

# Extending the Fisher metric to density matrices \*

D. Petz and Cs. Sudár

Department for Mathematical Analysis,  
Technical University of Budapest, H-1521 Budapest, Hungary

February 27, 2001

## Abstract

Chentsov studied Riemannian metrics on the set of probability measures from the point of view of decision theory. He proved that up to a constant factor the Fisher information is the only metric which is monotone under stochastic transformation. The present paper deals with monotone metrics on the space of finite density matrices on the basis of motivation provided by quantum mechanics. A characterization of those metrics is given in terms of operator monotone functions. Several concrete metrics are constructed and analyzed, in particular, instead of uniqueness in the probabilistic case, there is a large class of monotone metrics, some of which appeared long time ago in the physics literature. Moreover a limiting procedure to pure states is discussed.

## 1 Introduction

The idea of statistical distance between two probability distributions goes back to Fisher who was interested in a quantity which shows how difficult it is to decide

---

\*Published in *Geometry in Present Days Science*, eds. O.E. Barndorff-Nielsen and E.B. Vedel Jensen, 21–34 (World Scientific, 1999), written version of the conference talk at Aarhus University in 1997.

between two probability measures by statistical sampling. He found that the spherical representation of the probability simplex is adequate. The probability distributions  $(p_1, p_2, \dots, p_n)$  on  $n$  points form an  $(n - 1)$ -dimensional simplex  $\mathcal{S}_{n-1}$ , since  $p_i \geq 0$  and  $\sum_i p_i = 1$ . If we introduce the parameters  $z_i = 2\sqrt{p_i}$ , then  $\sum_i z_i^2 = 4$  and the probability simplex is parametrized as a portion of the  $n$ -sphere. Let  $z(t)$  be a curve on the sphere. The square of the length of the tangent is

$$\langle \partial_t z, \partial_t z \rangle = \sum_i (\partial_t z_i)^2 = \sum_i p_i(t) (\partial_t \log p_i(t))^2, \quad (1)$$

which is the Fisher information. The geodesic distance between two probability distributions  $Q$  and  $R$  can be computed along a great circle and it is

$$d(Q, R) = 2 \arccos \sum_{i=1}^n \sqrt{p_i r_i}.$$

One observes that the geodesic distance is a simple transform of the Hellinger distance. Namely,

$$d_H(Q, R) \equiv \sqrt{\sum_{i=1}^n (p_i^{1/2} - r_i^{1/2})^2} = 2 \sin(d(Q, R)/4).$$

In applications of mathematical statistics one often meets a family of distributions parametrized by a real number or more generally by  $\theta \in \mathbf{R}^m$ . An example is the family  $N(\mu, \sigma)$  of normal distributions with mean  $\mu \in \mathbf{R}$  and variance  $\sigma \in \mathbf{R}^+$ . An  $n$ -tuple  $(\xi_1, \xi_2, \dots, \xi_n)$  of random variables is called an unbiased estimator of the parameter  $\theta$  if  $E(\xi_i) = \theta_i$  for  $1 \leq i \leq n$ . In statistical problems an unbiased estimator can be used to estimate the true value of the parameter  $\theta$  on the basis of a sample. The variance of the estimator is desired to be small in order to have an effective estimation. The classical Cramér-Rao inequality is related to that point. The  $m \times m$  covariance matrix  $E(\xi_i \xi_j) - E(\xi_i)E(\xi_j)$  is always larger than the inverse of the Fisher information matrix. The latter is independent of the estimator  $(\xi_1, \xi_2, \dots, \xi_n)$  and one desirable property of an unbiased estimator is closeness of the covariance matrix to the inverse Fisher information matrix.

