Rev. Math. Phys. **8**(1996), 819–859

Quantum mechanics in AF C^*-systems

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Abstract. Motivated from the chemical potential theory, we study quantum statistical thermodynamics in AF C^*-systems generalizing ususal one-dimensional quantum lattice systems. Our systems are C^*-algebras $\mathcal{A}$ which have a localization $\{\mathcal{A}_{[i,j]}\}$ of finite-dimensional subalgebras indexed by finite intervals of $\mathbf{Z}$ and an automorphism γ acting as a right shift on the localization. Model examples are supplied from derived towers (string algebras) for type II_1 factor-subfactor pairs. Given a (γ-invariant) interaction and a specific tracial state, we formulate the Gibbs conditions and the variational principle for (γ-invariant) states on $\mathcal{A}$, and investigate the relationship among these conditions and the KMS condition for the time evolution generated by the interaction. Special attention is turned to C^*-systems of gauge invariance (typical model in the chemical potential theory) and to C^*-systems considered as quantum random walks on discrete groups. The CNT-dynamical entropy for the shift automorphism γ is also discussed.

Introduction

Let $\mathbf{Z}^\nu$ be the simple cubic lattice of dimension ν, and $\mathcal{A}_k$ ($k \in \mathbf{Z}^\nu$) be copies of $M_d(\mathbf{C})$, the $d\times d$ matrix algebra. Then the usual ν-dimensional quantum lattice system or quantum spin system is described as the infinite tensor product C^*-algebra $\mathcal{A} = \bigotimes_{k\in\mathbf{Z}^\nu} \mathcal{A}_k$ with the space translations γ_k ($k \in \mathbf{Z}^\nu$). As is fully presented in [9], the rigorous treatment of quantum lattice systems is one of major successes of the C^*-algebraic approach to quantum physics. An interaction Φ in the quantum spin C^*-algera $\mathcal{A}$ is given when a selfadjoint element $\Phi(X)$ in the local algebra $\mathcal{A}_X = \bigotimes_{k\in X} \mathcal{A}_k$ is specified for each finite $X \subset \mathbf{Z}^\nu$. Then the local Hamiltonian for a finite $\Lambda \subset \mathbf{Z}^\nu$ is $H_\Lambda = \sum_{X\subset\Lambda} \Phi(X)$, and the one-parameter dynamics (i.e. the time evolution) α_t^Φ ($t \in \mathbf{R}$) can be introduced as the strong limit of $e^{itH_\Lambda} \cdot e^{-itH_\Lambda}$ as $\Lambda \to \mathbf{Z}^\nu$ under a certain decay condition for Φ. The so-called variational principle is an essential ingredient in the translation-invariant theory of quantum lattice systems, where the mean entropy of a state and the pressure of an interaction play important roles. The main results concerning KMS or Gibbs states in the above setting are summarized as follows: Given a translation-invariant interaction Φ, the α^Φ-KMS condition, the Gibbs condition with respect to Φ, and the variational principle with respect to Φ are all equivalent for translation-invariant states on $\mathcal{A}$ ([46, 32, 3]). Another important result is concerning the uniqueness of α^Φ-KMS states (i.e. no phase transition) under the condition of bounded surface energies of Φ ([4, 28, 47]); this is typical in the one-dimensional quantum spin case. The aim of this paper is to extend quantum statistical thermodynamics from the setting of quantum spin (UHF) C^*-algebras to that of AF (non-UHF) C^*-algebras. An earlier attempt in this direction was made by Kishimoto [29, 30]. Also, Matsui [35] recently discussed ground states on the CAR algebra and on its gauge invariant part.

