SETS OF FINITE SETS SATISFYING UNION CONDITIONS

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Abstract. Let $\mathscr{F}$ denote a set of subsets of $X = \{1, 2, ..., n\}$. Let $\deg(i)$ be the number of members of $\mathscr{F}$ containing i and $\operatorname{val}(\mathscr{F}) = \min\{\deg(i) : i \in X\}$. Suppose no k members of $\mathscr{F}$ have union X. We conjecture $\operatorname{val}(\mathscr{F}) \leqslant 2^{n-k-1}$ for $k \geqslant 3$. This is known for $n \leqslant 2k$ and we prove it for $k \geqslant 25$. For $k = 2$ an example has $\operatorname{val}(\mathscr{F}) > 2^{n-2}(1-n^{-0\cdot 651})$ and we prove $\operatorname{val}(\mathscr{F}) \leqslant 2^{n-2}(1-n^{-1})$. We also prove that if the union of k sets one from each of $\mathscr{F}_1, ..., \mathscr{F}_k$ has cardinality at most $n-t$ then $\min\{\text{cardinality } \mathscr{F}_j\} < 2^n\alpha^t$ where $\alpha^k = 2\alpha - 1$ and $\frac{1}{2} < \alpha < 1$.

§1. *Introduction and Statement of Results.* Let k, n, t be integers $2 \leqslant k$ and $n \geqslant t \geqslant 1$. We shall always let $\mathscr{F}$ denote a set of subsets of the set $X = \{1, 2, ..., n\}$. We say $\mathscr{F}$ has property $P(n, k, t)$ if the union of any k members of $\mathscr{F}$ has cardinality at most $n-t$. Let s denote a non-negative integer with $t+ks \leqslant n$. Then $E(n, k, t, s)$ is the set of all $\mathscr{F}$, for which there exists a subset $Y = Y(\mathscr{F})$ of X with cardinality $|Y| = t+ks$, such that $\mathscr{F}$ has the form $\mathscr{F} = \{F \subset X : |F \cap Y| \leqslant s\}$. Clearly, if $\mathscr{F} \in E(n, k, t, s)$, then $\mathscr{F}$ has $P(n, k, t)$. We write $e(n, k, t, s)$ for the common cardinalities of members of $E(n, k, t, s)$. In particular $e(n, k, t, 0) = 2^{n-t}$ and $e(n, k, 1, 1) = (k+2)2^{n-k-1}$. Consider

CONJECTURE 1. (Erdős and Frankl [**6**]). *If $\mathscr{F}$ has $P(n, k, t)$ and $|\mathscr{F}|$ is maximal, then $\mathscr{F} \in E(n, k, t, s)$ for some s, unless $k = 2$, $t = 1$.*

The conjecture is true in the following cases.

Case 1. $t = 1$. Trivial (*cf.* [**5**] p. 319(ii)). The case $k = 2$, $t = 1$ is only excluded because there are examples not of the form E.

Case 2. (Katona [**8**]). $k = 2$.

Case 3. (Frankl [**6**]). $k = 3$, $t = 2$ and then $s = 0$.

Case 4. (Frankl [7]). $t \leqslant k2^k/150$.

Case 5. (Frankl [7]). $k \geqslant 6$, $t \leqslant (e+1)^{-1}(2^{k-1}-k+1)-1$ and then $s = 0$.

Consider k sets $\mathscr{F}_1, ..., \mathscr{F}_k$. We give

THEOREM 2. *Suppose $k \geqslant 3$ and the union of any k sets, one from each $\mathscr{F}_i$, has cardinality at most $n-t$. Let $\alpha = \alpha(k)$ be the unique root of $x^k - 2x + 1 = 0$ in $\frac{1}{2} < \alpha < 1$. Then* $\min|\mathscr{F}_i| < 2^n\alpha^t$. *Also there is an example with* $\min|\mathscr{F}_i| > c2^n\alpha^t/\sqrt{t}$, *where c is an absolute constant.*

We say $\mathscr{F}$ is *covering* if $\bigcup (F \in \mathscr{F})F = X$. There is

THEOREM 3. (Brace and Daykin [1]). *If $\mathscr{F}$ is covering and has $P(n,k,1)$ and $|\mathscr{F}|$ is maximal, then $\mathscr{F} \in E(n,k,1,1)$.*

This leads us to make

CONJECTURE 4. *Suppose $k \geqslant 3$ and the union of any k sets, one from each $\mathscr{F}_i$, has cardinality at most $n-1$. If each $\mathscr{F}_i$ is covering, then* $\min |\mathscr{F}_i| \leqslant e(n,k,1,1)$.