In quantum mechanics, the state space of an  $n$  level system is identified with the set of all  $n \times n$  positive semidefinite complex matrices of trace 1, they are the so-called density matrices. Let  $\mathcal{M}_n$  stand for the set of all positive definite density matrices. We can parametrize  $D = (D_{ij}) \in \mathcal{M}_n$  by the real numbers  $\text{Re } D_{ij}$ ,  $\text{Im } D_{ij}$  ( $1 \leq i < j \leq n$ ) and by the positive numbers  $D_{ii}$  ( $1 \leq i \leq n - 1$ ). In this way  $\mathcal{M}_n$  may be embedded into the Euclidean  $k$ -space with  $k = n^2 - 1$  and becomes a manifold. At each point  $D \in \mathcal{M}_n$  the tangent space  $T_D(\mathcal{M}_n)$  is identified with the set of all traceless selfadjoint matrices. One observes that the probability simplex is embedded into  $\mathcal{M}_n$ , since every probability distribution on

the  $n$ -point space gives a diagonal density matrix in the obvious way:

$$\mathcal{S}_{n-1} \ni (p_1, p_2, \dots, p_n) \mapsto \mathbf{Diag}(p_1, p_2, \dots, p_n) \in \mathcal{M}_n.$$

The aim of the present paper is a search for possible Riemannian metrics on the space of density matrices of a finite dimensional space. Without some restrictions this would be pointless, the emphasis is put on statistically relevant metrics which on the submanifold of probability distributions recover the Fisher information metric.

## 2 Chentsov's approach to the problem

Chentsov was led by decision theory when he considered a category whose objects are probability spaces and whose morphisms are Markov kernels. Although he worked in [3] with arbitrary probability spaces, his idea can be demonstrated very well on finite ones. In this case a Markov kernel from the probability  $(n - 1)$ -simplex  $\mathcal{S}_{n-1}$  to an  $(m - 1)$ -simplex  $\mathcal{S}_{m-1}$  is an  $m \times n$  stochastic matrix. If  $\Pi$  is such a matrix and  $P \in \mathcal{S}_n$  then  $\Pi P \in \mathcal{S}_m$  is considered more random than  $P$ . If we want to represent probability distributions as column vectors then the matrix  $\Pi$  has to be column-stochastic, that is,  $\sum_i \Pi_{ij} = 1$  for every  $j$ . An example of randomization comes from identification of two outcomes of our random experiment. This is described by a 0-1 matrix with one 1 in each row except for one where two 1's stand. In statistical physical literature the term coarse graining is more often used than randomization but they stand for the same concept.

Generally speaking, the parametrized family  $(Q_i)$  is more random than the parametrized family  $(P_i)$  (with the same parameter set) if there exists a stochastic matrix  $\Pi$  such that  $\Pi P_i = Q_i$  for every value of the parameter  $i$ . Two parametric families  $(P_i)$  and  $(Q_i)$  are equivalent in the theory of statistical inference if there are two stochastic matrices  $\Pi^{(12)}$  and  $\Pi^{(21)}$  such that

$$\Pi^{(12)} P_i = Q_i \quad \text{and} \quad \Pi^{(21)} Q_i = P_i \tag{2}$$

for every  $i$ . Chentsov defined a numerical function  $f$  given on pairs of measures to be invariant if

$$(P_1, P_2) \sim (Q_1, Q_2) \quad \text{implies} \quad f(P_1, P_2) = f(Q_1, Q_2) \tag{3}$$

and monotone if

$$f(P_1, P_2) \geq f(\Pi P_1, \Pi P_2). \tag{4}$$

for every stochastic matrix  $\Pi$ . A monotone function  $f$  is obviously invariant. Statistics and information theory know a lot of monotone functions, relative entropy

$$S(P, Q) = \sum_i p_i (\log p_i - \log q_i) \quad (5)$$

and its generalizations. If a Riemannian metric is given on all probability simplexes, then this family of metrics is called invariant (respectively, monotone) if the corresponding geodesic distance is an invariant (respectively, monotone) function. Chentsov's great achievement was to show that up to a constant factor the Fisher information (1) yields the only monotone family of Riemannian metrics on the class of finite probability simplexes ([3]).

