\nu$ is primitive. The simplest 0-designs have the following form

$$\nu = (x_{i_1} - x_{i_2})(x_{i_3} - x_{i_4}) \cdots (x_{i_{2t+1}} - x_{i_{2t+2}})x_{i_{2t+3}} \cdots x_{i_{k+t+1}}(x_{i_j} \neq x_{i_{j'}} \text{ for } j \neq j').$$

Such 0-designs are called *basic*. Graham, Li and Li [8] exhibited a basis for the space of all 0-designs consisting of *basic* 0-designs. We shall only use the following corollary (which was first proved by Graver and Jurkat [9]).

Corollary [8, 9]. *Every primitive* 0*-design can be expressed as an integral linear combination of* basic 0*-designs.*

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Our main theorem is the following.

THEOREM 1. *Let* $\nu=\sum_{F\in\binom{X}{k}}\alpha_F F$ *be a primitive* 0*-design. Let d be the g.c.d. of*

$$\left\{\binom{t+1}{k-t-1-i}, 0\le i<k-t-1\right\}(d=+\infty \text{ for } k=t+1).$$

Let $F_0\in\binom{X}{k}$ *be such that* $d|\alpha_{F_0}$ *then there exists an* $F_1\in\binom{X}{k}$ *such that* $d|\alpha_{F_1}$, *in particular* $\alpha_{F_1}\ne 0$, *and such that* $|F_0\cap F_1|=k-t-1$.

In [1] Erdös stated the following.

CONJECTURE 1. *Let* $0\le l<k$ *be fixed, and suppose* $\mathscr{F}\subseteq\binom{X}{k}$ *for any F, where* $F'\in\mathscr{F}|F\cap F'|\ne l$ *holds. Then for* $(k,l)\ne(3,1)$ *and* $n>n_0(k,l)$ *one has*

$$|\mathscr{F}|\le\max\left\{\binom{n-l-1}{k-l-1},\binom{n}{l}\Big/\binom{k}{l}\right\}.$$

The case $l=0$ is contained in Erdös–Ko–Rado [3]. The case $l=1$ was already earlier conjectured by Erdös and Sós (cf. [2]) and it was proved by Frankl [4]. Frankl [5] proved that for $k>3l+1$ one has

$$|\mathscr{F}|\le(1+o(1))\binom{n-l-1}{k-l-1}.$$

Most recently Frankl and Wilson [7] proved

$$|\mathscr{F}|\le\binom{n}{k-l-1}$$

for $k\ge 2l+1$ and $k-l$ is a prime power. We prove

THEOREM 2. *If* $k-l$ *has a prime power divisor which is greater than* l *then, with the conditions of Conjecture* 1, *one has*

$$|\mathscr{F}|\le\binom{n}{k-l-1},\ \text{moreover}\ \left|\left\{G\in\binom{X}{r}:\exists F\in\mathscr{F}, F\supset G\right\}\right|\ge|\mathscr{F}|,$$

for every $k-l-1\le r\le k$.

Notice that the condition of Theorem 2 is satisfied for all $k\ge 3$ if $l=1$ also if $k-l$ is a prime power and $k>2l$. In [1] Erdös also asked to determine $m(l,n)=\max\{|\mathscr{F}|:\mathscr{F}\subseteq 2^X|F\cap F'|\ne l$ for every $F,F'\in\mathscr{F}\}$. For $l=0$ trivially $m(l,n)=2^{n-1}$ and $m(1,n)$ was determined by Frankl [6], who made the following conjecture:

CONJECTURE 2. *For* $n>n_0(l)$

$$m(l,n)=\begin{cases}\sum_{i=0}^{l-1}\binom{n}{i}+\sum_{i>(n+l)/2}\binom{n}{i} & \text{if } n+l \text{ is odd}\\ \sum_{i=0}^{l-1}\binom{n}{i}+\binom{n-1}{(n+l)/2}\sum_{i>n+l/2}\binom{n}{i}, & \text{if } n+l \text{ is even.}\end{cases}$$