For $i \in X$ the degree of i is the number of members of $\mathscr{F}$ which contain i. The minimum of these degrees is called the valency $\operatorname{val}(\mathscr{F})$. For example if $\mathscr{F} \in E(n,k,1,1)$ then $\operatorname{val}(\mathscr{F}) = 2^{n-k-1}$. We say $\mathscr{F}$ has no *k-cover* if the union of any k members of $\mathscr{F}$ is not X. We make

CONJECTURE 5. *If $\mathscr{F}$ has no k-cover and $k \geqslant 3$, then* $\operatorname{val}(\mathscr{F}) \leqslant 2^{n-k-1}$.

This conjecture is true in

Case 1. (Daykin [4]). $n \leqslant 2k$.

We shall prove it true in

Case 2. $k \geqslant 25$.

When $\mathscr{F}$ has no 2-cover the problem† is harder and we present

THEOREM 6. *If $\mathscr{F}$ has no 2-cover then* $\operatorname{val}(\mathscr{F}) \leqslant 2^{n-2}(1-n^{-1})$. *There is an example with $n = 7^p$ and* $\operatorname{val}(\mathscr{F}) > 2^{n-2}(1-n^{-0\cdot 651})$.

§2. *The Proofs. Proof of Theorem* 2. This is almost a copy of Frankl's proof in [6], so we leave out some details. Let $1 \leqslant i < j \leqslant n$ and let $\mathscr{H}$ be a set of subsets of X. We define a set $A_{ij}(\mathscr{H})$ of subsets of X by

$$A_{ij}(\mathscr{H}) = \{A_{ij}(H) : H \in \mathscr{H}\},$$

where

$$A_{ij}(H) = \begin{cases} (H \setminus \{j\}) \cup \{i\}, & \text{if } j \in H, i \notin H, (H\setminus\{j\}) \cup \{i\} \notin \mathscr{H}, \\ H, & \text{otherwise}. \end{cases}$$

Notice that $|A_{ij}(\mathscr{H})| = |\mathscr{H}|$.

For convenience we replace all the sets in each $\mathscr{F}_g$ by their complements. The condition on $\mathscr{F}_1, \ldots, \mathscr{F}_k$ then becomes that the intersection of any k sets, one from each $\mathscr{F}_g$, has cardinality at least t. Call this the intersection condition. It is clear that

† This problem was discussed by Daykin at the Oxford Conference of 1972. There Paul Erdős offered a £5.00 prize for the best example. This prize was won by Daykin for the example given here. However he made mistakes when writing up [3] and these unfortunately make it appear that Erdős gave the prize for nothing.

$A_{ij}(\mathscr{F}_1), \ldots, A_{ij}(\mathscr{F}_k)$ satisfy this intersection condition. Repeating A_{ij} for various i,j we change $\mathscr{F}_1, \ldots, \mathscr{F}_k$ into $\mathscr{G}_1, \ldots, \mathscr{G}_k$, which satisfy both the intersection condition and

$$A_{ij}(\mathscr{G}_g) = \mathscr{G}_g \quad \text{for} \quad 1 \leqslant i < j \leqslant n \quad \text{and} \quad 1 \leqslant g \leqslant k\,. \tag{1}$$

Let $T = \{1, 2, \ldots, t-1\}$ and $K = \{0, 1, \ldots, k-1\}$. If f, h are integers with $0 \leqslant h \leqslant k-1$, put

$$f + K_h = \{f + j \ ; \ j \in K \setminus \{h\}\}\,,$$

$$S_h^\infty = T \cup (t + K_h) \cup (k + t + K_h) \cup (2k + t + K_h) \cup \ldots\,,$$

$$S_h = X \cap S_h^\infty\,.$$

If $|S_h| \neq |S_0|$ for some h, then $n \in S_h$ and we remove n from S_h without changing notation. Suppose $S_0 \in \mathscr{G}_g$ for some g. Then because (1) holds we have $S_h \in \mathscr{G}_g$ for all h. Hence, if $S_0 \in \mathscr{G}_g$ for every g, then $S_0 \in \mathscr{G}_1, S_1 \in \mathscr{G}_2, \ldots, S_{k-1} \in \mathscr{G}_k$. However this contradicts the intersection condition because $S_0 \cap S_1 \cap \ldots \cap S_{k-1} = T$ and $|T| = t-1$. Hence by symmetry we may assume $s_0 \notin \mathscr{G}_1$.