A decade later Chentsov turned to the quantum case, where the probability simplex is replaced by the set of density matrices. A linear mapping between two matrix spaces sends a density matrix into a density if the mapping preserves trace and positivity (i.e., positive semidefiniteness). By now it is well-understood that complete positivity is a natural and important requirement in the quantum case. Therefore, we call a trace preserving completely positive mapping stochastic. One of the equivalent forms of the complete positivity of a map  $T$  is the following.

$$\sum_{i=1}^n \sum_{j=1}^n a_i^* T(b_i^* b_j) a_j \geq 0$$

for all possible choice of  $a_i$ ,  $b_i$  and  $n$ . A completely positive mapping  $T$  satisfies the Schwarz inequality:  $T(a^* a) \geq T(a)^* T(a)$ .

Chentsov recognized that stochastic mappings are the appropriate morphisms in the category of quantum state spaces. (The monograph [1] contains more information about stochastic mappings, see also [10].) The above definitions of invariance and monotonicity make sense when stochastic matrices are replaced by stochastic mappings. Chentsov (with Morozova) aimed to find the invariant (or monotone) Riemannian metrics in the quantum setting as well. They obtained the following result ([12]). Assume that a family of Riemannian metrics is given on all spaces of density matrices which is invariant, then there exist a function  $c(x, y)$  and a constant  $C$  such that the squared length of a tangent vector  $A = (A_{ij})$  at a diagonal point  $D = \mathbf{Diag}(p_1, p_2, \dots, p_n)$  is of the form

$$C \sum_{k=1}^n p_k^{-1} A_{kk}^2 + 2 \sum_{j < k} c(p_j, p_k) |A_{jk}|^2. \quad (6)$$

Furthermore, the function  $c(x, y)$  is symmetric and  $c(\lambda x, \lambda y) = \lambda^{-1} c(x, y)$ . This result of Morozova and Chentsov was not complete. Although they had proposals

for the function  $c(x, y)$ , they did not prove monotonicity or invariance of any of the corresponding metrics. A complete result was obtained in [14] and [15] but before presenting it here we make a few comments on (6).

Both the function  $c(x, y)$  and the constant  $C$  are independent of the matrix size  $n$ . Restricting ourselves to diagonal matrices, which is in some sense a step back to the probability simplex, we can see that there is no ambiguity of the metric. Loosely speaking, the uniqueness result of the simplex case survives along the diagonal and the offdiagonal provides new possibilities for the definition of a stochastically invariant metric on the space  $\mathcal{M}$  of invertible density matrices. In other words, the tangent space  $T_D(\mathcal{M})$  at  $D$  decomposes as

$$T_D(\mathcal{M}) = T_D(\mathcal{M})^c \oplus T_D(\mathcal{M})^o, \quad (7)$$

where  $T_D(\mathcal{M})^c = \{A \in T_D(\mathcal{M}) : [A, D] = 0\}$  and  $T_D(\mathcal{M})^o$  is the orthogonal complement of  $T_D(\mathcal{M})^c$  with respect to the Hilbert-Schmidt inner product of matrices. The monotone metric is unique on  $T_D(\mathcal{M})^c$ ,

$$K_D(A, A) = C \text{Tr } D^{-1} A^2 \quad \text{if } A \in T_D(\mathcal{M})^c \quad (8)$$

and the function  $c(x, y)$  determines the metric on the orthogonal complement.

If a distance between density matrices expresses statistical distinguishability then this distance must decrease under coarse-graining. A good example of coarse-graining arises when a density matrix is partitioned in the form of a  $2 \times 2$  block matrix, and the coarse-graining forgets about the offdiagonal:

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \longmapsto \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$$

In the mathematical formulation, a coarse-graining is a completely positive mapping which preserves the trace and hence sends density matrix into density matrix. Such mapping will be called stochastic below. A Riemannian metric is defined to be monotone if the differential of any stochastic mapping is a contraction (in the sense that it is norm decreasing). If the affine parametrization is considered, then  $D_t = D + tA$  is a curve for an invertible density  $D$  and for a selfadjoint traceless  $A$ . Under a stochastic mapping  $\mathbf{T}$  this curve is transformed into  $\mathbf{T}(D_t) = \mathbf{T}(D) + t\mathbf{T}(A)$  provided that  $\mathbf{T}(D)$  is an invertible density and the real number  $t$  is small enough. The monotonicity condition for the Riemannian metric  $g$  on  $\mathcal{M}_n$  reads as

$$g_{\mathbf{T}(D)}(\mathbf{T}(A), \mathbf{T}(A)) \leq g_D(A, A), \quad (9)$$

for any invertible density  $D$ , for any traceless selfadjoint matrix  $A$  and for any stochastic mapping  $\mathbf{T}$ . Our goal is to show many examples of monotone metrics and to give their characterization in terms of operator monotone functions.

