PROPERTIES OF FREE ENTROPY RELATED TO POLAR DECOMPOSITION†

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Abstract. The free entropies $\hat{\chi}(a_1,\dots,a_N)$ of non-selfadjoint random variables and $\chi_u(u_1,\dots,u_N)$ of unitary random variables are introduced and discussed by the methods of Voiculescu's free analysis. The additivity $\chi_u(u_1,\dots,u_N)=\sum_i \chi_u(u_i)$ is shown to be equivalent to freeness. The relation among $\hat{\chi}$, χ_u and χ is investigated in the case when $a_i=u_ih_i$ is the polar decomposition. The subadditivity $\hat{\chi}(a_1,\dots,a_N)\le\chi_u(u_1,\dots,u_N)+\chi(h_1^2,\dots,h_N^2)+$ constant is proven and applications to some maximization problems for $\hat{\chi}$ are given.

Introduction

A highlight of free probability theory is the free entropy which has been extensively developed with several applications by Dan Voiculescu [12–17]. Up to now there are two kinds of free entropies χ and χ^* (for selfadjoint random variables). The free entropy $\chi(a_1,\dots,a_N)$ studied in [12–15, 17] is the matricial analogue of the classical Boltzmann-Gibbs entropy and is defined as the asymptotic growth rate of the volume of N-tuples of selfadjoint matrices approximating $(a_1,\dots,a_N)$ in the sense of joint moments when the matrix size is going to infinity. Its most significant property is the additivity: $\chi(a_1,\dots,a_N)=\chi(a_1)+\dots+\chi(a_N)$ in the case when (and only when) $a_1,\dots,a_N$ are in free relation. On the other hand, the free entropy $\chi^*(a_1,\dots,a_N)$ from [16] was defined as a certain integral of the free analogue of the Fisher information. The equality $\chi(a)=\chi^*(a)$ was shown in [16], but the coincidence of the two concepts in the general multi-variable case is not known.

Random matrices are often models of free random variables and the bridge between random matrix theory and free probability is the asymptotic freeness result of Voiculescu [11, 17]. The free entropy $\chi(a)$ of a single variable a coincides, up to the sign and an additive constant, with the so-called logarithmic energy of the distribution measure of a, which is familiar from potential theory. The one-variable free entropy is the main

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component of the rate function in large deviation theorems obtained first by Ben Arous and Guionnet [1] and subsequently by the present authors [4, 5, 10].

In this paper we discuss the matricial free entropies $\hat\chi$ of non-selfadjoint random variables and χ_u of unitary random variables. In fact, the non-selfadjoint version $\hat\chi$ of Voiculescu's entropy has been already implicitly used in several places in the literature. Our main goal is not really the entropy of non-selfadjoint variables but rather the study of the entropy of unitaries which is natural as well. Our motivation was the random matrix model of the Haar unitary distribution and the related large deviation results [4]. We make clear the intrinsic interrelation among different free entropies $\hat\chi$, χ_u and χ. This relation gives a useful method in the free entropy analysis.

The paper is organized as follows. In Sect. 1 we define the free entropies $\hat\chi(a_1,\dots,a_N)$ of non-selfadjoint random variables and $\chi_u(u_1,\dots,u_N)$ of unitary random variables, and their basic properties are given. The free entropy $\hat\chi(a_1,\dots,a_N)$ is equal to the free entropy of the real and imaginary parts of $a_1,\dots,a_N$, so the properties of $\hat\chi$ are direct translations from those of χ. If the polar decompositions $a_i=u_ih_i$ are taken, then the random variables $a_1,\dots,a_N$ may be considered as a combination of unitary $u_1,\dots,u_N$ and positive $h_1,\dots,h_N$. In Sect. 2 we introduce, as a technical device, the free entropy $\chi_{(u,+)}(u_1,\dots,u_N;h_1,\dots,h_N)$ of mixed strings of unitary and positive random variables. By making use of this mixed free entropy $\chi_{(u,+)}$ we obtain the following relation among $\hat\chi$, χ_u and χ when u_i is the unitary part of a_i:

$$\hat\chi(a_1,\dots,a_N)\le\chi_u(u_1,\dots,u_N)+\chi(a_1^*a_1,\dots,a_N^*a_N)+\frac{N}{2}\Big(\log\frac{\pi}{2}+\frac{3}{2}\Big).$$

In Sect. 3 we show that the equality holds true in the above inequality under a suitable freeness assumption. Furthermore, we show the additivity properties of χ_u and $\hat\chi$ based on the above analysis through $\chi_{(u,+)}$. The result on approximate freeness for matrices in [17] and the formula in a separate change of variables in [15] are useful for our purpose. Our method has applications also to the maximization problems for $\hat\chi(a)$, for instance, under the fixed distribution of a^*a. The R-diagonal element introduced by A. Nica and R. Speicher [9] appears as the maximizer in this type of maximization of $\hat\chi(a)$. Section 4 treats the similar maximization for the free entropy in the case of a matrix of noncommutative random variables. Our Sect. 4 is strongly inspired by a recent paper of Nica, Shlyakhtenko and Speicher [8] where the same problems were considered for χ^*. The authors are grateful for having access to [8] prior to its publication.

Just after this paper was completed, we received a preprint of Nica, Shlyakhtenko and Speicher [19] where the maximization problems of Section 4 were considered.

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1. Free entropies of non-selfadjoint and unitary random variables

In this section we introduce the free entropies $\hat\chi(a_1,\dots,a_N)$ of non-selfadjoint random variables and $\chi_u(u_1,\dots,u_N)$ of unitary random variables. Throughout the paper let $(\mathcal{M},\tau)$ be a tracial W*-probability space, that is, $\mathcal{M}$ is a von Neumann algebra with a faithful normal tracial state τ. Let $\mathcal{M}^{sa}$ be the set of selfadjoint elements in $\mathcal{M}$. The free entropy $\chi(a_1,\dots,a_N)$ of an N-tuple of $a_1,\dots,a_N$ in $\mathcal{M}^{sa}$ was introduced in [13]. One can define the free entropy of an N-tuple of (non-selfadjoint) elements in $\mathcal{M}$ with an appropriate slight modification of Voiculescu's original definition.

Let M_n denote the algebra of $n\times n$ complex matrices and M_n^{sa} the space of selfadjoint matrices in M_n. Let tr_n stand for the normalized trace on M_n while Tr_n is the usual trace. The Lebesgue measure $\hat\Lambda_n$ on M_n (resp. Λ_n on M_n^{sa}) is transformed from the usual Lebesgue measure on $\mathbb{R}^{2n^2}$ (resp. $\mathbb{R}^{n^2}$) via the natural isometry $M_n \cong \mathbb{R}^{2n^2}$ (resp. $M_n^{sa}\cong\mathbb{R}^{n^2}$) between the Hilbert-Schmidt norm of M_n (resp. M_n^{sa}) and the Euclidean norm of $\mathbb{R}^{2n^2}$ (resp. $\mathbb{R}^{n^2}$). Consider the map $A\mapsto(B,C)\in(M_n^{sa})^2$ given by the Descartes decomposition $A=B+\mathrm{i}C$. Since $\|A\|_{HS}=(\|B\|_{HS}^2+\|C\|_{HS}^2)^{1/2}$ for the Hilbert-Schmidt norm, the following is obvious.

Lemma 1.1. *Under the map $A\in M_n\mapsto(B,C)\in(M_n^{sa})^2$ above, $\hat\Lambda_n$ on M_n corresponds to $\Lambda_n\otimes\Lambda_n$ on $(M_n^{sa})^2$.*

Let $a_1,\dots,a_N\in\mathcal{M}$. For $n,r\in\mathbb{N}$, $\varepsilon>0$ and $R>0$ define

$$\begin{aligned}\hat\Gamma_R(a_1,\dots,a_N;n,r,\varepsilon) := \big\{(A_1,\dots,A_N)\in(M_n)^N : {}&\|A_i\|\le R,\\ &|\mathrm{tr}_n(A'_{i_1}\cdots A'_{i_k})-\tau(a'_{i_1}\cdots a'_{i_k})|\le\varepsilon\\ &\text{for all } 1\le i_1,\dots,i_k\le 2N,\ 1\le k\le r\big\}\end{aligned}$$

where

$$\begin{aligned}(a'_1,\dots,a'_{2N}) &:= (a_1,\dots,a_N,a_1^*,\dots,a_N^*)\,,\\ (A'_1,\dots,A'_{2N}) &:= (A_1,\dots,A_N,A_1^*,\dots,A_N^*)\,.\end{aligned}$$

Moreover,

$$\hat\chi_R(a_1,\dots,a_N;r,\varepsilon) := \limsup_{n\to\infty}\left[\frac{1}{n^2}\log\hat\Lambda(\hat\Gamma_R(a_1,\dots,a_N;n,r,\varepsilon)+N\log n\right]$$

($\hat\Lambda$ is used for $\hat\Lambda_n^{\otimes N}$ for brevity),

$$\hat\chi_R(a_1,\dots,a_N) := \lim_{\substack{r\to\infty\\ \varepsilon\to+0}}\hat\chi_R(a_1,\dots,a_N;r,\varepsilon)\,,$$

$$\hat\chi(a_1,\dots,a_N) := \sup_{R>0}\hat\chi_R(a_1,\dots,a_N)\,.$$

Then $\hat\chi(a_1,\dots,a_N)$ is called the free entropy of the N-tuple $(a_1,\dots,a_N)$.

Proposition 1.2. *Let $a_1,\dots,a_N\in\mathcal{M}$ and $b_i:=(a_i+a_i^*)/2$, $c_i:=(a_i-a_i^*)/2\mathrm{i}$. Then*

$$\chi_R(b_1,c_1,\dots,b_N,c_N)\ge\hat\chi_R(a_1,\dots,a_N)\ge\chi_{R/2}(b_1,c_1,\dots,b_N,c_N)\,,$$

$$\hat\chi(a_1,\dots,a_N)=\chi(b_1,c_1,\dots,b_N,c_N)\,.$$

Proof. The following are easy to check:

$$\begin{aligned}
&\Gamma_R(b_1,c_1,\dots,b_N,c_N;n,r,\varepsilon)\\
&\qquad\supset \bigl\{(B_1,C_1,\dots,B_N,C_N)\in (M_n^{sa})^{2N}:\\
&\qquad\qquad (B_1+\mathrm{i}\,C_1,\dots,B_N+\mathrm{i}\,C_N)\in \hat\Gamma_R(a_1,\dots,a_N;n,r,\varepsilon)\bigr\},
\end{aligned}$$

$$\begin{aligned}
&\hat\Gamma_{2R}(a_1,\dots,a_N;n,r,\varepsilon)\\
&\qquad\supset \bigl\{(B_1+\mathrm{i}\,C_1,\dots,B_N+\mathrm{i}\,C_N)\in (M_n)^{N}:\\
&\qquad\qquad (B_1,C_1,\dots,B_N,C_N)\in \Gamma_R(b_1,c_1,\dots,b_N,c_N;n,r,\varepsilon/2^r)\bigr\}.
\end{aligned}$$

By Lemma 1.1 these imply that

$$\chi_R(b_1,c_1,\dots,b_N,c_N;r,\varepsilon)\ge \hat\chi_R(a_1,\dots,a_N;r,\varepsilon),$$

$$\hat\chi_{2R}(a_1,\dots,a_N;r,\varepsilon)\ge \chi_R(b_1,c_1,\dots,b_N,c_N;r,\varepsilon/2^r).$$

Hence we get the conclusions. $\square$

The above proposition enables us to reformulate the results on free entropy of self-adjoint random variables as those of non-selfadjoint ones. For instance, $\hat\chi(a_1,\dots,a_N)$ is subadditive and upper semicontinuous similary to the selfadjoint case treated in [13].

Proposition 1.3. *Let $a_1,\dots,a_N\in\mathcal{M}$ and $C>0$. When $\tau(a_1^*a_1+\dots+a_N^*a_N)\le C$,*

$$\hat\chi(a_1,\dots,a_N)\le N\log\frac{\pi eC}{N},$$

*and the equality is attained if and only if $a_1,\dots,a_N$ are *-free circular elements of the same radius $2\sqrt{C/N}$.*

Proof. Let b_i,c_i be as above. Then $a_1,\dots,a_N$ are *-free circular elements of radius $2\sqrt{C/N}$ if and only if $b_1,c_1,\dots,b_N,c_N$ are free semicircular elements of radius $\sqrt{2C/N}$. Hence the result is just the translation of [15, Prop. 2.4] (also [3, Theorem 4.1]). $\square$

According to a strong result in [14] we know that $\chi(a_1,a_2)=-\infty$ if $a_1,a_2\in\mathcal{M}^{sa}$ commute. (This can be seen also by using the change of variable formula in [15].) By Proposition 1.2 and the subadditivity it follows that $\hat\chi(a_1,\dots,a_N)=-\infty$ whenever there is a normal element among $a_1,\dots,a_N$.