We associate with every subset T of the set $\{1, 2, \ldots\}$ a random walk in the real plane as follows. We start from the origin $(0, 0)$. If after q steps we are at (x, y), then we move to $(x, y+1)$, if $q+1 \in T$, otherwise, we move to $(x+1, y)$. We denote this walk by $W(T)$.

PROPOSITION 1. *If $G \in \mathscr{G}_1$, then $W(G)$ hits the line $y = (k-1)x + t$.*

Proof. The walk $W(S_0^\infty)$ does not hit the line but is as close as possible to it. Thus every random walk either hits the line or lies entirely under $W(S_0^\infty)$. If $W(G)$ lies under $W(S_0^\infty)$, we can obtain $W(S_0)$ from $W(G)$ by iteration of the operations: (i) replace $i+1$ by i in G; and (ii) adjoin the element i to G. Notice that operation (i) changes $\lrcorner$ to $\ulcorner$ in $W(G)$, while operation (ii) lifts $W(G)$. Applying these operations to G produces sets which are in $\mathscr{G}_1$ because $\mathscr{G}_1$ is an up-set and (1) holds. In this way we eventually deduce that $S_0 \in \mathscr{G}_1$ contradicting our assumption.

PROPOSITION 2. *Consider the random walk which starts from the origin and moves up or to the right with equal probabilities $\frac{1}{2}, \frac{1}{2}$. The probability that this walk hits the line $y = (k-1)x + t$ is $\alpha(k)^t$.*

Proof. The result is part of more general theorems in [**9**] Chapter XII.

The proof of Theorem 2 now follows by observing firstly, that the set of all subsets of X gives a model for random walks with probabilities $\frac{1}{2}, \frac{1}{2}$, and secondly, that the probability of hitting the given line in $\leqslant n$ steps is strictly less than the probability of ever hitting it.

Proof of Case 2 *of Conjecture* 5. We shall use Case 5 of Conjecture 1. So we let m be the least integer such that

$$k < (e+1)^{-1}(2^{m-1} - m + 1) - 1\,. \tag{2}$$

Since $m = 8$ when $k = 25$ we always have

$$m \geqslant 8 \quad \text{and} \quad 2 \leqslant k-2m+1 . \tag{3}$$

We may assume that if $F \in \mathscr{F}$ then all subsets of F are in $\mathscr{F}$, in other words that $\mathscr{F}$ is a down-set. We may also assume that all singletons are in $\mathscr{F}$. Let $M = M(m, \mathscr{F})$ be the maximum cardinality of the union of m members of $\mathscr{F}$.

Case. $M \leqslant n-k$. This means that $\mathscr{F}$ has $P(n, m, k)$. So by Case 5 of Conjecture 1 we have $|\mathscr{F}| \leqslant 2^{n-k}$. Since $\mathscr{F}$ is a down-set, every $i \in X$ is in at most half the sets in $\mathscr{F}$, so $\operatorname{val}(\mathscr{F}) \leqslant 2^{n-k-1}$ as required.

Case. $M \geqslant n-k+1$. We choose $F_1, \ldots, F_m$ in $\mathscr{F}$ with $|C| = M$ where $C = F_1 \cup \ldots \cup F_m$. We put $B_0 = X \setminus C$. If there is an $F \in \mathscr{F}$ with $|B_0 \cap F| \geqslant 2$, because $\mathscr{F}$ is a down-set, there is an $F_{m+1} \in \mathscr{F}$ with $|F_{m+1}| = 2$ and $F_{m+1} \subset B_0$. When this happens we put $B_1 = B_0 \setminus F_{m+1}$. Then we look for $F_{m+2} \subset B_1$ with $|F_{m+2}| = 2$ to put $B_2 = B_1 \setminus F_{m+2}$. We repeat this process as many times as possible until we get a B_z with

$$|B_z \cap F| \leqslant 1 \qquad \text{for all } F \in \mathscr{F} . \tag{4}$$

Of course $z = 0$ if we cannot find F_{m+1}. We used m sets to cover C, then z sets to cover $B_0 \setminus B_z$. Hence

$$k-m-z < b = |B_z| , \tag{5}$$

for otherwise a singleton for each point of B_z together with the m sets and the z sets would contradict the assumption that $\mathscr{F}$ has no k-cover.

Suppose $m \leqslant z$ and put $D = F_1 \cup \ldots \cup F_{2m}$. Then

$$|D| = M+2m \geqslant n-k+1+2m \quad \text{so} \quad |X \setminus D| \leqslant k-2m-1 .$$

Taking a singleton for each point of $X \setminus D$ together with the $2m$ sets would again contradict the fact that $\mathscr{F}$ has no k-cover. This proves that $z \leqslant m-1$.