### 3 Monotone metrics

Let us recall that a function  $f : \mathbf{R}^+ \rightarrow \mathbf{R}$  is called operator monotone if the relation  $0 \leq K \leq H$  implies  $0 \leq f(K) \leq f(H)$  for any matrices  $K$  and  $H$  (of any order). The theory of operator monotone functions was established in the 1930's by Löwner and there are several reviews on the subject, for example [2], [5] are suggested.

The following result was obtained in [15].

**Theorem 3.1.** *There exists a one-to-one correspondence between monotone metrics and operator monotone functions  $f : \mathbf{R}^+ \rightarrow \mathbf{R}^+$  such that  $f(t) = tf(t^{-1})$ . If  $D = \mathbf{Diag}(p_1, p_2, \dots, p_n)$ , then the metric corresponding to  $f$  is of the form*

$$\sum_{j=1}^n \sum_{k=1}^n c(p_j, p_k) |A_{jk}|^2, \quad (10)$$

where  $c(x, y) = 1/yf(x/y)$ .

The proof of this result is given in the original paper. Here we remark that the metric (6) can be written by means of a certain function  $f$  such that  $c(x, y) = 1/yf(x/y)$  holds. The point is to demonstrate, on the one hand that this function  $f$  must be operator monotone and, on the other hand that every operator monotone function provides a monotone metric. The symmetry condition  $f(t) = tf(t^{-1})$  is equivalent to the condition that the Riemannian inner product is real valued on the selfadjoint tangent vectors. It seems natural to normalize metrics such a way that on the submanifold of diagonal matrices the standard Fisher metric should appear. In this case one can say following Uhlmann that the metric is Fisher adjusted. This normalization is equivalent to the condition  $f(1) = 1$ . Below we always assume that  $f(1) = 1$ , that is, we restrict our discussion to Fisher adjusted metrics. Some examples of functions  $f$  satisfying the hypothesis of Theorem 3.1 are the following.

$$\frac{2x^{\alpha+1/2}}{1+x^{2\alpha}}, \quad \frac{x-1}{\log x}, \quad \frac{x-1}{\log x} \frac{2\sqrt{x}}{1+x}, \quad \left(\frac{x-1}{\log x}\right)^2 \frac{2}{1+x}, \quad \frac{1+x}{2} \quad (11)$$

where  $0 \leq \alpha \leq 1/2$ .

It is worthwhile to note that Kubo and Ando established a correspondence between operator monotone functions and means of positive operators. Our condition  $f(t) = tf(t^{-1})$  on the operator monotone function  $f$  is equivalent to the

symmetry of the corresponding operator mean. The smallest mean is the harmonic one. This corresponds to the function  $f(t) = 2t/(t + 1)$  and gives the metric

$$g_D^{\text{RL}}(A, B) = \frac{1}{2} \text{Tr } D^{-1}(AB + BA). \quad (12)$$

Since a larger function  $f$  yields a smaller metric, we have

**Theorem 3.2.** *The Riemannian metric (12) is monotone and it is the largest among all Fisher-adjusted monotone metrics.*

One can see monotonicity of (12) directly. The operator inequality

$$\mathbf{T}(K)\mathbf{T}(D)^{-1}\mathbf{T}(K)^* \leq \mathbf{T}(KD^{-1}K^*), \quad (13)$$

holds for positive invertible  $D$  for every stochastic mapping [4], [11]. Taking the trace of both sides of (13), we conclude monotonicity.

The arithmetic operator mean is the largest symmetric mean and it gives the smallest metric which is usually called the metric of the symmetric logarithmic derivative.