Next we turn to the entropy of unitary random variables. Let γ_n denote the Haar probability measure on the unitary group $\mathcal{U}(n)$. Let $u_1,\dots,u_N\in\mathcal{M}$ be unitaries. For $n,r\in\mathbb{N}$ and $\varepsilon>0$ define

$$\begin{aligned}
\Gamma_u(u_1,\dots,u_N;n,r,\varepsilon):=\bigl\{&(U_1,\dots,U_N)\in(\mathcal{U}(n))^N:\\
&|\mathrm{tr}_n(U'_{i_1}\cdots U'_{i_k})-\tau(u'_{i_1}\cdots u'_{i_k})|\le\varepsilon\\
&\text{for all } 1\le i_1,\dots,i_k\le 2N,\ 1\le k\le r\bigr\}
\end{aligned}$$

where

$$(u'_1, \dots, u'_{2N}) := (u_1, \dots, u_N, u_1^*, \dots, u_N^*)\,,$$
$$(U'_1, \dots, U'_{2N}) := (U_1, \dots, U_N, U_1^*, \dots, U_N^*)\,.$$

The free entropy $\chi_u(u_1, \dots, u_N)$ of the N-tuple $(u_1, \dots, u_N)$ is defined as follows:

$$\chi_u(u_1, \dots, u_N; r, \varepsilon) := \limsup_{n\to\infty} \frac{1}{n^2} \log \gamma(\Gamma_u(u_1, \dots, u_N; n, r, \varepsilon))$$

(γ is for $\gamma_n^{\otimes N}$ on $(\mathcal{U}(n))^N$),

$$\chi_u(u_1, \dots, u_N) := \lim_{\substack{r\to\infty \\ \varepsilon\to+0}} \chi_u(u_1, \dots, u_N; r, \varepsilon)\,.$$

For the case of a single unitary we have

Proposition 1.4. *Let $u \in \mathcal{M}$ be a unitary and μ be the distribution measure of u on $\mathbb{T}$. Then the limit*

$$\chi_u(u; r, \varepsilon) = \lim_{n\to\infty} \frac{1}{n^2} \log \gamma_n(\Gamma_u(u, n, r, \varepsilon)) \tag{1.1}$$

exists for every $r \in \mathbb{N}$ and $\varepsilon > 0$, and

$$\chi_u(u) = \Sigma(\mu) := \iint_{\mathbb{T}^2} \log|\zeta - \eta|\, d\mu(\zeta)\, d\mu(\eta)\,.$$

In particular, if $\chi_u(u) > -\infty$ then μ is non-atomic.

Proof. It is convenient to use large deviation theory, see [2] for basics on large deviations. Let $\mathcal{M}(\mathbb{T})$ be the space of all probability measures on $\mathbb{T}$ with the weak topology. Let P_n be the empirical eigenvalue distribution of γ_n which is a probability measure on $\mathcal{M}(\mathbb{T})$. It is known [4] that (P_n) satisfies the large deviation principle in the scale n^{-2} with rate function $I(\mu) := -\Sigma(\mu)$. For $r \in \mathbb{N}$ and $\varepsilon > 0$ set a closed neighborhood of $\mu \in \mathcal{M}(\mathbb{T})$

$$F(r, \varepsilon) := \{\nu \in \mathcal{M}(\mathbb{T}) : |m_k(\nu) - m_k(\mu)| \le \varepsilon,\ -r \le k \le r\}$$

and an open neighborhood $G(r, \varepsilon)$ by replacing $\le \varepsilon$ by $< \varepsilon$ in the above, where $m_k(\mu)$ denotes the k^{th} moment of μ. Then the above large deviation theorem implies that

$$\limsup_{n\to\infty} \frac{1}{n^2} \log P_n(F(r, \varepsilon)) \le \sup\{\Sigma(\nu) : \nu \in F(r, \varepsilon)\}\,,$$

$$\liminf_{n\to\infty} \frac{1}{n^2} \log P_n(G(r, \varepsilon)) \ge \sup\{\Sigma(\nu) : \nu \in G(r, \varepsilon)\}\,.$$

But it is straightforward to see that

$$P_n(G(r, \varepsilon)) \le P_n(F(r, \varepsilon)) = \gamma_n(\Gamma_u(u; n, r, \varepsilon))\,,$$

$$\sup\{\Sigma(\nu) : \nu \in F(r, \varepsilon)\} = \sup\{\Sigma(\nu) : \nu \in G(r, \varepsilon)\}\,.$$

Therefore,

$$\chi_u(u,r,\varepsilon)=\lim_{n\to\infty}\frac{1}{n^2}\log\gamma_n(\Gamma_u(u;n,r,\varepsilon))=\sup\{\Sigma(\nu):\nu\in F(r,\varepsilon)\}\,.$$

Since the latter tends to $\Sigma(\mu)$ as $r\to\infty$ and $\varepsilon\to+0$, we have $\chi_u(u)=\Sigma(\mu)$. □

Remark 1.5. When $a\in\mathcal{M}^{sa}$, similarly to the above proof one can use the large deviation technique to show that the limit

$$\chi_R(a;r,\varepsilon)=\lim_{n\to\infty}\left[\frac{1}{n^2}\log\Lambda_n(\Gamma_R(a;n,r,\varepsilon))+\frac{1}{2}\log n\right]$$

exists for every $r\in\mathbb{N}$, $\varepsilon>0$ and $R\ge\|a\|$. This slightly improves the result in [13]. The large deviation used here is concerned with the empirical eigenvalue distribution for the normalized Lebesgue measure on $\{A\in M_n^{sa}:\|A\|\le R\}$. (The details are in [6].)

The negativity $\chi_u(u_1,\dots,u_N)\le 0$ is obvious. The subadditivity and the upper semicontinuity of $\chi_u(u_1,\dots,u_N)$ are easily shown as in the selfadjoint case in [13]. The following is a unitary counterpart of [13, Prop. 3.8].

Proposition 1.6. *Let $u_1,\dots,u_N,v_1,\dots,v_N\in\mathcal{M}$ be unitaries. If $v_1=u_1$ and $v_iu_i^*\in\{u_1,\dots,u_{i-1}\}''$ for $2\le i\le N$, then*

$$\chi_u(u_1,\dots,u_N)=\chi_u(v_1,\dots,v_N)\,.$$

Proof. Since the assumption implies also that $u_iv_i^*\in\{v_1,\dots,v_{i-1}\}''$ for $2\le i\le N$, it suffices to show that $\chi_u(u_1,\dots,u_N)\le\chi_u(v_1,\dots,v_N)$. One can choose selfadjoint noncommutative polynomials $P_{m,i}(X_1,X_2,\dots,X_{2i-2})$ for $2\le i\le N$, $m\in\mathbb{N}$, such that

$$\exp\Bigl(\mathrm{i}\,P_{m,i}\Bigl(\frac{u_1+u_1^*}{2},\frac{u_1-u_1^*}{2\mathrm{i}},\dots,\frac{u_{i-1}+u_{i-1}^*}{2},\frac{u_{i-1}-u_{i-1}^*}{2\mathrm{i}}\Bigr)\Bigr)\to v_iu_i^*$$

strongly* as $m\to\infty$. Set $v_{m,1}:=v_1=u_1$ and for $2\le i\le N$,

$$v_{m,i}:=\exp\Bigl(\mathrm{i}\,P_{m,i}\Bigl(\frac{u_1+u_1^*}{2},\frac{u_1-u_1^*}{2\mathrm{i}},\dots,\frac{u_{i-1}+u_{i-1}^*}{2},\frac{u_{i-1}-u_{i-1}^*}{2\mathrm{i}}\Bigr)\Bigr)u_i\,.$$

Then $v_{m,i}\to v_i$ strongly* as $m\to\infty$. If a map $\Phi:\mathcal{U}(n)^N\to\mathcal{U}(n)^N$, $\Phi(U_1,\dots,U_N)=(V_1,\dots,V_N)$, is defined by $V_1:=U_1$ and for $2\le i\le N$,

$$V_i:=\exp\Bigl(\mathrm{i}\,P_{m,i}\Bigl(\frac{U_1+U_1^*}{2},\frac{U_1-U_1^*}{2\mathrm{i}},\dots,\frac{U_{i-1}+U_{i-1}^*}{2},\frac{U_{i-1}-U_{i-1}^*}{2\mathrm{i}}\Bigr)\Bigr)U_i\,,$$

then it is obvious that $\gamma\circ\Phi=\gamma$ holds due to the multiplication invariance of γ. For any $m,r\in\mathbb{N}$ and $\varepsilon>0$ one can easily see that there are $r_1\in\mathbb{N}$ and $\varepsilon_1>0$ such that

$$\Phi(\Gamma_u(u_1,\dots,u_N;n,r_1,\varepsilon_1))\subset\Gamma_u(v_{m,1},\dots,v_{m,N};n,r,\varepsilon)\qquad(n\in\mathbb{N}).$$

This yields

$$\chi_u(u_1,\dots,u_N;r_1,\varepsilon_1)\le\chi_u(v_{m,1},\dots,v_{m,N};r,\varepsilon)$$

so that $\chi_u(u_1,\dots,u_N)\le\chi_u(v_{m,1},\dots,v_{m,N})$. Hence the desired inequality follows as $m\to\infty$ thanks to the upper semicontinuity. □

2. Relation among different free entropies

Let $u_1, \dots, u_N \in \mathcal{M}$ be unitaries and $h_1, \dots, h_N \in \mathcal{M}^+$, where $\mathcal{M}^+$ denotes the set of positive elements in $\mathcal{M}$. The free entropy $\hat{\chi}(u_1h_1, \dots, u_Nh_N)$ may be also considered as the free entropy of the $2N$-tuple $(u_1, \dots, u_N, h_1, \dots, h_N)$ or rather $(u_1, \dots, u_N, h_1^2, \dots, h_N^2)$ of unitary and positive random variables mixed. In this section we will first introduce the free entropy of the mixed tuple of this kind and next obtain its connection with $\hat{\chi}(u_1h_1, \dots, u_Nh_N)$. In this way, we can construct a bridge between the free entropy of unitary random variables and that of non-selfadjoint ones (thus selfadjoint ones).

For a non-singular $A \in M_n$ (the singular case is negligible) one has a unique polar decomposition $A = UH$ with $U \in \mathcal{U}(n)$ and $H = |A| \in M_n^+$, where M_n^+ denotes the set of positive matrices in M_n. Let $\Lambda_{+,n}$ be the measure on M_n^+ induced from $\hat{\Lambda}_n$ on M_n via the map $A \mapsto A^*A$. (This measure is more convenient than that induced via $A \mapsto |A|$.) The next lemma shows that $\hat{\Lambda}_n$ on M_n corresponds (up to a constant) to the product of γ_n on $\mathcal{U}(n)$ and the restriction of Λ_n on M_n^+.