Using (3) and adding $z \leqslant m-1$ to (5) gives

$$2 \leqslant k-2m+1 < b . \tag{6}$$

Let $B_z = \{a_1, a_2, \ldots, a_b\}$ and put

$$\mathscr{R}_i = \{F \setminus \{a_i\} \ ; \ a_i \in F \in \mathscr{F}\} \qquad \text{for } 1 \leqslant i \leqslant b .$$

By (4) we know that the degree of a_i in $\mathscr{F}$ is $|\mathscr{R}_i|$. We let

$$V_i = \bigcup (F \in \mathscr{R}_i) F \qquad \text{for } 1 \leqslant i \leqslant b ,$$

then $V_i \subset X \setminus B_z$ for each i. We also assume that

$$|V_i| \geqslant n-k \qquad \text{for } 1 \leqslant i \leqslant b , \tag{7}$$

for otherwise $\operatorname{val}(\mathscr{F}) \leqslant 2^{n-k-1}$ as required.

Case. After renumbering $\{a_1, \ldots, a_b\}$ *there exist* $R_1 \in \mathscr{R}_1, R_2 \in \mathscr{R}_2, R_3 \in \mathscr{R}_3$ *with*

$$n-2k+2m+2 \leqslant |R_1 \cup R_2 \cup R_3| . \tag{8}$$

Suppose that in fact

$$n-k \leqslant |R_1 \cup R_2 \cup R_3| , \tag{9}$$

which is a stronger condition than (8) by (3). Then

$$n-k+3 \leqslant |(R_1 \cup \{a_1\}) \cup (R_2 \cup \{a_2\}) \cup (R_3 \cup \{a_3\})| ,$$

and we could choose $k-3$ singletons to contradict the fact that $\mathscr{F}$ has no k-cover. Hence (9) does not hold. It now follows by (7) that we can find an $R_4 \in \mathscr{R}_4$ containing a point of $X \setminus (B_z \cup R_1 \cup R_2 \cup R_3)$. By a similar argument to that used on (9) we cannot have

$$n-k \leqslant |R_1 \cup R_2 \cup R_3 \cup R_4| ,$$

and hence there is an $R_5 \in \mathscr{R}_5$ with a point in $X \setminus (B_z \cup R_1 \cup \ldots \cup R_4)$. We repeat the argument till we have $R_i \in \mathscr{R}_i$ for $1 \leqslant i \leqslant b$. Then

$$n-k+b-1 \geqslant |(R_1 \cup \{a_1\}) \cup \ldots \cup (R_b \cup \{a_b\})| = b + |R_1 \cup \ldots \cup R_b|$$

$$\geqslant b+(b-3)+|R_1 \cup R_2 \cup R_3| \geqslant 2b+n-2k+2m-1 ,$$

where the first inequality holds because $\mathscr{F}$ has no k-cover, and the last inequality is (8). However the result contradicts (6), so this case is impossible.

Case. Inequality (8) *is false for all* $R_1 \in \mathscr{R}_1, R_2 \in \mathscr{R}_2, R_3 \in \mathscr{R}_3$. Hence we apply to $\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3$ the case $k = 3$ of Theorem 2 which has $\alpha = \frac{1}{2}(-1+\sqrt{5})$. We recall that $\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3$ are sets of subsets of $X \setminus B_z$ which has cardinality $n-b \leqslant n-k+2m-2$ by (6). Hence

$$\operatorname{val}(\mathscr{F}) \leqslant \min\{|\mathscr{R}_1|, |\mathscr{R}_2|, |\mathscr{R}_3|\} \leqslant 2^{n-k+2m-2}(\tfrac{1}{2}(-1+\sqrt{5}))^{k-3} ,$$

and this last expression is $\leqslant 2^{n-k-1}$ by choice of m. This completes the proof of Case 2 of Conjecture 5.

Proof of Theorem 6. Firstly we can add sets till $\mathscr{F}$ is a down-set and $\mathscr{F}$ will still have no 2-cover. Further we can assume that for each $U \subset X$ either U or its complement is in $\mathscr{F}$. Let f_i be the number of sets in $\mathscr{F}$ of cardinality i. Then

$$mf_m = \frac{m}{2}\binom{2m}{m} = \frac{n}{2}\binom{n-1}{\frac{1}{2}n}, \qquad \text{if } n = 2m \text{ is even} .$$