**Theorem 3.3.** *Among all Fisher-adjusted monotone metrics the smallest one is given as*

$$g_D^{\text{SL}}(A, B) = \text{Tr } AG, \quad (14)$$

where  $G$  is the unique solution of the equation

$$DG + GD = 2B. \quad (15)$$

The metrics  $g^{\text{RL}}$  and  $g^{\text{SL}}$  appeared in connection with generalizations of the Cramér-Rao inequality and  $g^{\text{SL}}$  play important role in the work of Uhlmann when he extends Berry phase to mixed states from the pure ones. It is rather instructive to have a look at the simple  $2 \times 2$  case.

Dealing with  $2 \times 2$  density matrices, we conveniently use the so-called Stokes parametrization.

$$D_x = \frac{1}{2}(I + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) \equiv \frac{1}{2}(I + x \cdot \sigma) \quad (16)$$

where  $\sigma_1, \sigma_2, \sigma_3$  are the Pauli matrices and  $(x_1, x_2, x_3) \in \mathbf{R}^3$  with  $x_1^2 + x_2^2 + x_3^2 \leq 1$ . The monotone metrics on  $\mathcal{M}_2$  are rotation invariant in the sense that they depend only on  $r = \sqrt{x^2 + y^2 + z^2}$  and split into radial and tangential components as follows.

$$ds^2 = \frac{1}{1-r^2} dr^2 + \frac{1}{1+r} g\left(\frac{1-r}{1+r}\right) dn^2 \quad \text{where} \quad g(t) = \frac{1}{f(t)}. \quad (17)$$

The radial component is independent of the function  $f$ . In case of  $f(t) = (t+1)/2$  we have constant tangential component. In the case of  $f(t) = 2t/(1+t)$ ,  $ds^2 = (1-r^2)^{-1}(dr^2 + dn^2)$ . Hence both the smallest and the largest metrics possess a rather particular form.

The limit of the tangential component exists when  $r \rightarrow 1$  if  $f(0) \neq 0$ . In this way the standard metric is obtained on the set of pure states, up to a constant factor. In case of larger density matrices, pure states form a small part of the topological boundary of the invertible density matrices. Hence, in order to speak about the extension of a Riemannian metric on invertible densities to pure states, a rigorous meaning of the extension should be given. This is the subject of the paper [16] and will be discussed in the next section.

It is remarkable that quantum statistical mechanics seems to prefer another metric, different from the smallest and from the largest one. This is termed the Kubo-Mori, or Bogoliubov metric and sometimes canonical correlation. In the above used affine parametrization of the state space the Kubo-Mori metric takes the form

$$g_D^{\text{KM}}(A, B) = \int_0^\infty \text{Tr} (D+t)^{-1} A (D+t)^{-1} B dt.$$

In order to see that this is the usual Kubo-Mori inner product, we rewrite it in the logarithmic coordinate system instead of the affine one. In terms of the inverse Kubo transforms

$$A' = \int_0^\infty (D+s)^{-1} A (D+s)^{-1} ds, \quad (18)$$

$$B' = \int_0^\infty (D+s)^{-1} B (D+s)^{-1} ds \quad (19)$$

we have

$$g_D^{\text{KM}}(A, B) = \int_0^1 D^t A' D^{1-t} B' dt. \quad (20)$$

**Theorem 3.4.** *Assume that a Fisher adjusted monotone metric  $g$  is obtained from a smooth function  $G : \mathbf{R}^+ \rightarrow \mathbf{R}$  by*

$$g(A, B)(D) = \left. \frac{\partial}{\partial t \partial s} \right|_{t=s=0} \text{Tr} G(D + tA + sB).$$

*Then  $g(A, B)$  is the Kubo-Mori inner product.*

*Proof.* When  $A, B$  and  $D$  commute, we have

$$\left. \frac{\partial}{\partial t \partial s} \right|_{t=s=0} \text{Tr} G(D + tA + sB) = \text{Tr} G''(D) AB.$$

Since we assumed that the metric is Fisher-adjusted,  $G''(t) = t^{-1}$  and we have  $G(t) = t \log t + Ct + D$  and the differentiation gives the Kubo-Mori metric.  $\square$