Lemma 2.1. *The measure $\hat{\Lambda}_n$ is transformed to the product measure $\gamma_n \otimes \Lambda_{+,n}$ under the map $A \in M_n \mapsto (U, A^*A) \in \mathcal{U}(n) \times M_n^+$ (U is the unitary part of A). Furthermore, the measure $\Lambda_{+,n}$ is a constant multiple of the restriction of Λ_n on M_n^+:*

$$\Lambda_{+,n} = C_n \Lambda_n|_{M_n^+} \quad \textit{with} \quad C_n = 2^{-n(n-1)/2}\pi^{n(n+1)/2}\left(\prod_{j=1}^{n-1} j!\right)^{-1}.$$

Proof. We consider under the coordinate change $H \in M_n^+ \leftrightarrow (V, D) \in \mathcal{U}(n)/T \times (\mathbb{R}^+)^n_\leq$ by the diagonalization $H = VDV^*$, where T is the diagonal unitaries and $(\mathbb{R}^+)^n_\leq := \{(t_1, \dots, t_n) : 0 \leq t_1 \leq \cdots \leq t_n\}$. Let $\dot{\gamma}_n$ be the probability measure on $\mathcal{U}(n)/T$ induced from γ_n. Write $A^*A = VDV^*$ and $A = UVD^{1/2}V^*$ with $U, V \in \mathcal{U}(n)$ and $D = \mathrm{Diag}(t_1, \dots, t_n)$. Differentiating $A = UVD^{1/2}V^*$ and using the standard method for random matrices (see [7]) one can easily see that $\hat{\Lambda}_n$ is transformed to the measure

$$\gamma_n \otimes \dot{\gamma}_n \otimes \left(C_n' \prod_{i<j}(t_i - t_j)^2 \prod_{i=1}^n dt_i\right)$$

on $\mathcal{U}(n) \times \mathcal{U}(n)/T \times (\mathbb{R}^+)^n_\leq$ under the map $A \mapsto (U, V, D)$. To determine the normalizing constant C_n', use the standard $n \times n$ non-selfadjoint Gaussian matrix having the distribution $(\pi/n)^{-n^2} \exp(-n^2 \mathrm{tr}_n(A^*A))$ and compute

$$\begin{aligned} 1 &= \left(\frac{\pi}{n}\right)^{-n^2} C_n' \int_{(\mathbb{R}^+)^n_\geq} \exp\left(-n\sum_{i=1}^n t_i\right) \prod_{i<j}(t_i - t_j)^2 \, dt \\ &= \pi^{-n^2} C_n' \left(\prod_{j=1}^{n-1} j!\right)^2 \end{aligned}$$

by the Selberg integral [7, p. 354]. Hence, under the coordinate change $H \leftrightarrow (V,D)$, the measure $\Lambda_{+,n}$ is written as

$$\dot\gamma_n \otimes \left(\pi^{n^2}\left(\prod_{j=1}^{n-1} j!\right)^{-2} \prod_{i<j}(t_i-t_j)^2 \prod_{i=1}^n dt_i\right).$$

On the other hand, Λ_n is given as

$$\dot\gamma_n \otimes \left((2\pi)^{n(n-1)/2}\left(\prod_{j=1}^{n-1} j!\right)^{-1} \prod_{i<j}(t_i-t_j)^2 \prod_{i=1}^n dt_i\right).$$

(See the proof of [13, Lemma 4.2] for the latter normalizing constant.) Comparing the above two we get the conclusion □

The above constant C_n satisfies

$$\lim_{n\to\infty}\left(\frac{1}{n^2}\log C_n + \frac{1}{2}\log n\right) = \frac{1}{2}\log\frac{\pi}{2} + \frac{3}{4}, \tag{2.1}$$

as is readily checked from the Stirling formula.

Let $u_1,\dots,u_N \in \mathcal{M}$ be unitaries and $h_1,\dots,h_L \in \mathcal{M}^+$. For $n,r \in \mathbb{N}$, $\varepsilon > 0$ and $R > 0$ we define

$$\begin{aligned}
&\Gamma_{(u,+),R}(u_1,\dots,u_N;h_1,\dots,h_L;n,r,\varepsilon)\\
&\quad := \big\{(U_1,\dots,U_N;H_1,\dots,H_L) \in (\mathcal{U}(n))^N \times (M_n^+)^L : \|H_i\| \le R,\\
&\qquad\qquad |\mathrm{tr}_n(B_{i_1}\cdots B_{i_k}) - \tau(b_{i_1}\cdots b_{i_k})| \le \varepsilon\\
&\qquad\qquad \text{for all } 1 \le i_1,\dots,i_k \le 2N+L,\ 1 \le k \le r\big\},
\end{aligned}$$

where

$$\begin{aligned}
(b_1,\dots,b_{2N+L}) &:= (u_1,\dots,u_N,u_1^*,\dots,u_N^*,h_1,\dots,h_L),\\
(B_1,\dots,B_{2N+L}) &:= (U_1,\dots,U_N,U_1^*,\dots,U_N^*,H_1,\dots,H_L),
\end{aligned}$$

and further define

$$\begin{aligned}
&\chi_{(u,+),R}(u_1,\dots,u_N;h_1,\dots,h_L;r,\varepsilon)\\
&\quad := \limsup_{n\to\infty}\left[\frac{1}{n^2}\log(\gamma\otimes\Lambda_+)(\Gamma_{(u,+),R}(u_1,\dots,u_N;h_1,\dots,h_L;n,r,\varepsilon)) + L\log n\right],
\end{aligned}$$

where $\gamma\otimes\Lambda_+$ is the abbreviation of $\gamma_n^{\otimes N}\otimes\Lambda_{+,n}^{\otimes L}$. Now the definition of the free entropy $\chi_{(u,+)}(u_1,\dots,u_N;h_1,\dots,h_L)$ is as before. When $L=0$ or $(h_1,\dots,h_L)$ is void, this is nothing but $\chi_u(u_1,\dots,u_N)$ in the previous section. On the other hand, when no unitaries are present, we write $\Gamma_{+,R}(h_1,\dots,h_L;n,r,\varepsilon)$, $\chi_+(h_1,\dots,h_L)$, etc.

It is obvious that the free entropy $\chi_{(u,+)}$ has the subadditivity property as χ and χ_u. Moreover, the following upper semicontinuity of $\chi_{(u,+)}$ can be shown in a way similar to [13, Props. 2.4 and 2.6].

Proposition 2.2. *Let $u_1, \dots, u_N$ and $u_{m,1}, \dots, u_{m,N}$ ($m \in \mathbb{N}$) be unitaries in $\mathcal{M}$. Let $h_1, \dots, h_L$ and $h_{m,1}, \dots, h_{m,L}$ ($m \in \mathbb{N}$) be in $\mathcal{M}^+$. If $(u_{m,1}, \dots, u_{m,N}, h_{m,1}, \dots, h_{m,L}) \to (u_1, \dots, u_N, h_1, \dots, h_L)$ in *-distribution and* $\sup_m \|h_{m,i}\| < +\infty$ $(1 \le i \le L)$, *then*

$$\chi_{(u,+)}(u_1, \dots, u_N; h_1, \dots, h_L) \ge \limsup_{m\to\infty} \chi_{(u,+)}(u_{m,1}, \dots, u_{m,N}; h_{m,1}, \dots, h_{m,L}) .$$

The next proposition says that the free entropy χ_+ is nothing but χ restricted on positive random variables up to additive constants.

Proposition 2.3. *The equality*

$$\chi_+(h_1, \dots, h_N) = \chi(h_1, \dots, h_N) + \frac{N}{2}\Big(\log\frac{\pi}{2} + \frac{3}{2}\Big)$$

holds for every $N \in \mathbb{N}$ and $h_1, \dots, h_N \in \mathcal{M}^+$.

Proof. For $r \in \mathbb{N}$, $\varepsilon > 0$ and $R > 0$, it is obvious that

$$\Gamma_{+,R}(h_1, \dots, h_N; n, r, \varepsilon) \subset \Gamma_R(h_1, \dots, h_N; n, r, \varepsilon)$$

(the right-hand side is taken in $(M_n^{sa})^N \supset (M_n^+)^N$). Hence it immediately follows from Lemma 2.1 and (2.1) that

$$\chi_+(h_1, \dots, h_N) \le \chi(h_1, \dots, h_N) + \frac{N}{2}\Big(\log\frac{\pi}{2} + \frac{3}{2}\Big) .$$

To show the reverse inequality, we choose $(h_1 + \delta\mathbf{1}, \dots, h_N + \delta\mathbf{1})$ instead of $(h_1, \dots, h_N)$ and also $R > \max_i \|h_i\| + \delta$ for $\delta > 0$. From [13, Prop. 2.4] and the translation invariance of Λ_n we can estimate

$$\begin{aligned}
&\chi(h_1, \dots, h_N) \\
&\quad = \chi(h_1 + \delta\mathbf{1}, \dots, h_N + \delta\mathbf{1}) \\
&\quad = \lim_{\substack{r\to\infty \\ \varepsilon\to+0}} \limsup_{n\to\infty} \bigg[\frac{1}{n^2}\log\Lambda_n(\Gamma_{+,R}(h_1 + \delta\mathbf{1}, \dots, h_N + \delta\mathbf{1}; n, r, \varepsilon)) + \frac{N}{2}\log n\bigg] \\
&\quad = \lim_{\substack{r\to\infty \\ \varepsilon\to+0}} \limsup_{n\to\infty} \bigg[\frac{1}{n^2}\log\Lambda_{+,n}(\Gamma_{+,R}(h_1 + \delta\mathbf{1}, \dots, h_N + \delta\mathbf{1}; n, r, \varepsilon)) + N\log n \\
&\qquad\qquad - N\Big(\frac{1}{n^2}\log C_n + \frac{1}{2}\log n\Big)\bigg] \\
&\quad \le \chi_+(h_1 + \delta\mathbf{1}, \dots, h_N + \delta\mathbf{1}) - \frac{N}{2}\Big(\log\frac{\pi}{2} + \frac{3}{2}\Big) .
\end{aligned}$$

Using the upper semicontinuity (Proposition 2.2) as $\delta \to +0$ we obtain the result. □

Remarks 2.4. (1) For $\chi_+(h)$ of a single $h \in \mathcal{M}^+$ we have

$$\chi_+(h) = \chi(h) + \frac{1}{2}\log\frac{\pi}{2} + \frac{3}{4} = \Sigma(\mu) + \log\pi + \frac{3}{2} ,$$

where μ is the distribution of h and $\Sigma(\mu) := \iint \log|s-t|\,d\mu(s)\,d\mu(t)$. Moreover, by Remark 1.5 and Lemma 2.1 we observe that the limit

$$\chi_{+,R}(h;r,\varepsilon) = \lim_{n\to\infty}\left[\frac{1}{n^2}\log\Lambda_{+,n}(\Gamma_{+,R}(h;n,r,\varepsilon)) + \log n\right] \tag{2.2}$$

exists for every $r \in \mathbb{N}$, $\varepsilon > 0$ and $R \geq \|h\|$.

(2) Note (see [3]) that, among $h \in \mathcal{M}^+$ with $\tau(h) \leq C$, the free entropy $\chi_+(h)$ attains the maximal value $\log(\pi e C)$ when (and only when) h has the distribution

$$\frac{\sqrt{4Ct - t^2}}{2\pi Ct}\chi_{[0,4C]}(t)\,dt$$

or equivalently $h^{1/2}$ is a quarter-circular element of radius $2\sqrt{C}$.

The following relation between two free entropies $\hat{\chi}$ and $\chi_{(u,+)}$ is naturally expected from the definitions in the light of Lemma 2.1.

Theorem 2.5. *If $u_1,\dots,u_N \in \mathcal{M}$ are unitaries and $h_1,\dots,h_N \in \mathcal{M}^+$, then*

$$\begin{aligned}\hat{\chi}(u_1h_1,\dots,u_Nh_N) &= \chi_{(u,+)}(u_1,\dots,u_N;h_1^2,\dots,h_N^2)\\ &\leq \chi_u(u_1,\dots,u_N) + \chi(h_1^2,\dots,h_N^2) + \frac{N}{2}\left(\log\frac{\pi}{2} + \frac{3}{2}\right).\end{aligned}$$

To prove the theorem, we need to approximate the unitary part of A by polynomials of A, A^*. The approximation here must be uniform for $A \in M_n$ with $\|A\| \leq R$ in some sense. The next lemma provides the right approximation procedure for our purpose.

Let $a \in \mathcal{M}$ and assume that the distribution of $|a|$ is non-atomic. Let $a = u|a|$ be the polar decomposition. Note that $u \in \mathcal{M}$ must be a unitary because $\ker a = \{0\}$ from the assumption (and $\mathcal{M}$ is a finite von Neumann algebra). Let $\|\cdot\|_p$ denote the Schatten p-norm with respect to τ or tr_n.

Lemma 2.6. *With the above assumption and notation, for every $p \geq 1$, $\varepsilon > 0$ and $R \geq \|a\|$, there exist $n_0, r \in \mathbb{N}$, $\delta > 0$ and a real polynomial $P(t)$ such that $\|u - aP(a^*a)\|_p \leq \varepsilon$, and such that, for each $n \geq n_0$, if $A \in M_n$ with $\|A\| \leq R$ is non-singular and U is the unitary part of A and if*

$$|\mathrm{tr}_n((A^*A)^k) - \tau((a^*a)^k)| \leq \delta \qquad (1 \leq k \leq r), \tag{2.3}$$

then $\|U - AP(A^*A)\|_p \leq \varepsilon$.