The Erdős–Ko–Rado Theorem [**5**] says that

$$f_{n-i} \leqslant \binom{n-1}{i-1} \qquad \text{for } \tfrac{1}{2}n < n-i .$$

Hence for $\frac{1}{2}n < n-i$ we have

$$if_i+(n-i)f_{n-i} = i\left\{\binom{n}{i}-f_{n-i}\right\}+(n-i)f_{n-i}$$

$$= i\binom{n}{i}+(n-2i)f_{n-i} \leqslant i\binom{n}{i}+(n-2i)\binom{n-1}{n-i}$$

$$= i\binom{n-1}{i}+(n-i)\binom{n-1}{n-i}.$$

Finally

$$\sum_{1\leqslant j\leqslant n} \text{degree}(j) = \sum_{0\leqslant i\leqslant n} if_i$$

$$= mf_m + \sum_{\frac{1}{2}n<n-i} \{if_i+(n-i)f_{n-i}\}$$

$$\leqslant mf_m + \sum_{\frac{1}{2}n<n-i} \left\{i\binom{n-1}{i}+(n-i)\binom{n-1}{n-i}\right\}$$

$$= \sum_{0\leqslant i\leqslant n} i\binom{n-1}{i} = (n-1)2^{n-2}$$

and so $\text{val}(\mathscr{F}) \leqslant 2^{n-2}(1-n^{-1})$.

Remark. Actually the proof shows that in a set without a 2-cover the average degree cannot exceed $2^{n-2}(1-n^{-1})$. This is best possible because one can take $\mathscr{F}$ to be all the 2^{n-1} sets missing a given $i \in X$.

Finally we must construct the example for Theorem 6. Let us say that $\mathscr{F}$ is *good* if it is a vertex-transitive up-set on X containing $2^{|X|-1}$ members no two of which have an empty intersection. This implies that, if $\mathscr{F}^c = \{X\setminus F \;:\; F \in \mathscr{F}\}$, then $\mathscr{F}^c$ is a vertex-transitive down-set with no 2-cover and

$$|\mathscr{F}^c| = |\mathscr{F}| = 2^{|X|-1} = \text{val}(\mathscr{F})+\text{val}(\mathscr{F}^c).$$

Our example will be an $\mathscr{F}^c$ but it is more convenient to work with $\mathscr{F}$. For $p = 1, 2, \ldots$ we shall define a good family $\mathscr{F}_p$ on $X_p = \{1, \ldots, 7^p\}$.

Let S be the Steiner triple system on X_1. Then $\mathscr{F}_1$ is the up-set generated by S so $\mathscr{F}_1 = \{F \;;\; \exists G \in S \text{ with } G \subset F \subset X_1\}$. A calculation shows that $\mathscr{F}_1$ is good with $\text{val}(\mathscr{F}_1) = 41$ so $\text{val}(\mathscr{F}_1^c) = 23$. We inductively define $\mathscr{F}_{p+1}$ in terms of $\mathscr{F}_p$ and $\mathscr{F}_1$. For $1 \leqslant i \leqslant 7$ let $X_{p,i} = (i-1)7^p + X_p$ and let $\mathscr{F}_{p,i} = (i-1)7^p + \mathscr{F}_p$ so $\mathscr{F}_{p,i}$ is a copy of $\mathscr{F}_p$. Notice that X_{p+1} is the disjoint union $X_{p,1} \cup \ldots \cup X_{p,7}$. For $F \subset X_{p+1}$ put

$$\pi(F) = \{i \;;\; 1 \leqslant i \leqslant 7, F \cap X_{p,i} \in \mathscr{F}_{p,i}\}$$

and let $\mathscr{F}_{p+1} = \{F \;;\; F \subset X_{p+1}, \pi(F) \in \mathscr{F}_1\}$.

It is not hard to show that $\mathscr{F}_{p+1}$ is good with

$$\operatorname{val}(\mathscr{F}_{p+1}) = \big(41\operatorname{val}(\mathscr{F}_p)+23\operatorname{val}(\mathscr{F}_p^c)\big)(\tfrac{1}{2}\theta_p)^6 \text{ where } \theta_p = 2^{7^p}.$$

If we make the substitution $1+\delta_p = 4\operatorname{val}(\mathscr{F}_p)/\theta_p$ this equation reduces to $\delta_{p+1} = (9/32)\delta_p$ and $\delta_1 = 9/32$ so $\delta_p = (9/32)^p$ for all p. We conclude that

$$\operatorname{val}(\mathscr{F}_p^c) = \big(1-(9/32)^p\big)\theta_p/4\,.$$

If $(9/32)^p = (7^p)^\beta$, then $\beta = -0{\cdot}651$ and this completes the proof of Theorem 6. We conjecture that this example is best possible.

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