Here we prove

THEOREM 3. *Set* $q(l)=\prod_{p^\alpha\le l<p^{\alpha+1}} p^\alpha$ *(p prime). Then*

$$m(l,n)=\begin{cases}\sum_{i>(n+l)/2}\binom{n}{i}+0(n^{q(l)+l}) & \text{if } n+l \text{ is odd,}\\ \binom{n-1}{(n+l)/2}+\sum_{i>(n+l/2)}\binom{n}{i}+0(n^{q(l)+l}), & \text{if } n+l \text{ is even.}\end{cases}$$

REMARK. Conjecture 1 is the analogue of the Erdös–Ko–Rado [3] theorem

$$\left(\text{if } |F\cap F'|>l,\ n>n_0(k,l) \text{ then } |\mathcal{F}|\le\binom{n-l-1}{k-l-1}\right),$$

Conjecture 2 is an analogue of a theorem of Katona [10] (if $|F\cap F'|>l$ and $|F|$ is not restricted then

$$|\mathcal{F}|\le\sum_{i>(n+l)/2}\binom{n}{l} \text{ if } n+l \text{ is odd and } |\mathcal{F}|\le\binom{n-1}{(n+l)/2}+\sum_{i>(n+1)/2}\binom{n}{i}$$

if $n+l$ is even.) In the case $l=0$ there are many ways to have equality in Katona's inequality. However, for $l\ge 1$, Katona proved that the only families which achieve equality are the following.

(a) $n+l$ is odd and

$$\mathcal{F}=\left\{F\subseteq X: |F|>\frac{n+l}{2}\right\},$$

(b) $n+l$ is even, $x\in X$ is fixed,

$$\mathcal{F}=\left\{F\subseteq X: |F|>\frac{n+l}{2}\right\}\cup\binom{X-\{x\}}{(n+l)/2}.$$

2. THE PROOF OF THEOREM 1

We start with a lemma.

LEMMA 1. *Let* $F_0\in\binom{X}{k}$, $0\le s\le k$, *and* $U=\sum\alpha_F F$, *F a basic* 0*-design, i.e.*

$$F=(x_{i_1}-x_{i_2})\cdots(x_{i_{2t+1}}-x_{i_{2t+2}})x_{i_{2t+3}}\cdots x_{i_{k+t+1}}.$$

Then

$$\sum_{|F\cap F_0|=s}\alpha_F=0, \text{ unless } |F_0\cap\{x_{i_{2l+1}},x_{i_{2l+2}}\}|=1 \quad \text{for all } 0\le l\le t.$$

PROOF. Suppose by symmetry $|F_0\cap\{x_{i_1},x_{i_2}\}|\ne 1$. Let $\pm x_{i_1}y_2y_3\cdots y_k$ be a term in the expansion of u. Then $\pm x_{i_2}y_2\cdots y_k$ is also a term in the expansion of u, moreover it is of exactly opposite sign. One has

$$|\{x_{i_1},y_2,\ldots,y_k\}\cap F_0|=|\{x_{i_2},y_2,\ldots,y_k\}\cap F_0|,$$

and all the terms in the expansion of u contain either x_1 or x_2. Thus the statement of the lemma follows.

To prove the theorem it is sufficient to show that (under the hypothesis of the theorem).

$$\sum_{\substack{F\in\binom{X}{k}\\ |F\cap F_0|=k-t-1}} \alpha_F \quad \text{is not divisible by } d. \tag{1}$$