Proof. Let μ be the distribution of $|a|$. For every $\alpha, \beta > 0$, since

$$u - a(|a| + \alpha\mathbf{1})^{-1} = u(\mathbf{1} - |a|(|a| + \alpha\mathbf{1})^{-1}) = \alpha u(|a| + \alpha\mathbf{1})^{-1},$$

we have

$$\begin{aligned}\|u - a(|a| + \alpha\mathbf{1})^{-1}\|_p^p &= \|\alpha(|a| + \alpha\mathbf{1})^{-1}\|_p^p\\ &= \int_0^\infty \left(\frac{\alpha}{t+\alpha}\right)^p d\mu(t) \leq \mu([0,\beta]) + \left(\frac{\alpha}{\beta}\right)^p.\end{aligned}$$

Similarly for any non-singular $A \in M_n$ with $A = U|A|$, we have

$$\|U - A(|A| + \alpha I)^{-1}\|_p^p = \frac{1}{n}\sum_{i=1}^n \Big(\frac{\alpha}{\lambda_i + \alpha}\Big)^p \le \frac{1}{n}\#\{i : \lambda_i \le \beta\} + \Big(\frac{\alpha}{\beta}\Big)^p,$$

where $(0 <)\, \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are the eigenvalues of $|A|$.

Now for each $n \in \mathbb{N}$, since μ is non-atomic, one can choose $0 < \xi_1^{(n)} < \xi_2^{(n)} < \cdots < \xi_n^{(n)} = \|a\|$ such that $\mu([0, \xi_i^{(n)}]) = i/n$ $(1 \le i \le n)$. Then it immediately follows that

$$\tau((a^*a)^k) = \int_0^\infty t^{2k}\, d\mu(t) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n (\xi_i^{(n)})^{2k} \qquad (k \in \mathbb{N}).$$

Let $\beta > 0$ be fixed so that $\mu([0, 2\beta]) < \varepsilon^p/2$. By [13, Lemma 4.3] there are $r \in \mathbb{N}$ and $\delta > 0$ such that, for every $n \in \mathbb{N}$, if $(\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^+)^n_\le$ satisfies

$$\Big|\frac{1}{n}\sum_{i=1}^n \lambda_i^{2k} - \frac{1}{n}\sum_{i=1}^n (\xi_i^{(n)})^{2k}\Big| \le 2\delta \qquad (1 \le k \le r),$$

then

$$\frac{1}{n}\sum_{i=1}^n (\lambda_i^2 - (\xi_i^{(n)})^2)^2 \le \beta^4 \varepsilon^p. \tag{2.4}$$

Next, choose $n_0 \in \mathbb{N}$ such that

$$\Big|\frac{1}{n}\sum_{i=1}^n (\xi_i^{(n)})^{2k} - \tau((a^*a)^k)\Big| \le \delta \qquad (1 \le k \le r)$$

whenever $n \ge n_0$. Then, for any $n \ge n_0$, (2.4) is valid if $A \in M_n$ satisfies (2.3). Furthermore, when (2.3) is satisfied, we have

$$\frac{1}{n}\#\{i : \lambda_i \le \beta\} \le \frac{11}{18}\varepsilon^p. \tag{2.5}$$

Indeed, put $l := \#\{i : \lambda_i \le \beta\}$ and $m := \#\{i : \xi_i^{(n)} \le 2\beta\}$. If $m < i \le l$, then $\lambda_i \le \lambda_l \le \beta$, but $2\beta < \xi_{m+1}^{(n)} \le \xi_i^{(n)}$, so $(\lambda_i^2 - (\xi_i^{(n)})^2)^2 \ge (4\beta^2 - \beta^2)^2 = 9\beta^4$. Hence (2.4) implies $\frac{1}{n}(l - m) \cdot 9\beta^4 \le \beta^4\varepsilon^p$, so that $l/n \le m/n + \varepsilon^p/9$. Since

$$\frac{m}{n} = \mu([0, \xi_m^{(n)}]) \le \mu([0, 2\beta]) \le \frac{\varepsilon^p}{2},$$

we have $l/n \le \varepsilon^p/2 + \varepsilon^p/9$, showing (2.5).

By the above estimates altogether, we infer that, for each $\alpha > 0$ and $n \ge n_0$, if $A \in M_n$ with $\|A\| \le R$ is non-singular and satisfies (2.3), then

$$\|U - A(|A| + \alpha I)^{-1}\|_p^p \le \frac{11}{18}\varepsilon^p + \Big(\frac{\alpha}{\beta}\Big)^p$$

as well as

$$\|u - a(|a| + \alpha\mathbf{1})^{-1}\|_p^p \le \frac{\varepsilon^p}{2} + \left(\frac{\alpha}{\beta}\right)^p.$$

Choose $\alpha > 0$ such that $(\alpha/\beta)^p \le \varepsilon^p/18$, and next choose a polynomial $P(t)$ such that

$$|P(t) - (\sqrt{t} + \alpha)^{-1}| \le \frac{1}{R}\left(1 - \left(\frac{2}{3}\right)^{1/p}\right)\varepsilon \quad \text{on} \quad [0, R^2].$$

Then for each $n \ge n_0$ and A as above, we obtain

$$\|U - A(|A| + \alpha I)^{-1}\|_p \le \left(\frac{2}{3}\right)^{1/p}\varepsilon\,,$$

$$\begin{aligned}\|AP(A^*A) - A(|A| + \alpha I)^{-1}\|_p &\le \|A\|\,\|P(A^*A) - (|A| + \alpha I)^{-1}\| \\ &\le \left(1 - \left(\frac{2}{3}\right)^{1/p}\right)\varepsilon\,,\end{aligned}$$

so $\|U - AP(A^*A)\|_p \le \varepsilon$ holds, and similarly $\|u - aP(a^*a)\|_p \le \varepsilon$. □

Proof of Theorem 2.5. First, the inequality in the theorem is a consequence of the subadditivity of $\chi_{(u,+)}$ and Proposition 2.3. Define $\Psi : M_n \to \mathcal{U}(n) \times M_n^+$ by $\Psi(A) := (U, A^*A)$ where U is the unitary part of A. This is bijective except the negligible singular elements (in M_n and M_n^+). Put $a_i := u_i h_i$ so that $h_i^2 = a_i^* a_i$. Let $n, r \in \mathbb{N}$, $\varepsilon > 0$ and $R > \max\{1, \|h_1\|, \dots, \|h_N\|\}$. It is straightforward to see that there are $r_1 \in \mathbb{N}$ and $\varepsilon_1 > 0$ such that

$$\Psi(\hat{\Gamma}_R(a_1, \dots, a_N; n, r_1, \varepsilon_1)) \subset \mathcal{U}(n) \times \Gamma_{+,R^2}(h_1^2, \dots, h_N^2; n, r, \varepsilon)$$

and by Lemma 2.1

$$\hat{\Lambda}(\hat{\Gamma}_R(a_1, \dots, a_N; n, r_1, \varepsilon_1)) \le \Lambda_+(\Gamma_{+,R^2}(h_1^2, \dots, h_N^2; n, r, \varepsilon))$$

for all $n \in \mathbb{N}$. This yields $\hat{\chi}(a_1, \dots, a_N) \le \chi_+(h_1^2, \dots, h_N^2)$. Hence we may assume that $\chi(h_1^2, \dots, h_N^2) > -\infty$, so the distribution of each h_i is non-atomic.

Let $\varepsilon_0 > 0$ be such that $r\varepsilon_0(R^2 + \varepsilon_0)^{r-1} \le \varepsilon/3$. By Lemma 2.6 there exist $n_0, r_0 \in \mathbb{N}$, $\delta > 0$ and real polynomials $P_i(t)$ $(1 \le i \le N)$ such that $\|u_i - a_i P_i(a_i^* a_i)\|_r \le \varepsilon_0$, and such that, for each $1 \le i \le N$ and $n \ge n_0$, if $A_i \in M_n$ is non-singular with $A_i = U_i|A_i|$, $\|A_i\| \le R$, and

$$|\mathrm{tr}_n((A_i^* A_i)^k) - \tau((a_i^* a_i)^k)| \le \delta \qquad (1 \le k \le r_0),$$

then $\|U_i - A_i P_i(A_i^* A_i)\|_r \le \varepsilon_0$. For $A_i \in M_n$ $(1 \le i \le N)$ satisfying the above conditions, we set

$$\begin{aligned}(B_1, \dots, B_{3N}) &:= (U_1, \dots, U_N, U_1^*, \dots, U_N^*, A_1^*A_1, \dots, A_N^*A_N)\,, \\ (B_1', \dots, B_{3N}') &:= (A_1P_1(A_1^*A_1), \dots, A_NP_N(A_N^*A_N), \\ &\qquad P_1(A_1^*A_1)A_1^*, \dots, P_N(A_N^*A_N)A_N^*, A_1^*A_1, \dots, A_N^*A_N)\,,\end{aligned}$$

as well as

$$
\begin{aligned}
(b_1, \dots, b_{3N}) &:= (u_1, \dots, u_N, u_1^*, \dots, u_N^*, a_1^* a_1, \dots, a_N^* a_N)\,, \\
(b_1', \dots, b_{3N}') &:= (a_1 P_1(a_1^* a_1), \dots, a_N P_N(a_N^* a_N), \\
&\qquad P_1(a_1^* a_1) a_1^*, \dots, P_N(a_N^* a_N) a_N^*, a_1^* a_1, \dots, a_N^* a_N)\,.
\end{aligned}
$$

Then for any $n \ge n_0$ and $1 \le i_1, \dots, i_k \le 3N$ $(1 \le k \le r)$, by using the Hölder inequality, it is checked that

$$
\begin{aligned}
|\mathrm{tr}_n(B_{i_1} \cdots B_{i_k}) - \mathrm{tr}_n(B'_{i_1} \cdots B'_{i_k})| &\le \|B_{i_1} \cdots B_{i_k} - B'_{i_1} \cdots B'_{i_k}\|_1 \\
&\le k\varepsilon_0 (R^2 + \varepsilon_0)^{k-1} \le \frac{\varepsilon}{3}\,,
\end{aligned}
$$

and similarly $|\tau(b_{i_1} \cdots b_{i_k}) - \tau(b'_{i_1} \cdots b'_{i_k})| \le \varepsilon/3$. Now choose r_1 $(\ge 2r_0)$ large enough and ε_1 $(\le \delta)$ small enough such that if $(A_1, \dots, A_N) \in \hat{\Gamma}_R(a_1, \dots, a_N; n, r_1, \varepsilon_1)$ then $|\mathrm{tr}_n(B'_{i_1} \cdots B'_{i_k}) - \tau(b'_{i_1} \cdots b'_{i_k})| \le \varepsilon/3$ for all $1 \le i_1, \dots, i_k \le 3N$ $(1 \le k \le r)$. Therefore, for $n \ge n_0$ we obtain

$$
\Psi(\hat{\Gamma}_R(a_1, \dots, a_N; n, r_1, \varepsilon_1)) \subset \Gamma_{(u,+),R^2}(u_1, \dots, u_N; h_1^2, \dots, h_N^2; n, r, \varepsilon)
$$

(up to negligible sets) and hence by Lemma 2.1

$$
\hat{\Lambda}(\hat{\Gamma}_R(a_1, \dots, a_N; n, r_1, \varepsilon_1)) \le (\gamma \otimes \Lambda_+)(\Gamma_{(u,+),R^2}(u_1, \dots, u_N; h_1^2, \dots, h_N^2; n, r, \varepsilon))\,.
$$

This implies that

$$
\hat{\chi}(a_1, \dots, a_N) \le \chi_{(u,+)}(u_1, \dots, u_N; h_1^2, \dots, h_N^2)\,.
$$

Conversely, given $r \in \mathbb{N}$, $\varepsilon > 0$ and $R > 0$, by approximating $\sqrt{t}$ on $[0, R^2]$ by a polynomial, it is seen that there are $r_1 \in \mathbb{N}$ and $\varepsilon_1 > 0$ such that

$$
\Gamma_{(u,+),R^2}(u_1, \dots, u_N; h_1^2, \dots, h_N^2; n, r_1, \varepsilon_1) \subset \Psi(\hat{\Gamma}_R(a_1, \dots, a_N; n, r, \varepsilon))
$$

(up to negligible sets) for all $n \in \mathbb{N}$. This gives the reverse inequality. $\square$

Theorem 2.5 gives

$$
\hat{\chi}(a_1, \dots, a_N) \le \chi_u(u_1, \dots, u_N) + \chi(a_1^* a_1, \dots, a_N^* a_N) + \frac{N}{2}\left(\log \frac{\pi}{2} + \frac{3}{2}\right)
$$

for every $a_1, \dots, a_N \in \mathcal{M}$ and all unitaries $u_1, \dots, u_N \in \mathcal{M}$ satisfying $a_i = u_i |a_i|$. In particular, we have the following corollary. Its proof was indeed given in the first paragraph of the proof of Theorem 2.5.