The above proof also gives that the Kubo-Mori metric is the negative Hessian of the von Neumann entropy functional on the state space. Recall that the von Neumann entropy is the Boltzmann-Shannon entropy of the eigenvalues, that is,

$$S(D) := -\text{Tr}(D \log D).$$

Differentiation of entropy-like functional is a good method to obtain monotone metrics. In one variable Theorem 3.4 does not allow many possibilities but in the two variable case one can get more metrics. A typical two-variable-entropy is the relative entropy  $\text{Tr}(D_1(\log D_1 - \log D_2))$  which is a member of the family of  $\alpha$ -entropies. If  $-2 < \alpha < 2$ , then

$$S_\alpha(D_1, D_2) = \frac{4}{1 - \alpha^2} \text{Tr}(I - D_2^{\frac{1+\alpha}{2}} D_1^{-\frac{1+\alpha}{2}}) D_1 \quad (21)$$

is jointly convex. The metric

$$\frac{\partial^2}{\partial t \partial u} S_\alpha(D + tA, D + uB) \Big|_{t=u=0} = K_D^\alpha(A, B) \quad (22)$$

was studied first by Hasegawa [6], [7] and its monotonicity was proved in [9] and [8]. Note that the limit  $\alpha \rightarrow \pm 1$  in the formulas recovers the usual relative entropy and the Kubo-Mori metric. Since (22) is a monotone metric, it is really interesting on tangent vectors orthogonal to the commutator of  $D$ :

$$K_D^\alpha(i[D, X], i[D, X]) = \frac{2}{1 - \alpha^2} \text{Tr}([D^{\frac{1-\alpha}{2}}, X][D^{\frac{1+\alpha}{2}}, X]), \quad (23)$$

where  $X$  is selfadjoint. It is worthwhile to point out the similarity to the skew information proposed by Wigner, Yanase and Dyson (apart from a constant factor), see [17] or p. 49 in [13]. The operator monotone functions corresponding to (22) are

$$f_\alpha(x) = \beta(1 - \beta) \frac{(x - 1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)}.$$

where  $\beta = (1 - \alpha)/2$ .

The following characterization of the  $\alpha$ -metrics was obtained in [8].

**Theorem 3.5.** *In the class of symmetric monotone metrics, the Wigner-Yanase-Dyson skew information (i.e. the  $\alpha$ -metric (22)) is characterized by the property that*

$$K_\rho(A, B) = \frac{\partial^2}{\partial t \partial s} \text{Tr} g(\rho + tA) g^*(\rho + sB) \Big|_{t=s=0}, \quad A = i[\rho, X], B = i[\rho, Y].$$

for some smooth functions  $g$  and  $g^*$ .

To prove this theorem we compute the Morozova-Chentsov function for the metric determined by  $g$  and  $g^*$  and we get

$$c(\lambda, \mu) = \frac{(g(\lambda) - g(\mu))(g^*(\lambda) - g^*(\mu))}{(\lambda - \mu)^2}$$

From the property  $c(t\lambda, t\mu) = t^{-1}c(\lambda, \mu)$  we deduce that, under the condition  $g(0)g^*(0) = 0$ ,  $g(t\lambda)g^*(t\lambda) = tg(\lambda)g^*(\lambda)$  must hold. This implies that

$$g(x)g^*(x) = cx \quad (x \in \mathbf{R}^+).$$

Another necessary condition comes from the property that  $\lim_{\lambda \rightarrow \mu} c(\lambda, \mu) = \mu^{-1}$ . In this way, we arrive at the condition

$$g'(x)g^*(x) = x^{-1} \quad (x > 0)$$

and the equations (24) and (25) together have the solution  $g(x) = ax^p$  and  $g^*(x) = bx^{1-p}$ ,  $ab = c = 1/p(1-p)$ , and the possible limit  $\lim_{p \rightarrow 0, or 1}$  allowing  $x$  and  $\log x$ .