AF C^*-systems considered in this paper naturally arise from a recent development of the subfactor theory initiated by the celebrated Jones index theory [24] for type II_1 subfactors. In fact, our discussions are strongly motivated from our previous study [13, 19, 20] succeeding to [38, 11] on the subfactor theory from the entropic viewpoint. When $N \subset M$ is an inclusion of type II_1 factors with the Jones index $[M : N] < +\infty$, we obtain the Jones tower $\cdots \subset M_{-2} \subset M_{-1} = N \subset M_0 = M \subset M_1 \subset M_2 \subset \cdots$ by iterating the Jones basic construction both upward and downward. Then the relative commutant algebras or the string algebras $\mathcal{A}_{[i,j]} = M'_{i-1} \cap M_j$ are attached to intervals $[i, j]$ of $\mathbf{Z}$, which are finite-dimensional and generate the AF C^*-algebra $\mathcal{A} = \overline{\bigcup_n \mathcal{A}_{[-n,n]}}$. Furthermore, the Markov trace τ on $\bigcup_n M_n$ restricts on $\mathcal{A}$ and the canonical shift γ (a 2-shift on the localization $\{\mathcal{A}_{[i,j]}\}$) is defined on $\mathcal{A}$. This way, we can get an AF C^*-system $(\mathcal{A}, \{\mathcal{A}_{[2i,2j]}\}, \gamma, \tau)$ which is a model example of our C^*-systems. Recall that the double sequence $\{M' \cap M_n \subset N' \cap M_n\}_{n=1}^\infty$ attached the weights from the Markov trace τ is crucial in classification of type II_1 subfactors; in fact, it provides a complete conjugacy invariant among strongly amenable type II_1 inclusions ([42] and also [36]). Although the Jones theory for type II_1 subfactors was generalized to the type III ones (see e.g. [31, 33, 23]), it is enough from the viewpoint of this paper to confine ourselves to the type II_1 case because no new C^*-systems appear when general inclusions $N \subset M$ are considered beyond type II_1 ones. Also it should be mentioned that we are rather concerned with the infinite depth case in the subfactor model, while the finite depth case might be more interested from the subfactor theory. Indeed, when $N \subset M$ has finite depth, the resulting C^*-algebra $\mathcal{A}$ has a unique traical state so that the situation is almost the same as the quantum spin case.

Another motivation of our study comes from the chemical potential theory [7, 6], where the observable algebra $\mathcal{A}$ is the fixed point subalgebra of the field C^*-algebra $\mathcal{F}$ by some gauge action of a compact group and a one-parameter dynamics α_t on $\mathcal{F}$ is commuting with the gauge action so that α restricts on $\mathcal{A}$. The main concern in [6] is to extend an extremal α-KMS state on $\mathcal{A}$ to an extremal KMS state for a modified dynamics on $\mathcal{F}$. Here the notion of chemical potentials enters into the theory. A gauge action of a compact group G is sometimes given as a product action $\beta_g = \bigotimes_{\mathbf{Z}} \mathrm{Ad}\, \sigma_g$ $(g \in G)$ on the field algebra $\mathcal{F} = \bigotimes_{\mathbf{Z}} M_d(\mathbf{C})$ where σ is a unitary representation of G on $V = \mathbf{C}^d$. In this case, the observable algebra $\mathcal{A} = \mathcal{F}^\beta$ having the localization $\mathcal{A}_{[i,j]} = (\bigotimes_{k=i}^j M_d(\mathbf{C}))^\beta$ and the usual shift γ is essentially the same as the C^*-system arising from the subfactor of

Wassermann's type [53] defined by the representation σ. It is worth noting that, in several examples, the chemical potentials in [6] bijectively corresponds to the extremal faithful (γ-invariant) tracial states on $\mathcal{A}$. So it seems reasonable that we consider such tracial states on $\mathcal{A}$ as substitutes of chemical potentials in the abstract setup of AF C^*-systems.