Corollary 2.7. *Let $a_1, \dots, a_N \in \mathcal{M}$. If $\hat{\chi}(a_1, \dots, a_N) > -\infty$, then the distribution of $a_i^* a_i$ is non-atomic (hence $\ker a_i = \{0\}$) for every $1 \le i \le N$.*

3. Additivity of free entropies

In this section we first show that the inequality in Theorem 2.5 can be replaced by the equality in some cases of the free relation. Second, we discuss the additivity properties of the free entropies χ_u and $\hat{\chi}$. The characterization of the additivity of χ_u is completely analogous to the case of χ.

First, we take a free family $\{h_1,\dots,h_N\}$ which is also free from $\{u_1,\dots,u_N, u_1^*,\dots,u_N^*\}$. Then an exact relation among $\hat{\chi}$, χ_u and χ is obtained as follows. Hence we have a formula for χ_u in terms of $\hat{\chi}$ (hence χ).

Theorem 3.1. *Let $u_1,\dots,u_N \in \mathcal{M}$ be unitaries and $h_1,\dots,h_N \in \mathcal{M}^+$. If $\{u_1,\dots,u_N, u_1^*,\dots,u_N^*\}$, $h_1,\dots,h_N$ are free, then*

$$\hat{\chi}(u_1h_1,\dots,u_Nh_N) = \chi_u(u_1,\dots,u_N) + \sum_{i=1}^N \chi(h_i^2) + \frac{N}{2}\Big(\log\frac{\pi}{2} + \frac{3}{2}\Big).$$

In particular, if $h_1,\dots,h_N$ are free standard (i.e. of radius 2) quarter-circular elements and they are free from $\{u_1,\dots,u_N,u_1^*,\dots,u_N^*\}$, then

$$\begin{aligned}\chi_u(u_1,\dots,u_N) &= \hat{\chi}(u_1h_1,\dots,u_Nh_N) - N\log(\pi e)\\ &= \chi(b_1,c_1,\dots,b_N,c_N) - N\log(\pi e),\end{aligned}$$

where $u_ih_i = b_i + \mathrm{i}\,c_i$ with selfadjoint b_i, c_i.

In the proof below we use Voiculescu's result on approximate freeness for standard unitary random matrices. The notion of approximate freeness for matrices was introduced in [17]. Let $(M_n^{\star N}, \mathrm{tr}_n^{\star N})$ be the free product of N-copies of (M_n, tr_n) and j_i the injection of M_n into the $i^{\rm th}$ copy in $M_n^{\star N}$. When $\Omega_i \subset M_n$ $(1 \le i \le N)$, $r \in \mathbb{N}$ and $\varepsilon > 0$ are given, the subsets $\Omega_1,\dots,\Omega_N$ are said to be (r,ε)-free if

$$|\mathrm{tr}_n(A_1\dots A_k) - \mathrm{tr}_n^{\star N}(\tilde{A}_1\cdots\tilde{A}_k)| \le \varepsilon$$

for all $A_1,\dots,A_k \in \bigsqcup_{i=1}^N \Omega_i$, $1 \le k \le r$, where $\tilde{A} := j_i(A)$ for $A \in \Omega_i$.

Lemma 3.2. *Let $u_1,\dots,u_N,h_1,\dots,h_N$ be as in Theorem 3.1, and assume that $\chi_u(u_1,\dots,u_N) > -\infty$ and $\chi_+(h_i^2) > -\infty$ $(1 \le i \le N)$. Then, for every $r \in \mathbb{N}$, $\varepsilon > 0$ and $R > \max_i \|h_i\|^2$, there exists $\varepsilon_1 > 0$ such that*

$$\lim_{n\to\infty} \frac{(\gamma\otimes\Lambda_+)(\Xi_n(r,\varepsilon_1)\cap\Theta_n(r,\varepsilon))}{(\gamma\otimes\Lambda_+)(\Xi_n(r,\varepsilon_1))} = 1,$$

where

$$\Xi_n(r,\varepsilon_1) := \Gamma_u(u_1,\dots,u_N;n,r,\varepsilon_1) \times \prod_{i=1}^N \Gamma_{+,R}(h_i^2;n,r,\varepsilon_1),$$

$$\Theta_n(r,\varepsilon) := \Gamma_{(u,+),R}(u_1,\dots,u_N;h_1^2,\dots,h_N^2;n,r,\varepsilon).$$

Proof. Thanks to the freeness of $\{u_1,\dots,u_N,u_1^*,\dots,u_N^*\}$, $h_1^2,\dots,h_N^2$, one can choose $\varepsilon_1>0$ such that if $(U_1,\dots,U_N;H_1,\dots,H_N)\in\Xi_n(r,\varepsilon_1)$ and $\{U_1,\dots,U_N,U_1^*,\dots,U_N^*\}$, $\{H_1\},\dots,\{H_N\}$ are (r,ε_1)-free, then $(U_1,\dots,U_N;H_1,\dots,H_N)\in\Theta_n(r,\varepsilon)$. For every $\theta>0$, according to [17, Cor. 2.13], there exists $n_0\in\mathbb{N}$ such that

$$\gamma\big(\{(V_1,\dots,V_N)\in(\mathcal{U}(n))^N:\{U_1,\dots,U_N,U_1^*,\dots,U_N^*\},\{V_1H_1V_1^*\},\\ \dots,\{V_NH_NV_N^*\}\text{ are }(r,\varepsilon_1)\text{-free}\}\big)\ge 1-\theta \qquad (3.1)$$

whenever $n\ge n_0$ independently of the choice of any $U_i\in\mathcal{U}(n)$ and $H_i\in M_n^+$ with $\|H_i\|\le R$ $(1\le i\le N)$. By the assumption that $\chi_u(u_1,\dots,u_N)>-\infty$ and $\chi_+(h_i^2)>-\infty$, it follows that the $\gamma\otimes\Lambda_+$-measure of $\Xi_n(r,\varepsilon_1)$ is positive (at least if n is large). So, for any large n $(\ge n_0)$ one can define the probability measure σ_n on $\Xi_n(r,\varepsilon_1)$ by normalizing the restriction of $\gamma\otimes\Lambda_+$ to $\Xi_n(r,\varepsilon_1)$. Then, since σ_n is invariant under the action of $(\mathcal{U}(n))^N$ on $\Xi_n(r,\varepsilon_1)$ given by $(U_1,\dots,U_N;H_1,\dots,H_N)\mapsto(U_1,\dots,U_N;V_1H_1V_1^*,\dots,V_NH_NV_N^*)$ for $(V_1,\dots,V_N)\in(\mathcal{U}(n))^N$, we have

$$\frac{(\gamma\otimes\Lambda_+)(\Xi_n(r,\varepsilon_1)\cap\Theta_n(r,\varepsilon))}{(\gamma\otimes\Lambda_+)(\Xi_n(r,\varepsilon_1))}\\ =\int_{\Xi_n(r,\varepsilon_1)}\bigg(\int_{(\mathcal{U}(n))^N}\psi(U_1,\dots,U_N;V_1H_1V_1^*,\dots,V_NH_NV_N^*)\,d\gamma(V_1,\dots,V_N)\bigg)\,d\sigma_n\,,$$

where ψ is the characteristic function of $\Xi_n(r,\varepsilon_1)\cap\Theta_n(r,\varepsilon)$. The choice of ε_1 and (3.1) show that

$$\int_{(\mathcal{U}(n))^N}\psi(U_1,\dots,U_N;V_1H_1V_1^*,\dots,V_NH_NV_N^*)\,d\gamma(V_1,\dots,V_N)\ge 1-\theta$$

for all $(U_1,\dots,U_N;H_1,\dots,H_N)\in\Xi_n(r,\varepsilon_1)$. Therefore, we infer that

$$\frac{(\gamma\otimes\Lambda_+)(\Xi_n(r,\varepsilon_1)\cap\Theta_n(r,\varepsilon))}{(\gamma\otimes\Lambda_+)(\Xi_n(r,\varepsilon_1))}\ge 1-\theta$$

whenever n is large enough, and the result follows. □

Proof of Theorem 3.1. By Theorem 2.5 and Proposition 2.3 it suffices to show that

$$\chi_{(u,+)}(u_1,\dots,u_N;h_1^2,\dots,h_N^2)\ge\chi_u(u_1,\dots,u_N)+\sum_{i=1}^N\chi_+(h_i^2)\,, \qquad (3.2)$$

so we may assume that $\chi_u(u_1,\dots,u_N)>-\infty$ and $\chi_+(h_i^2)>-\infty$ $(1\le i\le N)$. For any $r\in\mathbb{N}$, $\varepsilon>0$ and $R>\max_i\|h_i\|^2$, let $\varepsilon_1>0$ be as in Lemma 3.2. Then we have

$$\begin{aligned}
&\chi_{(u,+),R}(u_1,\dots,u_N;h_1^2,\dots,h_N^2;r,\varepsilon)\\
&\quad\ge\limsup_{n\to\infty}\bigg[\frac{1}{n^2}\log(\gamma\otimes\Lambda_+)(\Xi_n(r,\varepsilon_1))+N\log n\bigg]\\
&\quad=\limsup_{n\to\infty}\bigg[\frac{1}{n^2}\log\gamma(\Gamma_u(u_1,\dots,u_N;n,r,\varepsilon_1))\\
&\qquad\qquad+\sum_{i=1}^N\Big(\frac{1}{n^2}\log\Lambda_{+,n}(\Gamma_{+,R}(h_i^2;n,r,\varepsilon_1))+\log n\Big)\bigg]\\
&\quad=\chi_u(u_1,\dots,u_N;r,\varepsilon_1)+\sum_{i=1}^N\chi_{+,R}(h_i^2;r,\varepsilon_1)\,.
\end{aligned}$$

Above we used the fact that lim sup becomes limit in (2.2). Thus (3.2) is shown. The second part is clear from Remark 2.4 (2) and Proposition 1.2. ☐

When the roles of $u_1, \dots, u_N$ and $h_1, \dots, h_N$ are exchanged in Theorem 3.1, we have

Theorem 3.3. *Let $u_1, \dots, u_N \in \mathcal{M}$ be unitaries and $h_1, \dots, h_N \in \mathcal{M}^+$. If $\{u_1, u_1^*\}$, $\dots, \{u_N, u_N^*\}$, $\{h_1, \dots, h_N\}$ are free, then*

$$\hat{\chi}(u_1h_1, \dots, u_Nh_N) = \sum_{i=1}^{N} \chi_u(u_i) + \chi(h_1^2, \dots, h_N^2) + \frac{N}{2}\left(\log\frac{\pi}{2} + \frac{3}{2}\right).$$

If $u_1, \dots, u_N$ are Haar unitaries in addition, then

$$\hat{\chi}(u_1h_1, \dots, u_Nh_N) = \chi(h_1^2, \dots, h_N^2) + \frac{N}{2}\left(\log\frac{\pi}{2} + \frac{3}{2}\right).$$

Proof. By Theorem 2.5 and Proposition 2.3 we may show that

$$\chi_{(u,+)}(u_1, \dots, u_N; h_1^2, \dots, h_N^2) \ge \sum_{i=1}^{N} \chi_u(u_i) + \chi_+(h_1^2, \dots, h_N^2), \tag{3.3}$$

and we may assume $\chi_u(u_i) > -\infty$ and $\chi_+(h_1^2, \dots, h_N^2) > -\infty$. For $n, r \in \mathbb{N}$, $\varepsilon > 0$ and $R > 0$ set

$$\Xi_n(r,\varepsilon) := \prod_{i=1}^{N} \Gamma_u(u_i; n, r, \varepsilon) \times \Gamma_{+,R}(h_1^2, \dots, h_N^2; n, r, \varepsilon),$$

and $\Theta_n(r,\varepsilon)$ is the same as in Lemma 3.2. By the freeness assumption there is $\varepsilon_1 > 0$ such that if $(U_1, \dots, U_N; H_1, \dots, H_N) \in \Xi_n(r, \varepsilon_1)$ and $\{U_1, U_1^*\}, \dots, \{U_N, U_N^*\}$, $\{H_1, \dots, H_N\}$ are (r, ε_1)-free, then $(U_1, \dots, U_N; H_1, \dots, H_N) \in \Theta_n(r, \varepsilon)$. For every $\theta > 0$ by [17, Cor. 2.13] there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\gamma\big(\{(V_1, \dots, V_N) \in (\mathcal{U}(n))^N : \{V_1U_1V_1^*, V_1U_1^*V_1^*\}, \dots, \{V_NU_NV_N^*, V_NU_N^*V_N^*\},\\ \{H_1, \dots, H_N\} \text{ are } (r, \varepsilon_1)\text{-free}\}\big) \ge 1 - \theta\end{aligned}$$

whenever $n \ge n_0$ independently of the choice of any $U_i \in \mathcal{U}(n)$ and $H_i \in M_n^+$ with $\|H_i\| \le R$ $(1 \le i \le N)$. Then as in the proof of Lemma 3.2 we have

$$\frac{(\gamma \otimes \Lambda_+)(\Xi_n(r, \varepsilon_1) \cap \Theta_n(r, \varepsilon))}{(\gamma \otimes \Lambda_+)(\Xi_n(r, \varepsilon_1))} \ge 1 - \theta$$

for large n. Therefore,

$$\lim_{n\to\infty} \frac{(\gamma \otimes \Lambda_+)(\Xi_n(r, \varepsilon_1) \cap \Theta_n(r, \varepsilon))}{(\gamma \otimes \Lambda_+)(\Xi_n(r, \varepsilon_1))} = 1.$$