## 4 Radial extension to pure states

The idea behind the radial extension comes from the  $2 \times 2$  case when the Stokes parametrization given by (16) identifies  $\mathcal{M}_2$  with the open unit ball in  $\mathbf{R}^3$  and the pure states form the unit sphere. Let us fix a point  $P$  in the unit sphere (i.e.  $P$  is a pure state) and a tangent vector  $A$  at  $P$ . Moreover, let  $D$  be an element of the open unit ball except the origin such that  $P$  and  $D$  lie on the same radial line  $r$ .  $P$  can be thought as the radial projection of  $D$  to the boundary of the unit ball. Define a tangent vector  $\hat{A}$  at  $D$  such that  $\hat{A}$  is orthogonal to  $r$  and the endpoints of  $A$  and  $\hat{A}$  lie on the same radial line.  $\hat{A}$  can be thought as a lift of  $A$  with respect to the radial projection. Differential geometers call such lifted vectors 'horizontal vectors' and vectors tangent to the radius at  $D$  are called 'vertical vectors'. Now one can take the inner product  $g_D(\hat{A}, \hat{B})$  of two lifts  $\hat{A}, \hat{B}$  of  $A, B$  at  $D$  with respect to a monotone Riemannian metric  $g$  and ask for conditions of the existence of the limit of  $g_D(\hat{A}, \hat{B})$  whenever  $D$  goes to  $P$  on the radius  $r$ .

In the general case the radial projection is defined on an open and dense subset  $\mathcal{M}'_n$  of  $\mathcal{M}_n$  where  $\mathcal{M}'_n$  is formed by the non-degenerate elements of  $\mathcal{M}_n$ , i.e. matrices whose eigenvalues are all distinct. Now the radial projection  $\pi$  is a smooth mapping from  $\mathcal{M}'_n$  into the pure states  $\mathcal{P}$  such that  $\pi(D)$  is the projection to the one-dimensional eigenspace corresponding to the largest eigenvalue of  $D$ .

The idea of this projection is that if  $D$  is “near” to a pure state then the largest eigenvalue of  $D$  is near to 1 and the corresponding eigenspace is one dimensional.

It can be proved that  $\mathcal{M}'_n$  is a fibre bundle over  $\mathcal{P}$  with projection  $\pi$  and in the  $2 \times 2$  case the fibers are exactly the radiuses. If  $\pi_{*,D}$  denotes the tangent map of  $\pi$  at  $D$  then the vertical space is  $\text{Ker } \pi_{*,D}$  and the horizontal space  $H_D$  is the orthogonal complement of  $\text{Ker } \pi_{*,D}$  with respect to a fixed monotone Riemannian metric  $g$ . Since  $\pi_{*,D}$  is surjective, the restriction of  $\pi_{*,D}$  to the horizontal space gives a linear isomorphism between  $H_D$  and the tangent space of  $\mathcal{P}$  at  $\pi(D)$  thus for any tangent vector  $A$  at  $\pi(D)$  there exist a unique lift  $\hat{A}$  at  $D$  such that  $\pi_{*,D}(\hat{A}) = A$ .

If  $D = \mathbf{Diag}(\lambda_1, \dots, \lambda_n)$  where  $\lambda_1$  is the largest eigenvalue then the vertical vectors at  $D$  are identified with vectors of the following form

$$\begin{pmatrix} x_{11} & 0 & \dots & 0 \\ 0 & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_{n2} & \dots & x_{nn} \end{pmatrix}$$

and the horizontal vectors have the form

$$\begin{pmatrix} 0 & \bar{u}_2 & \dots & \bar{u}_n \\ u_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_n & 0 & \dots & 0 \end{pmatrix}. \quad (24)$$

The tangent vectors at the pure state  $\pi(D) = \mathbf{Diag}(1, 0, \dots, 0)$  also have the same form and the lift of a tangent vector is given by

$$\begin{pmatrix} 0 & (\lambda_1 - \lambda_2)\bar{u}_2 & \dots & (\lambda_1 - \lambda_n)\bar{u}_n \\ (\lambda_1 - \lambda_2)u_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_1 - \lambda_n)u_n & 0 & \dots & 0 \end{pmatrix} \quad (25)$$

which is independent of the choice of  $g$ . Now the precise definition of the radial extension is the following