The paper is organized as follows. To begin with we fix the formalism of C^*-systems $(\mathcal{A}, \{\mathcal{A}_{[i,j]}\}, \gamma)$ treated throughout and justify it from the subfactor model. In §2 we introduce, given an interaction Φ, the notions (in the strong and weak senses) of Gibbs condition for states on $\mathcal{A}$. These notions are defined in terms of a chosen tracial state ϕ on $\mathcal{A}$ as well as the interaction Φ. Then it is proved that a state on $\mathcal{A}$ satisfies the KMS condition (at $\beta = 1$) for the one-parameter dynamics α^Φ generated by Φ if and only if it satisfies the Gibbs condition in the strong sense for Φ with respect to some tracial state ϕ. In §3 we formulate the variational principle given a γ-invariant interaction Φ in an AF C^*-system. In the course of doing so, we show the existence of the mean relative entropy $S_{\mathrm{M}}(\omega, \phi)$ of a γ-invariant state ω and that of the pressure $p(\Phi, \phi)$ of Φ with respect to a fixed faithful γ-invariant tracial state ϕ having some multiplicativity property. The variational principle was obtained in [30] in a similar but somewhat different setting of C^*-systems. Furthermore, as in usual quantum lattice systems, given Φ and ϕ as above we show the following implications for γ-invariant states on $\mathcal{A}$: The Gibbs condition in the weak sense implies the variational principle and the latter implies the α^Φ-KMS condition. Our final goal might be to establish the abstract version of the chemical potential theory with use of extremal faithful tracial states on $\mathcal{A}$ instead of chemical potentials (see Problems 2.5 and 3.12 below). Although this problem in the abstract setting seems difficult, we can get rather satisfactory results in §4 and §5 in some C^*-systems with additional structures. A C^*-system in §4 is given as the fixed point subalgebra of the product action determined by a unitary representation of a compact group, which is a typical example from the original chemical potential theory. A C^*-system in §5 is obtained from a discrete group with a finite number of generators, which is considered as a quantum version of random walks on groups. The final §6 is devoted to discussions on the CNT-dynamical entropy of γ in connection with its topological entropy, which are on similar lines of [11, 13, 19, 20].

Our C^*-systems are restricted to AF C^*-systems indexed to one-dimensional $\mathbf{Z}$, while we can similarly consider those indexed to multi-dimensional $\mathbf{Z}^\nu$. But this restriction makes some results considerably simpler because of bounded surface energies as in the one-dimensional spin case.

1. Setting and examples

For $i, j \in \mathbf{Z}$ let $[i, j]$ denote the interval $\{i, i+1, \dots, j\}$ of $\mathbf{Z}$ with convention $[i, j] = \emptyset$ if $i > j$. Assume that finite-dimensional C^*-algebras (or finite direct sums of matrix algebras) $\mathcal{A}_{[i,j]}$ are given for all intervals $[i, j]$ of $\mathbf{Z}$, $i \le j$, and they satisfy the following:

(I) $\mathcal{A}_{[i,j]} \subset \mathcal{A}_{[i',j']}$ with the common unit 1 if $[i, j] \subset [i', j']$, i.e. $i' \le i \le j \le j'$,

(II) $\mathcal{A}_{[i,j]}$ and $\mathcal{A}_{[j+1,k]}$ commute if $i \le j < k$.

Let $\mathcal{A}$ denote the AF C^*-algebra generated by $\{\mathcal{A}_{[i,j]}\}$, that is, the C^*-completion of $\bigcup_{n=1}^\infty \mathcal{A}_{[-n,n]}$. For any $K \subset \mathbf{Z}$ we denote by $\mathcal{A}_K$ the C^*-subalgebra of $\mathcal{A}$ generated by $\mathcal{A}_{[i,j]}$ with $[i, j] \subset K$. Also set $\mathcal{A}_\emptyset = \mathbf{C}1$.

We further assume that there exists an automorphism γ of $\mathcal{A}$ such that

(III) $\gamma(\mathcal{A}_{[i,j]}) = \mathcal{A}_{[i+1,j+1]}$ for all $i \le j$.

This means that $\mathcal{A}_{[i,j]} \cong \mathcal{A}_{[i+k,j+k]}$ for any $i,j,k \in \mathbf{Z}$ and γ acts on the local algebras $\mathcal{A}_{[i,j]}$ as the (bilateral) right shift. So $\gamma(\mathcal{A}_{[1,\infty)}) = \mathcal{A}_{[2,\infty)}$ and $\gamma|\mathcal{A}_{[1,\infty)}$ is an endomorphism of $\mathcal{A}_{[1,\infty)}$ like the unilateral right shift.

We write $\mathcal{S}_\gamma(\mathcal{A})$ for the set of all γ-invariant states on $\mathcal{A}$. Since $(\mathcal{A},\gamma)$ is asymptotically abelian in the norm sense, i.e.

$$\lim_{|n|\to\infty} \|\,[a,\gamma^n(b)]\,\| = 0, \qquad a,b \in \mathcal{A},$$

it follows [9, 4.3.11] that $\mathcal{S}_\gamma(\mathcal{A})$ forms a (Choquet) simplex. The simplex of all tracial states on $\mathcal{A}$ is denoted by $\mathcal{T}(\mathcal{A})$. Also, we write $\mathcal{T}_\gamma(\mathcal{A})$ for the set of all γ-invariant $\phi \in \mathcal{T}(\mathcal{A})$. The following fact might be well known and easily shown; so we omit the proof.