This implies that

$$
\begin{aligned}
&\chi_{(u,+),R}(u_1,\dots,u_N;h_1^2,\dots,h_N^2;r,\varepsilon) \\
&\quad\ge \limsup_{n\to\infty}\left[\frac{1}{n^2}\log(\gamma\otimes\Lambda_+)(\Xi_n(r,\varepsilon_1))+N\log n\right] \\
&\quad= \limsup_{n\to\infty}\left[\frac{1}{n^2}\sum_{i=1}^N\log\gamma_n(\Gamma_u(u_i;n,r,\varepsilon_1)) \right.\\
&\qquad\qquad\left. +\frac{1}{n^2}\log\Lambda_+(\Gamma_{+,R}(h_1^2,\dots,h_N^2;n,r,\varepsilon_1))+N\log n\right] \\
&\quad= \sum_{i=1}^N\chi_u(u_i;r,\varepsilon_1)+\chi_{+,R}(h_1^2,\dots,h_N^2;r,\varepsilon_1)
\end{aligned}
$$

thanks to (1.1), so (3.3) is obtained. □

Next, we apply the relation shown above to get the additivity properties of the free entropies χ_u and $\hat{\chi}$. We first give the change of variable formula similar to [15, Prop. 3.1] for $\chi_{(u,+)}$. To do so, we need a smoothing technique like [15, Lemma 4.1]. We denote by $\mathcal{F}_{\mathbb{T}}$ the set of all functions $f:\mathbb{T}\to\mathbb{T}$ which is given as $f(e^{\mathrm{i}\,t})=e^{\mathrm{i}\,\phi(t)}$ by a continuous increasing function ϕ on $[0,2\pi]$ with $\phi(0)=0$, $\phi(2\pi)=2\pi$. An $f\in\mathcal{F}_{\mathbb{T}}$ is said to be C^∞ if so is ϕ. Note that if ϕ is differentiable at $t\in[0,2\pi]$, then

$$
\lim_{\eta\to\zeta}\left|\frac{f(\eta)-f(\zeta)}{\eta-\zeta}\right|=\phi'(t)\quad\text{for}\quad\zeta=e^{\mathrm{i}\,t}.
$$

In this case we write $|f'(e^{\mathrm{i}\,t})|$ instead of $\phi'(t)$. For each unitary $u\in\mathcal{M}$ and $f\in\mathcal{F}_{\mathbb{T}}$ one can define the unitary $f(u)$ by functional calculus, that is, $f(u):=\int_{\mathbb{T}}f(\zeta)\,de(\zeta)$ for the spectral decomposition $u=\int_{\mathbb{T}}\zeta\,de(\zeta)$.

Lemma 3.4. *Let $u\in\mathcal{M}$ be a unitary with $\chi_u(u)>-\infty$, and let $f\in\mathcal{F}_{\mathbb{T}}$. Then there exists a sequence (f_m) of C^∞-functions in $\mathcal{F}_{\mathbb{T}}$ such that $|f_m'|>0$ on $\mathbb{T}$, $\|f_m(u)-f(u)\|\to 0$ and $\chi_u(f_m(u))\to\chi_u(f(u))$.*

On the other hand, we denote by $\mathcal{F}_{\mathbb{R}^+}$ the set of all continuous increasing functions $g:\mathbb{R}^+\to\mathbb{R}^+$ with $g(0)=0$.

Lemma 3.5. *Let $h\in\mathcal{M}^+$, $\chi(h)>-\infty$, and $g\in\mathcal{F}_{\mathbb{R}^+}$. Then there exists a sequence (g_m) of C^∞-functions in $\mathcal{F}_{\mathbb{R}^+}$ such that $g_m'>0$ on $\mathbb{R}^+$, $\|g_m(h)-g(h)\|\to 0$ and $\chi(g_m(h))\to\chi(g(h))$.*

Lemma 3.5 is essentially included in [15, Lemma 4.1]. The proof of Lemma 3.4 is similar with some modifications, and it may be omitted here.

Lemma 3.6. *Let $u_1,\dots,u_N\in\mathcal{M}$ be unitaries with $\chi_u(u_i)>-\infty$ and $h_1,\dots,h_L\in\mathcal{M}^+$ with $\chi_+(h_j)>-\infty$. Then*

$$
\begin{aligned}
&\chi_{(u,+)}(f_1(u_1),\dots,f_N(u_N);g_1(h_1),\dots,g_L(h_L)) \\
&\quad\ge\chi_{(u,+)}(u_1,\dots,u_N;h_1,\dots,h_L) \\
&\qquad+\sum_{i=1}^N\bigl[\chi_u(f_i(u_i))-\chi_u(u_i)\bigr]+\sum_{j=1}^L\bigl[\chi(g_j(h_j))-\chi(h_j)\bigr]
\end{aligned}
$$

for every $f_1, \ldots, f_N \in \mathcal{F}_{\mathbb{T}}$ *and* $g_1, \ldots, g_L \in \mathcal{F}_{\mathbb{R}+}$.

Proof. By Lemmas 3.4 and 3.5 together with Proposition 2.2 we may show the following two cases:

(a) If f is a C^∞-function in $\mathcal{F}_{\mathbb{T}}$ with $|f'| > 0$ on $\mathbb{T}$, then

$$\begin{aligned}&\chi_{(u,+)}(f(u_1), u_2, \ldots, u_N; h_1, \ldots, h_N)\\ &\qquad \geq \chi_{(u,+)}(u_1, \ldots, u_N; h_1, \ldots, h_N) + \chi_u(f(u_1)) - \chi_u(u_1).\end{aligned}$$

(b) If g is a C^∞-function in $\mathcal{F}_{\mathbb{R}+}$ with $g' > 0$ on $\mathbb{R}^+$, then

$$\begin{aligned}&\chi_{(u,+)}(u_1, \ldots, u_N; g(h_1), h_2, \ldots, h_N)\\ &\qquad \geq \chi_{(u,+)}(u_1, \ldots, u_N; h_1, \ldots, h_N) + \chi(g(h_1)) - \chi(h_1).\end{aligned}$$

The proof of (b) is the same as [15, Prop. 3.1]. We sketch the similar proof of (a). For $\zeta, \eta \in \mathbb{T}$ define

$$K(\zeta, \eta) := \begin{cases} \left|\frac{f(\zeta) - f(\eta)}{\zeta - \eta}\right| & \text{if } \zeta \neq \eta, \\ |f'(\zeta)| & \text{if } \zeta = \eta, \end{cases}$$

then $L(\zeta, \eta) := \log K(\zeta, \eta)$ is continuous on $\mathbb{T}^2$ and

$$\chi_u(f(u_1)) - \chi_u(u_1) = (\tau \otimes \tau)(L(u_1 \otimes \mathbf{1}, \mathbf{1} \otimes u_1)).$$

Write $F(U_1, \ldots, U_N; H_1, \ldots, H_L) := (f(U_1), U_2, \ldots, U_N; H_1, \ldots, H_L)$ on $(\mathcal{U}(n))^N \times (M_n^+)^L$. For every $r \in \mathbb{N}$ and $\varepsilon > 0$, by approximating f by a trigonometric polynomial, we notice that

$$\begin{aligned}&F(\Gamma_{(u,+),R}(u_1, \ldots, u_N; h_1, \ldots, h_L; n, r_1, \varepsilon_1))\\ &\qquad \subset \Gamma_{(u,+),R}(f(u_1), u_2, \ldots, u_N; h_1, \ldots, h_L; n, r, \varepsilon) \qquad (n \in \mathbb{N})\end{aligned}$$

for some $r_1 \in \mathbb{N}$ and $\varepsilon_1 > 0$. Since

$$\begin{aligned}\frac{d(\gamma_n \circ f)}{d\gamma_n}(U_1) &= \prod_{i<j} \left|\frac{f(\zeta_i) - f(\zeta_j)}{\zeta_i - \zeta_j}\right|^2 \prod_{i=1}^n |f'(\zeta_i)|\\ &= \exp(\mathrm{Tr}_n \otimes \mathrm{Tr}_n)(L(U_1 \otimes I, I \otimes U_1))\end{aligned}$$

($\zeta_1, \ldots, \zeta_N$ are the eigenvalues of U_1), we can show as in the proof of [15, Prop. 3.1] that for any $\delta > 0$ there are $r_1 \in \mathbb{N}$ and $\varepsilon_1 > 0$ such that

$$\left|\frac{1}{n^2} \log \frac{d(\gamma_n \circ f)}{d\gamma_n}(U_1) - \left[\chi_u(f(u_1)) - \chi_u(u_1)\right]\right| \leq 3\delta$$

for all $(U_1, \ldots, U_N; H_1, \ldots, H_L) \in \Gamma_{(u,+),R}(u_1, \ldots, u_N; h_1, \ldots, h_L; n, r_1, \varepsilon_1)$, $n \in \mathbb{N}$, and the inequality in (a) is obtained. □

If $f_1, \ldots, f_N \in \mathcal{F}_{\mathbb{T}}$ and $g_1, \ldots, g_L \in \mathcal{F}_{\mathbb{R}+}$ are strictly increasing (in terms of angle for f_i), then the inequality in Lemma 3.6 can be replaced by the equality.

Proposition 3.7. *If $u_1, \dots, u_N \in \mathcal{M}$ are unitaries, then $\chi_u(u_1, \dots, u_N) = 0$ if and only if $u_1, \dots, u_N$ are *-free Haar unitaries.*

Proof. Choose free standard quarter-circular elements $h_1, \dots, h_N$ which are free from $\{u_1, \dots, u_N, u_1^*, \dots, u_N^*\}$. Theorem 3.1 says that $\chi_u(u_1, \dots, u_N) = 0$ if and only if $\hat{\chi}(u_1h_1, \dots, u_Nh_N) = N \log(\pi e)$. According to Proposition 1.3 the latter equality holds if and only if $u_1h_1, \dots, u_Nh_N$ are *-free circular elements, which is equivalent to saying that $u_1, \dots, u_N$ are *-free Haar unitaries. □

Theorem 3.8. *Let $u_1, \dots, u_N \in \mathcal{M}$ be unitaries. If $u_1, \dots, u_N$ are *-free, then*

$$\chi_u(u_1, \dots, u_N) = \chi_u(u_1) + \dots + \chi_u(u_N)\,.$$

*Conversely, if $\chi_u(u_i) > -\infty$ for $1 \le i \le N$ and the above equality holds, then $u_1, \dots, u_N$ are *-free.*

Proof. When $(h_1, \dots, h_N)$ is void in the proof of (3.3), it can read as a proof of the first part here. (This part can be also shown in a way similar to the selfadjoint case in [13].)