**Definition.** We say that a smooth metric  $k$  on  $\mathcal{P}$  is the radial extension of  $g$  if for every  $P \in \mathcal{P}$ , for every pair of tangent vectors  $A, B$  at  $P$  and for every sequence  $D_m$  such that  $\pi(D_m) = P$

$$\lim_{m \rightarrow \infty} g_{D_m}(\hat{A}, \hat{B}) = k_P(A, B).$$

Using (25) one can compute  $g_D(\widehat{A}, \widehat{B})$ :

$$g_D(\widehat{A}, \widehat{B}) = 2\operatorname{Re} \sum_{i=2}^n \frac{(\lambda_1 - \lambda_i)^2}{f(\lambda_i/\lambda_1)\lambda_1} u^i \bar{v}^i$$

where  $f$  is the operator monotone function corresponding to the metric and  $u_i, v_i$  for  $i = 2, \dots, n$  are the matrix elements of horizontal vectors  $A, B$  as in (24). Now from this expression it can be easily obtained the following

**Theorem 4.1.** *Let  $g$  be a monotone Riemannian metric on  $\mathcal{M}_n$  and let  $f: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  be the corresponding operator monotone function. The radial extension  $k$  of  $g$  exists if and only if  $f(0) \neq 0$ . In this case  $k = h/f(0)$  where  $h$  is the canonical Riemannian metric on  $\mathcal{P}$ , the so called Fubini-Study metric.*

## References

- [1] P. M. Alberti and A. Uhlmann, *Stochasticity and partial order, doubly stochastic maps and unitary mixing*, VEB Deutscher Verlag Wiss, 1981.
- [2] T. Ando, *Concavity of certain maps and positive definite matrices and applications to Hadamard products*, Linear Algebra Appl. **26** (1979), 203–241.
- [3] N. N. Cencov, *Statistical decision rules and optimal inferences*, Translation of Math. Monog., vol. 53, Amer. Math. Society, Providence, 1982.
- [4] M. D. Choi, *Some assorted inequalities for positive linear maps on  $C^*$ -algebras*, J. Operator Theory **4** (1980), 271–285.
- [5] F. Hansen and G. K. Pedersen, *Jensen's inequality for operators and Löwner's theorem*, Math. Ann. **258** (1982), 229–241.
- [6] H. Hasegawa,  *$\alpha$ -divergence of the non-commutative information geometry*, Rep. Math. Phys. **33** (1993), 87–93.
- [7] ———, *Non-commutative extension of the information geometry*, Quantum Communication and Measurement (V. P. Belavkin, O. Hirota, and R. I. Hudson, eds.), Plenum, 1995, pp. 327–337.
- [8] H. Hasegawa and D. Petz, *Non-commutative extension of the information geometry II*, Quantum Communication and Measurement (O. Hirota, ed.), Hakone, 1996, pp. 109–118.

- [9] ———, *On the Riemannian metric of  $\alpha$ -entropies of density matrices*, Lett. Math. Phys. **38** (1996), 221–225.
- [10] K. Kraus, *States, effects, and operations*, Lecture Notes in Physics, vol. 190, Springer-Verlag, Berlin Heidelberg New York Inc., 1983.
- [11] M. D. Lieb and M. B. Ruskai, *Some operator inequalities of the Schwarz type*, Adv. in Math. **12** (1974), 269–273.
- [12] E. A. Morozova and N. N. Chentsov, *Markov invariant geometry on state manifolds*, Itogi Nauki i Tehniki **36** (1990), 69–102 (Russian).
- [13] M. Ohya and D. Petz, *Quantum entropy and its use*, Texts and Monographs in Physics, Springer-Verlag, Berlin Heidelberg New York Inc., 1993.
- [14] D. Petz, *Geometry of canonical correlation on the state space of a quantum system*, J. Math. Phys. **35** (1994), 780–795.
- [15] ———, *Monotone metrics on matrix spaces*, Linear Algebra Appl. **244** (1996), 81–96.
- [16] C. Sudár, *Radial extension of monotone riemannian metrics on density matrices*, Publ. Math. Debrecen **49** (1996), 243–250.
- [17] E. P. Wigner and M. M. Yanase, *Information content of distributions*, Proc. Nat. Acad. Sci. U.S.A. **49** (1963), 910–918.