October 25, 2006

The CCR algebra

Let $\mathcal{H}$ be a Hilbert space and for $f, g \in \mathcal{H}$ let $\sigma(f,g) := \mathrm{Im}\langle f, g\rangle$, The C^*-algebra of the **canonical commutation relation** over $\mathcal{H}$, written as $CCR(\mathcal{H})$, is by definition a C^*-algebra generated by elements $\{W(f) : f \in \mathcal{H}\}$ such that

(i) $\quad W(-f) = W(f)^* \qquad (f \in \mathcal{H})$

(ii) $\quad W(f)W(g) = \exp(i\,\sigma(f,g))W(f+g) \qquad (f, g \in \mathcal{H})$

Condition (ii) tells us that $W(f)W(0) = W(0)W(f) = W(f)$. Hence $W(0)$ is the unit of the algebra and it follows that $W(f)$ is a unitary for every $f \in \mathcal{H}$.

Theorem 1 *For any Hilbert space $\mathcal{H}$ the C^*-algebra $CCR(\mathcal{H})$ exists and unique up to isomorphism.*

The finite sums

$$\sum_i \lambda_i W(f_i)$$

form a dense *-subalgebra of $CCR(\mathcal{H})$, therefore a linear mapping defined on $CCR(\mathcal{H})$ is determined by its values on the unitaries $W(f)$. It is not trivial the positivity of the linear extension.

Let X be an arbitrary (nonempty) set. A function $F : X \times X \to \mathbb{C}$ is called a **positive definite kernel** if and only if

$$\sum_{j,k=1}^{n} c_j\, \overline{c}_k\, \varphi(x_j, x_k) \geq 0$$

for all $n \in \mathbb{N}$, $\{x_1, x_2, \ldots, x_n\} \subset X$ and $\{c_1, c_2, \ldots, c_n\} \subset \mathbb{C}$.

Theorem 2 *Let $\mathcal{H}$ be a Hilbert space and $G : \mathcal{H} \to \mathbb{C}$ be a function. There exists a state φ on $CCR(\mathcal{H})$ such that*

$$\varphi(W(f)) = G(f) \qquad (f \in \mathcal{H})$$

if and only if $G(0) = 1$ and the kernel

$$(f, g) \mapsto G(f - g)\, \exp(-i\,\sigma(f,g))$$

is positive definite.

Exercise 1 Show that the functions

$$G_1(f)\begin{cases} 1 & \text{if } f=0, \\ 0 & \text{otherwise} \end{cases}, \qquad \text{and} \qquad G_2(f)=\exp(-\|f\|^2/2)$$

satisfy the above conditions and therefore they determine the states τ and ψ.

τ is called **tracial state** and ψ is called **Fock state**.

Exercise 2 Show that τ satisfy the condition $\tau(ab)=\tau(ba)$. Use the inequality $|\tau(a)|\le \|a\|$, to show that

$$\|W(f)-W(g)\|^2 \ge 2 \tag{1}$$

if f and g are different vectors.

The inequality (1) implies that the mapping $f\mapsto W(f)$ is not norm continuous.

Let $(\Psi,\pi_\psi,\mathcal{H}_\psi)$ be the GNS construction from $(CCR(\mathcal{H}),\psi)$. The vectors $\pi_\psi(W(f))\Psi$ form a complete system in $\mathcal{H}_\psi$.

Exercise 3 Show that $t\mapsto \langle \pi_\psi(W(f))\Psi, \pi_\psi(W(th))\pi_\psi(W(g))\Psi\rangle$ is a continuous function of $t\in\mathbb{R}$ for every $f,g,h\in\mathcal{H}$.

Therefore

$$t\mapsto \pi_\psi(W(th))$$

is an so-continuous 1-parameter group of unitaries, and according to the Stone theorem

$$\pi_\psi(W(th))=\exp(\mathrm{i}tB(h))$$

for a selfadjoint operator $B(h)$, called **field operator**.

Exercise 4 Show that $B(h)$ is unbounded if $h\neq 0$.

Exercise 5 Show that $B(tf)=tB(f)$, $B(f+g)=B(f)+B(g)$ and $[B(f),B(g)]=-2\mathrm{i}\sigma(f,g)$.

Let

$$B^{\pm}(f)=\frac{1}{2}(B(f)\mp iB(if))\,. \tag{2}$$

Then

$$[B^-(f),B^+(g)]=\langle g,f\rangle \qquad (f,g\in\mathcal{H}) \tag{3}$$

is the canonical commutation relation for the **creation operator** $B^+(g)$ and the **annihilation operator** $B^-(f)$.

Exercise 6 Show that the distribution of $B(f)$ in the vector state Ψ is Gaussian. What is its variance?

Theorem 3 *Assume that* $g_1, g_2, \dots, g_k$ *are pairwise orthogonal vectors in* $\mathcal{H}$. *Then*

$$B^+(g_1)^{m_1} B^+(g_2)^{m_2} \dots B^+(g_k)^{m_k} \Phi$$

and

$$B^+(g_1)^{n_1} B^+(g_2)^{n_2} \dots B^+(g_k)^{n_k} \Phi$$

are orthogonal whenever $m_j \neq n_j$ *for at least one* $1 \leq j \leq k$.

The Hilbert space $\mathcal{H}_\psi$ is identical with the **symmetric Fock space** $\mathcal{F}_s(\mathcal{H})$ over $\mathcal{H}$.