Now we prove the second part. Assume that $\chi_u(u_i) > -\infty$ for $1 \le i \le N$ and the additivity holds. For each i, since the distribution of u_i is non-atomic, there is a (unique) $f_i \in \mathcal{F}_\mathbb{T}$ such that the distribution of $f_i(u_i)$ is the Haar probability measure on $\mathbb{T}$, so $\chi_u(f_i(u_i)) = 0$. Then, by Lemma 3.6 (in case of $L = 0$) and the additivity assumption, we get

$$\chi_u(f_1(u_1), \dots, f_N(u_N)) \ge \sum_{i=1}^N \chi_u(f_i(u_i)) = 0\,.$$

So Proposition 3.7 implies that $f_1(u_1), \dots, f_N(u_N)$ are *-free, and hence $u_1, \dots, u_N$ are *-free because $u_i \in \{f_i(u_i)\}''$. □

Theorem 3.9. *Let $a_1, \dots, a_N \in \mathcal{M}$ be such that $a_i = u_ih_i$ with a *-free pair of a unitary $u_i \in \mathcal{M}$ and $h_i \in \mathcal{M}^+$. If $a_1, \dots, a_N$ are *-free, then*

$$\hat{\chi}(a_1, \dots, a_N) = \hat{\chi}(a_1) + \dots + \hat{\chi}(a_N)\,.$$

*Conversely, if $\hat{\chi}(a_i) > -\infty$ for $1 \le i \le N$ and the above equality holds, then $a_1, \dots, a_N$ are *-free.*

Proof. If $a_1, \dots, a_N$ are *-free, then $u_1, \dots, u_N, h_1, \dots, h_N$ are *-free due to the *-freeness of u_i, h_i. Hence Theorems 3.1 and 3.8 imply that

$$\hat{\chi}(a_1, \dots, a_N) = \sum_{i=1}^N \chi_u(u_i) + \sum_{i=1}^N \chi(h_i^2) + \frac{N}{2}\left(\log\frac{\pi}{2} + \frac{3}{2}\right) = \sum_{i=1}^N \hat{\chi}(a_i)\,.$$

Conversely, assume that $\hat{\chi}(a_i) > -\infty$ for $1 \le i \le N$ and the additivity holds. Since $\chi_u(u_i) > -\infty$ and $\chi(h_i^2) > -\infty$, one can choose $f_i \in \mathcal{F}_\mathbb{T}$ and $g_i \in \mathcal{F}_{\mathbb{R}^+}$ such that $f_i(u_i)$ is a Haar unitary and $g_i(h_i)^2$ is a standard quarter-circular. Then, letting $b_i := f_i(u_i)g_i(h_i)$ and using Theorem 2.5, Lemma 3.6 (applied to f_i, $g_i(t^{1/2})^2$) and Theorem 3.1, we get

$$\begin{aligned}\hat{\chi}(b_1,\dots,b_N) &= \chi_{(u,+)}(f_1(u_1),\dots,f_N(u_N);g_1(h_1)^2,\dots,g_N(h_N)^2)\\ &\geq \hat{\chi}_{(u,+)}(u_1,\dots,u_N;h_1^2,\dots,h_N^2)\\ &\quad+\sum_{i=1}^N\bigl[\chi_u(f_i(u))-\chi_u(u_i)\bigr]+\sum_{i=1}^N\bigl[\chi(g_i(h_i)^2)-\chi(h_i^2)\bigr]\\ &=\hat{\chi}(a_1,\dots,a_N)-\sum_{i=1}^N\hat{\chi}(a_i)+N\log(\pi e)=N\log(\pi e).\end{aligned}$$

So Proposition 1.3 implies that $b_1,\dots,b_N$ are *-free standard circulars. Hence $a_1,\dots,a_N$ are *-free because of $a_i \in \{b_i, b_i^*\}''$. $\square$

In [9] the notion of R-diagonal pairs was introduced in connection with two-variable R-transform. Instead of giving its definition here, we remark the following characterization shown in [9]: If $a \in \mathcal{M}$ and $\ker a = \{0\}$, then a is an R-diagonal element (i.e. (a,a^*) is an R-diagonal pair) if and only if a is written as uh by a *-free pair of a Haar unitary $u \in \mathcal{M}$ and $h \in \mathcal{M}^+$. It was also shown in [9] that an R-diagonal element a is circular if and only if the real and imaginary parts of a are free. Theorem 3.9 can be applied in particular when $a_1,\dots,a_N$ are R-diagonal.

Specialized to the case $\hat{\chi}(a)$ of a single non-selfadjoint $a \in \mathcal{M}$ we state

Proposition 3.10. *Let $a \in \mathcal{M}$ with $\hat{\chi}(a) > -\infty$, and let $a = uh$ be the polar decomposition. Then*

$$\hat{\chi}(a) \leq \chi_u(u) + \chi(a^*a) + \frac{1}{2}\log\frac{\pi}{2} + \frac{3}{4}$$

*and the equality is attained if and only if u,h are *-free. Moreover, $\hat{\chi}(a) = \chi(a^*a) + \frac{1}{2}\log\frac{\pi}{2} + \frac{3}{4}$ if and only if a is R-diagonal.*

Proof. Theorem 3.1 includes the "if" part of the first assertion. To see the "only if", choose $f \in \mathcal{F}_{\mathbb{T}}$ and $g \in \mathcal{F}_{\mathbb{R}^+}$ such that $f(u)$ is a Haar unitary and $g(h)^2$ is a standard quarter-circular. Then the equality $\hat{\chi}(uh) = \chi_u(u) + \chi_+(h^2)$ implies $\hat{\chi}(f(u)g(h)) = \log(\pi e)$ as in the proof of Theorem 3.9, and this means that $f(u)g(h)$ is a standard circular and so u,h are *-free. The second assertion is immediate from the first. $\square$

The above proposition shows

Corollary 3.11. *Let μ be a probability measure on $\mathbb{R}^+$ with compact support and $\Sigma(\mu) > -\infty$. When $a \in \mathcal{M}$ is such that a^*a has the distribution μ,*

$$\hat{\chi}(a) \leq \Sigma(\mu) + \log\pi + \frac{3}{2},$$

and the equality is attained if and only if a is R-diagonal.

Example 3.12. For each $\lambda \geq 1$ the free analogue of the Poisson distribution (see [18]) is given by

$$\mu_\lambda := \frac{\sqrt{4\lambda-(t-1-\lambda)^2}}{2\pi t}\chi(t)\,dt\,,$$

where $\chi(t)$ is the characteristic function of the interval $\big[(\sqrt{\lambda}-1)^2, (\sqrt{\lambda}+1)^2\big]$. This measure is also called the Marchenko-Pastur distribution. From a computation in [5] (also [6]) we have

$$\Sigma(\mu_\lambda) = -1 + \frac{1}{2}(\lambda + \log\lambda + (\lambda-1)^2\log(1-\lambda^{-1}))\,.$$

If a is an R-diagonal element such that a^*a has the distribution μ_λ, then Corollary 3.11 gives

$$\hat{\chi}(a) = \log\pi + \frac{1}{2}(1 + \lambda + \log\lambda + (\lambda-1)^2\log(1-\lambda^{-1}))\,.$$

The case $\lambda = 1$ is a circular element of radius 2.

4. Maximization of free entropy for a matrix of random variables

A maximization result similar to Corollary 3.11 was recently shown in [8] for the version χ^* of free entropy introduced in [16]. Moreover, this maximization result for χ^* was extended to the case of a matrix $[a_{ij}]_{i,j=1}^d$ of random variables. In this section we consider the χ-version of the maximization problem from [8].

For each $d \in \mathbb{N}$ we have a tracial W^*-probability space $(M_d(\mathcal{M}) \equiv \mathcal{M}\otimes M_d, \tau\otimes \mathrm{tr}_d)$. Let a_{ij} $(1 \le i,j \le d)$ be a family of (non-selfadjoint) elements of $\mathcal{M}$, and set $b := [a_{ij}]_{i,j}^d \in M_d(\mathcal{M})$. We have the free entropy $\hat{\chi}((a_{ij})_{1\le i,j\le d})$ of the d^2-tuple of a_{ij}. On the other hand, following [14], one can define the (conditional) free entropy of b in the presence of $M_d(\mathbb{C}\mathbf{1})$ $(\equiv \mathbb{C}\mathbf{1}\otimes M_d \subset M_d(\mathcal{M}))$. Let $(e_{ij})_{1\le i,j\le d}$ be the usual matrix units of $M_d(\mathbb{C}\mathbf{1})$. For $n, r \in \mathbb{N}$, $\varepsilon > 0$ and $R > 0$ define

$$\begin{aligned}
&\hat{\Gamma}_R(b, (e_{ii})_{1\le i\le d}, (e_{ij})_{1\le i<j\le d}; n, r, \varepsilon)\\
&\quad := \big\{(B, (B_{ii})_{1\le i\le d}, (B_{ij})_{1\le i<j\le d}) \in M_n \times (M_n^{sa})^d \times (M_n)^{d(d-1)/2} :\\
&\qquad\qquad \|B\|, \|B_{ij}\| \le R \ (1 \le i \le j \le d),\\
&\qquad\qquad |\mathrm{tr}_n(Y_{i_1}\cdots Y_{i_k}) - (\tau\otimes\mathrm{tr}_d)(y_{i_1}\cdots y_{i_k})| \le \varepsilon\\
&\qquad\qquad \text{for all } 0 \le i_1, \dots, i_k \le d^2,\ 1 \le k \le r\big\},
\end{aligned}$$

where

$$\begin{aligned}
(y_0, y_1, \dots, y_{d^2}) &:= (b, (e_{ii})_{1\le i\le d}, (e_{ij})_{1\le i<j\le d}, (e_{ji})_{1\le i<j\le d})\,,\\
(Y_0, Y_1, \dots, Y_{d^2}) &:= (B, (B_{ii})_{1\le i\le d}, (B_{ij})_{1\le i<j\le d}, (B_{ij}^*)_{1\le i<j\le d})\,,
\end{aligned}$$

and further define

$$\begin{aligned}
&\hat{\Gamma}_R(b : (e_{ii})_{1\le i\le d}, (e_{ij})_{1\le i<j\le d}; n, r, \varepsilon)\\
&\quad := \big\{B \in M_n : (B, (B_{ii})_{1\le i\le d}, (B_{ij})_{1\le i<j\le d}) \in \hat{\Gamma}_R(b, (e_{ii}), (e_{ij})_{i<j}; n, r, \varepsilon)\\
&\qquad\qquad \text{for some } ((B_{ii})_{1\le i\le d}, (B_{ij})_{1\le i<j\le d}) \in (M_n^{sa})^d \times (M_n)^{d(d-1)/2}\big\}\,.
\end{aligned}$$

Then $\hat{\chi}(b : (e_{ii})_{1\le i\le d}, (e_{ij})_{1\le i<j\le d})$ is defined as before, which we denote by $\hat{\chi}(b : M_d(\mathbb{C}\mathbf{1}))$.

In fact, choose any family $\{c_1, \dots, c_l\}$ in $M_d(\mathbb{C}\mathbf{1})^{sa}$ which generates $M_d(\mathbb{C}\mathbf{1})$ as $*$-algebra, and define $\hat{\chi}(b : c_1, \dots, c_l)$ in the same way by taking $(M_n^{sa})^l$ in place of $(M_n^{sa})^d \times (M_n)^{d(d-1)/2}$ in the above. Then it follows (see [14]) that $\hat{\chi}(b : M_d(\mathbb{C}\mathbf{1})) = \hat{\chi}(b : c_1, \dots, c_l)$.

Proposition 4.1. *With the above notations,*

$$\begin{aligned}\hat{\chi}((a_{ij})_{1\le i,j\le d}) &\le d^2\hat{\chi}(b : M_d(\mathbb{C}\mathbf{1})) - d^2\log d\\ &\le d^2\hat{\chi}(b) - d^2\log d\,.\end{aligned}$$

Furthermore, if $\{b, b^*\}$ *is free from* $M_d(\mathbb{C}\mathbf{1})$ *then*

$$\hat{\chi}((a_{ij})_{1\le i,j\le d}) = d^2\hat{\chi}(b) - d^2\log d\,. \tag{4.1}$$

Proof. First, the second inequality is trivial. Define a linear map $\Phi : (M_n)^{d^2} \to M_{nd}$ by $\Phi((A_{ij})_{1\le i,j\le d}) := [A_{ij}]_{i,j=1}^d$. Since $\|[A_{ij}]_{i,j=1}^d\|_{HS} = (\sum_{i,j=1}^d \|A_{ij}\|_{HS}^2)^{1/2}$, it is obvious that

$$\hat{\Lambda}_n^{\otimes d^2} = \hat{\Lambda}_{nd}\circ\Phi\,. \tag{4.2}$$

Let $(E_{ij})_{1\le i,j\le d}$ be the usual matrix units of $M_d = I_n \otimes M_d \subset M_{nd}$. Now let $(A_{ij})_{1\le i,j\le d} \in \hat{\Gamma}_R((a_{ij})_{1\le i,j\le d}; n, r, \varepsilon)$, and set $B := \Phi((A_{ij})_{1\le i,j\le d})$ and

$$\begin{aligned}(x_1,\dots,x_{d^2}) &:= (a_{ij})_{1\le i,j\le d}\,,\\ (X_1,\dots,X_{d^2}) &:= (A_{ij})_{1\le i,j\le d}\,,\\ (y_0,y_1,\dots,y_{d^2}) &:= (b,(e_{ii})_{1\le i\le d},(e_{ij})_{1\le i<j\le d},(e_{ji})_{1\le i<j\le d})\,,\\ (Y_0,Y_1,\dots,Y_{d^2}) &:= (B,(E_{ii})_{1\le i\le d},(E_{ij})_{1\le i<j\le d},(E_{ji})_{1\le i<j\le d})\,.\end{aligned}$$