To calculate this sum we write ν as an integer linear combination of basic 0-designs:

$$\nu = \sum_{\langle x_{i_1},\dots x_{i_{k+t+1}}\rangle} \alpha(i_1, i_2, \dots, i_{k+t+1})(x_{i_1}-x_{i_2})\dots(x_{i_{2t+1}}-x_{i_{2t+2}})x_{i_{2t+3}}\dots x_{i_{k+t+1}}$$

$$= \sum_{F\in\binom{X}{k}} \alpha_F F$$

In view of Lemma 1, to count (1) it is sufficient to consider the 0-designs for which $F_0\cap\{x_{i_1}, x_{i_2}, \dots, x_{i_{2t+2}}\}=\{x_{i_1}, x_{i_3}, \dots, x_{i_{2t+1}}\}$. Let $\sum_{F\in\binom{X}{k}} \beta_F F$ be such a basic 0-design. Then one has

$$\sum_{\substack{|F\cap F_0|=k-t-1\\ F\in\binom{X}{k}}} \beta_F = (-1)^{k-i}\binom{t+1}{k-t-1-i}, \text{where } i=|F_0\cap\{x_{2t+3},\dots,x_{k+t+1}\}| \tag{2}$$

Notice that $i=k-t-1$ corresponds to the basic 0-designs in which F_0 occurs as a term, and it has coefficient 1. Thus (1) extended over such basic 0-designs only gives $\alpha_{F_0}(-1)^{t+1}$, which is of course not divisible by d.

As by definition $d\,\Big|\binom{t+1}{k-t-1-i}$ for $0\le i<k-t-1$, thus taking into consideration (2) we deduce for (1).

$$\sum_{\substack{|F\cap F_0|=k-t-1\\ F\in\binom{X}{k}}} \equiv (-1)^{t+1}\alpha_{F_0} \pmod d \text{ which yields the statement.}$$

3. The Proof of Theorem 2

Suppose $\mathcal{F}\subset\binom{X}{k}$ has the desired property, i.e., for $F, F'\in\mathcal{F}$, $|F\cap F'|\ne l$ holds. Let us set $t=k-l-1$. Recall that $V(n, t)$ is the free vector space of all the formal rational linear combinations of t-subsets of X. Let us associate with every $F\in\mathcal{F}$ a vector

$$u(F)=\sum_{G\subseteq F, |G|=t} G \text{ of } V(n,t).$$

As $\dim V(n,t)=\binom{n}{t}$ the first statement of Theorem 2 will follow if we prove that the vectors $u(F)$ are linearly independent.

Suppose the contrary and let

$$\sum_{F\in\mathcal{F}} \alpha_F u(F)=0 \text{ be a nontrivial linear combination,}$$

but this means that

$$\sum_{F\in\mathcal{F}} \alpha_F F \text{ is a nontrivial 0-design.}$$

By multiplying, if necessary, all the coefficients α_F with the same rational number we may suppose the α_F's are integers with g.c.d. equal to 1 i.e. the 0-design is primitive.

Let $q=p^\alpha$ be the prime power which divides $k-l=t+1$ but which is greater than $l=k-t-1$. Then it is easy to check that $p\,\Big|\binom{t+1}{i}$ for $1\leq i\leq l=k-t-1$. Now with the notation of Theorem 1 we have $p|d$. As $\sum \alpha_F F$ is a primitive 0-design, there exists an $F_0\in\mathscr{F}$ such that $p\nmid\alpha(F_0)$. Now Theorem 1 implies the existence of $F_1\in\mathscr{F}$ with $|F_0\cap F_1|=k-t-1=l$, a contradiction, proving

$$|\mathscr{F}|\leq\binom{n}{k-l-1}.$$

We have proved that the vectors $u(F)\in V(n,t)$ are independent. This implies the independence of the vectors $u_r(F)\in V(n,r)$:

$$U_r(F)=\sum_{H\in\binom{X}{r},H\subseteq F} H \quad \text{for } k\geq r\geq k-l-1.$$

In fact any linear dependence $\sum\alpha(F)u_r(F)=0$ is also a linear dependence for the vectors $u_{r'}(F)$, $r\geq r'$, i.e., $\sum\alpha(F)u_{r'}(F)=0$. Thus for every $k-l-1\leq r\leq k$ the $u_r(F)$s span a vector space of dimension $|\mathscr{F}|$, and consequently its support $\{G\in\binom{X}{r}:\exists F\in\mathscr{F}, G\subseteq F\}$ has cardinality at least $|\mathscr{F}|$.