For any $0 \le i_1,\dots,i_k \le d^2$ $(1\le k\le r)$, one can write

$$\mathrm{tr}_{nd}(Y_{i_1}\cdots Y_{i_k}) = \frac{1}{d}\sum \mathrm{tr}_n(X_{j_1}\cdots X_{j_l})\,, \tag{4.3}$$

$$(\tau\otimes\mathrm{tr}_d)(y_{i_1}\cdots y_{i_k}) = \frac{1}{d}\sum \tau(x_{j_1}\cdots x_{j_l})\,, \tag{4.4}$$

where the summations in (4.3) and (4.4) are in the same pattern, the number of terms in sum is at most d^k and $1\le j_1,\dots,j_l\le d^2$, $1\le l\le k$. Hence we obtain

$$|\mathrm{tr}_{nd}(Y_{i_1}\cdots Y_{i_k}) - (\tau\otimes\mathrm{tr}_d)(y_{i_1}\cdots y_{i_k})| \le \frac{1}{d}\cdot d^k\cdot\varepsilon \le d^{r-1}\varepsilon\,.$$

Therefore, noting $\|B\|\le dR$, we infer that

$$\Phi(\hat{\Gamma}_R((a_{ij})_{1\le i,j\le d}; n,r,\varepsilon)) \subset \hat{\Gamma}_{dR}(b : (e_{ii})_{1\le i\le d},(e_{ij})_{1\le i<j\le d}; nd, r, d^{r-1}\varepsilon)\,.$$

Thanks to (4.2) this yields

$$\hat{\Lambda}_n^{\otimes d^2}(\hat{\Gamma}_R((a_{ij})_{1\le i,j\le d}; n,r,\varepsilon)) \le \hat{\Lambda}_{nd}(\Gamma_{dR}(b : (e_{ii})_{1\le i\le d},(e_{ij})_{1\le i<j\le d}; nd, r, d^{r-1}\varepsilon)$$

and hence

$$\begin{aligned}&\frac{1}{n^2}\log\hat{\Lambda}_n^{\otimes d^2}(\hat{\Gamma}_R((a_{ij})_{1\le i,j\le d}; n,r,\varepsilon)) + d^2\log n\\ &\quad\le d^2\left[\frac{1}{n^2d^2}\log\hat{\Lambda}_{nd}(\hat{\Gamma}_{dR}(b : (e_{ii})_{1\le i\le d},(e_{ij})_{1\le i<j\le d}; nd, r, d^{r-1}\varepsilon)) + \log(nd)\right]\\ &\qquad\quad - d^2\log d\,.\end{aligned}$$

So we have

$$\hat\chi_R((a_{ij})_{1\le i,j\le d};r,\varepsilon)\le d^2\hat\chi_{dR}(b:(e_{ii})_{1\le i\le d},(e_{ij})_{1\le i<j\le d};r,d^{r-1}\varepsilon)-d^2\log d\,.$$

Take the limits as $r\to\infty$ and $d^{r-1}\varepsilon\to 0$ to obtain

$$\hat\chi_R((a_{ij})_{1\le i,j\le d})\le d^2\hat\chi_{dR}(b:M_d(\mathbb{C}\mathbf{1}))-d^2\log d\,,$$

so we have $\hat\chi((a_{ij})_{1\le i,j\le d})\le d^2\hat\chi(b:M_d(\mathbb{C}\mathbf{1}))-d^2\log d$.

Next, assume that $\{b,b^*\}$ is free from $M_d(\mathbb{C}\mathbf{1})$. Let $r\in\mathbb{N}$, $\varepsilon>0$ and $R\ge 1$ be given, and let $r_1:=2r+1$. For each $m\in\mathbb{N}$ let n be the integer part of m/d, and let $(E_{ij})_{1\le i,j\le d}$ be the usual matrix units of M_d, where M_d is embedded in M_m as $M_d=(I_n\otimes M_d)\oplus 0_{m-nd}\subset M_m$ so that the rank of E_{ii}'s is n. It is clear that the joint distribution of $(E_{ij})_{1\le i,j\le d}$ in M_m converges to that of $(e_{ij})_{1\le i,j\le d}$ in $M_d(\mathcal{M})$ as $m\to\infty$. Hence, thanks to the freeness of $\{b,b^*\}$ and $\{e_{ij}\}_{1\le i,j\le d}$, one can choose $\varepsilon_1>0$ and $m_0\in\mathbb{N}$ such that if $m\ge m_0$, $B\in\hat\Gamma_R(b;m,r_1,\varepsilon_1)$ and $\{B\}$, $\{E_{ij}\}_{1\le i,j\le d}$ are (r_1,ε_1)-free, then $(B,(E_{ii})_{1\le i\le d},(E_{ij})_{1\le i<j\le d})\in\hat\Gamma_R(b,(e_{ii})_{1\le i\le d},(e_{ij})_{1\le i<j\le d};m,r_1,\varepsilon)$. Set

$$\begin{aligned}\Xi_m(r_1,\varepsilon_1)&:=\hat\Gamma_R(b;m,r_1,\varepsilon_1)\,,\\ \Theta_m(r_1,\varepsilon)&:=\bigl\{B\in M_m:(B,(E_{ii})_{1\le i\le d},(E_{ij})_{1\le i<j\le d})\\ &\qquad\qquad\in\hat\Gamma_R(b,(e_{ii})_{1\le i\le d},(e_{ij})_{1\le i<j\le d};m,r_1,\varepsilon)\bigr\}\,.\end{aligned}$$

Then, as in the proofs of Lemma 3.2 and Theorem 3.3, one can show that

$$\lim_{m\to\infty}\frac{\hat\Lambda(\Xi_m(r_1,\varepsilon_1)\cap\Theta_m(r_1,\varepsilon))}{\hat\Lambda(\Xi_m(r_1,\varepsilon_1))}=1\,,$$

and this implies that

$$\hat\chi_R(b;r_1,\varepsilon_1)\le\limsup_{m\to\infty}\left[\frac{1}{m^2}\log\hat\Lambda_m(\Theta_m(r_1,\varepsilon))+\log m\right].\tag{4.5}$$

Now define a linear map $\Psi:M_m\to(M_n)^{d^2}\times\mathbb{C}^q$ $(q:=m^2-n^2d^2)$ by

$$\Psi(B):=\Bigl((B_{ij})_{1\le i,j\le d},(b_{ij})_{(i,j)\in R_m}\Bigr)\quad\text{for}\quad B=[b_{ij}]_{i,j=1}^m,$$

where

$$B_{ij}:=[b_{kl}]_{(i-1)d+1\le k\le id,(j-1)d+1\le l\le jd}\,,\quad R_m:=\{1,\dots,m\}^2\setminus\{1,\dots,nd\}^2\,.$$

Since Ψ is an isometry with respect to the Euclidean norm, we have

$$\hat\Lambda_m=\bigl(\hat\Lambda_n^{\otimes d^2}\otimes\lambda_q\bigr)\circ\Psi\tag{4.6}$$

where λ_q is the usual Lebesgue measure on $\mathbb{C}^q$. Let $1\le i_1,\dots,i_k,j_1,\dots,j_k\le d$ with $1\le k\le r$. One can write

$$\begin{aligned}\mathrm{tr}_n(B_{i_1j_1}B_{i_2j_2}\cdots B_{i_kj_k})&=\frac{m}{n}\mathrm{tr}_m(E_{1i_1}BE_{j_1i_2}BE_{j_2i_3}\cdots E_{j_{k-1}i_k}BE_{j_k1})\,,\\ \tau(a_{i_1j_1}a_{i_2j_2}\cdots a_{i_kj_k})&=d(\tau\otimes\mathrm{tr}_d)(e_{1i_1}be_{j_1i_2}be_{j_2i_3}\cdots e_{j_{k-1}i_k}be_{j_k1})\,.\end{aligned}$$

For every $B \in \Theta_m(r_1, \varepsilon)$, since

$$|\mathrm{tr}_m(E_{1i_1}BE_{j_1i_2}\cdots BE_{j_k1}) - (\tau\otimes\mathrm{tr}_d)(e_{1i_1}be_{j_1i_2}\cdots be_{j_k1})| \le \varepsilon\,,$$

we have

$$|\mathrm{tr}_n(B_{i_1j_1}\cdots B_{i_kj_k}) - \tau(a_{i_1j_1}\cdots a_{i_kj_k})| \le 2d\varepsilon$$

whenever m is large enough. In this way, we infer that for large m

$$\Psi(\Theta_m(r_1,\varepsilon)) \subset \hat\Gamma_R((a_{ij})_{1\le i,j\le d}; n, r, 2d\varepsilon) \times \{\zeta\in\mathbb{C} : |\zeta|\le R\}^q$$

so that thanks to (4.6)

$$\hat\Lambda_m(\Theta_m(r_1,\varepsilon)) \le \hat\Lambda_n^{\otimes d^2}(\hat\Gamma_R((a_{ij})_{1\le i,j\le d}; n, r, 2d\varepsilon)) \times (\pi R^2)^q\,.$$

Since $q < 2m(d-1)$, this and (4.5) yield

$$\begin{aligned}\hat\chi_R(b; r_1, \varepsilon_1) &\le \limsup_{m\to\infty}\left[\frac{1}{m^2}\log\hat\Lambda_n^{\otimes d^2}\big(\hat\Gamma_R((a_{ij})_{1\le i,j\le d}; n, r, 2d\varepsilon)\big) + \log m\right]\\ &= \frac{1}{d^2}\hat\chi_R((a_{ij})_{1\le i,j\le d}; r, 2d\varepsilon) + \log d\,.\end{aligned}$$

Therefore, we obtain

$$\hat\chi_R(b) \le \frac{1}{d^2}\hat\chi_R((a_{ij})_{1\le i,j\le d}) + \log d\,,$$

completing the proof. □

In particular, $\hat\chi(b : M_d(\mathbb{C}\mathbf{1})) = \hat\chi(b)$ if $\{b, b^*\}$ is free from $M_d(\mathbb{C}\mathbf{1})$. In fact, this is a consequence of [17, Prop. 3.10].

Let μ be as in Corollary 3.11. In the situation of Proposition 4.1 we observe that if b^*b has the distribution μ, then

$$\hat\chi((a_{ij})_{1\le i,j\le d}) \le d^2\Big(\Sigma(\mu) + \log\pi + \frac{3}{2}\Big) - d^2\log d\,, \tag{4.7}$$

and the equality is attained when b is R-diagonal and $\{b, b^*\}$ is free from $M_d(\mathbb{C}\mathbf{1})$. An example of $(a_{ij})_{1\le i,j\le d}$ for which b satisfies these conditions is easy to construct by using the free product (see [8]).

It may be possible that the equality

$$\hat\chi((a_{ij})_{1\le i,j\le d}) = d^2\hat\chi(b : M_d(\mathbb{C}\mathbf{1})) - d^2\log d$$

holds in general. Also it is interesting to know whether the freeness of $\{b, b^*\}$ from $M_d(\mathbb{C}\mathbf{1})$ is necessary for the equality (4.1) (or at least for the equality case in (4.7)).

Let $a_{ii} \in \mathcal{M}^{sa}$ for $1\le i\le d$ and $a_{ij}\in\mathcal{M}$ for $1\le i<j\le d$. Set $a_{ji} := a_{ij}^*$ and $b := [a_{ij}]_{i,j=1}^d \in M_d(\mathcal{M})^{sa}$. Then one can define $\hat\chi((a_{ii})_{1\le i\le d}, (a_{ij})_{1\le i<j\le d}) = \chi((a_{ii})_{1\le i\le d}, (b_{ij}, c_{ij})_{1\le i<j\le d})$ where $a_{ij} = b_{ij} + \mathrm{i}\,c_{ij}$ with selfadjoint b_{ij}, c_{ij}, and also $\chi(b : M_d(\mathbb{C}\mathbf{1}))$ as before. Then the proof of the following is a slight modification of that of Proposition 4.1, and we omit it.

Proposition 4.2. *With the above notations,*

$$\begin{aligned}\hat{\chi}((a_{ii})_{1\le i\le d},(a_{ij})_{1\le i<j\le d}) &\le d^2\chi(b:M_d(\mathbb{C}\mathbf{1}))-\frac{d^2}{2}\log d\\ &\le d^2\chi(b)-\frac{d^2}{2}\log d\,.\end{aligned}$$

Furthermore, if b is free from $M_d(\mathbb{C}\mathbf{1})$ then

$$\hat{\chi}((a_{ii})_{1\le i\le d},(a_{ij})_{1\le i<j\le d})=d^2\chi(b)-\frac{d^2}{2}\log d\,.$$

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