4. The Proof of Theorem 3

Set $\mathscr{F}_s=\{F\in\mathscr{F}:|F|=s\}$, where $\mathscr{F}\subseteq 2^X$ is s.t. $|\mathscr{F}|=m(l,n)$ and $|F\cap F'|\neq l$ for every $F,F'\in\mathscr{F}$. For $s>q(l)+l$ of course $s-l$ has a prime power divisor which is greater than l. Set

$$\mathscr{F}_s^l=\left\{G\in\binom{X}{s-l}:\exists F\in\mathscr{F}_s, G\subset F\right\}.$$

Then by Theorem 2 we have $|\mathscr{F}_s^l|\geq|\mathscr{F}_s|$.

Proposition 1. *For $q(l)+l<s<(n+l)/2$ we have*

$$|\mathscr{F}_s|+|\mathscr{F}_{n+l-s}|\leq\binom{n}{s-l}=\binom{n}{n+l-s}. \tag{3}$$

The Proof of the Proposition. Let us set $\bar{\mathscr{F}}_{n+l-s}=\{X-F:F\in\mathscr{F}_{n+l-s}\}$. Suppose $\mathscr{F}_s^l\cap\bar{\mathscr{F}}_{n+l-s}\neq\emptyset$. Let G be an element of the intersection, thus there exist F, F' such that $G\subseteq F\in\mathscr{F}_s$, $X-G=F'\in\mathscr{F}_{n+l-s}$. But this implies $|F\cap F'|=l$, a contradiction. Thus $\mathscr{F}_s\cap\bar{\mathscr{F}}_{n+l-s}=\emptyset$ yielding $|\mathscr{F}_s^l|+|\bar{\mathscr{F}}_{n+l-s}|\leq\binom{n}{s-l}$. As $|\mathscr{F}_s|\leq|\mathscr{F}_s^l|$ and $|\bar{\mathscr{F}}_{n+l-s}|=|\mathscr{F}_{n+l-s}|$ (3) follows.

Now sum up (3) for $q(l)+l<s<(n+l)/2$ to obtain

$$\sum_{\substack{q(l)+l<s<n-q(l)\\ s\neq(n+l)/2}}\leq\sum_{(n+l)/2<s<n-q(l)}\binom{n}{s}.$$

Taking into consideration that for every $0\leq s\leq n$ we have $|\mathscr{F}_s|\leq\binom{n}{s}$ and $|\mathscr{F}|=\sum_{0\leq s\leq n}|\mathscr{F}_s|$ the statement of Theorem 3 follows for $n+l$ odd and for $n+l$ even as well as soon as we show

$$|\mathscr{F}_{(n+l)/2}| \leqslant \binom{n-1}{(n+l)/2}. \tag{4}$$

To show (4) observe that for $F, F' \in \mathscr{F}_{(n+l)/2}$, $|F \cap F'| = l$ is equivalent to $|F \cup F'| = X$ i.e. to $(X-F) \cap (X-F') = \varnothing$. Applying the Erdös–Ko–Rado theorem (case $l=0$ of Conjecture 1) to $\bar{\mathscr{F}}_{(n+l)/2} = \{X-F: F \in \mathscr{F}_{(n+l)/2}\}$ we obtain

$$|\mathscr{F}_{(n+l)/2}| = |\bar{\mathscr{F}}_{(n+l)/2}| \leqslant \binom{n-1}{(n-l)/2-1} = \binom{n-1}{(n+l)/2}.$$

NOTE ADDED IN PROOF. Frankl and Füredi—using the methods of this paper and a stronger version of Theorem 2—have proved Conjecture 2. By completely different methods they proved Conjecture 1 for $k \geqslant 2l+2$. Examples of the first author show that it is not true for $k \leqslant 2l+1$. Both these results appeared or will appear in Journal of Combinatorial Theory A.

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