Proposition 1.1 $\mathcal{T}_\gamma(\mathcal{A})$ is a face of $\mathcal{S}_\gamma(\mathcal{A})$ and so $\mathcal{T}_\gamma(\mathcal{A})$ becomes a simplex.

The quantum system $(\mathcal{A},\gamma)$ described above sometimes possesses a particular (not unique in general) trace τ having the following properties:

(IV) τ is faithful and γ-invariant,

(V) τ is *multiplicative* in the sense that $\tau(ab) = \tau(a)\tau(b)$ for all $a \in \mathcal{A}_{[i,j]}$ and $b \in \mathcal{A}_{[j+1,k]}$, $i \le j < k$.

Let Tr_n denote the canonical trace of $\mathcal{A}_{[1,n]}$, where the term "canonical" means that $\mathrm{Tr}_n(e) = 1$ for all minimal projections $e \in \mathcal{A}_{[1,n]}$. For $\tau \in \mathcal{T}_\gamma(\mathcal{A})$ let $d\tau_n/d\mathrm{Tr}_n$ be the Radon-Nikodym derivative of $\tau_n = \tau|\mathcal{A}_{[1,n]}$ with respect to Tr_n (so it belongs to the center of $\mathcal{A}_{[1,n]}$). Then the following rather strong condition for τ is considered, which means that the McMillan type convergence in the uniform norm holds for τ with respect to the localization $\{\mathcal{A}_{[1,n]}\}$:

(VI) For some constant $\lambda \in (0,1)$,

$$\lim_{n\to\infty} \left\| \frac{1}{n} \log \frac{d\tau_n}{d\mathrm{Tr}_n} - (\log\lambda)1 \right\| = 0.$$

In the rest of this section let us explain that the subfactor model provides many examples of $(\mathcal{A}, \{\mathcal{A}_{[i,j]}\}, \gamma)$ together with a distinguished trace τ. More examples will be presented in §4 and §5.

Example 1.2. Let $\mathcal{A}_0$ be a finite-dimensional C^*-algebra, $\mathcal{A}_k$ ($k \in \mathbf{Z}$) copies of $\mathcal{A}_0$, and $\mathcal{A}_{[i,j]} = \bigotimes_{k=i}^{j} \mathcal{A}_k$ ($i \le j$). The AF C^*-algebra $\mathcal{A}$ is the infinite C^*-tensor product $\bigotimes_{k\in\mathbf{Z}} \mathcal{A}_k$ and γ is the usual right shift. Set $\tau = \bigotimes_{\mathbf{Z}} \tau_0$, the product of $\tau_0 = \mathrm{Tr}/\mathrm{Tr}(1)$, where Tr is the canonical trace of $\mathcal{A}_0$. Then all (I)–(VI) are satisfied. Concerning (VI) we have $n^{-1}\log d\tau_n/d\mathrm{Tr}_n = \lambda 1$ for all n with $\lambda = \mathrm{Tr}(1)^{-1}$. When $\mathcal{A}_0 = M_d(\mathbf{C})$, the $d \times d$ matrix algebra, this $\mathcal{A}$ is the so-called *quantum spin* C^*-algebra.

Example 1.3. Let $\{e_i\}_{i\in\mathbf{Z}}$ be a two-sided sequence of the *Jones projections* ([24, 38]), that is, e_i are projections such that for some $0 < \lambda < 1$

$$\begin{cases} e_i e_{i\pm1} e_i = \lambda e_i, & i \in \mathbf{Z}, \\ e_i e_j = e_j e_i, & |i-j| \ge 2. \end{cases} \tag{1.1}$$

Then it is well known [24, 54] that $\lambda^{-1} \in \{4\cos^2 \pi/(m+1) : m \geq 2\} \cup [4, \infty)$. Let $\mathcal{A}_{[i,j]} = \mathrm{Alg}\{1, e_{i+1}, \dots, e_j\}$, the algebra generated by $\{1, e_{i+1}, \dots, e_j\}$, and $\mathcal{A}$ be the C^*-completion of $\bigcup_{n=1}^{\infty} \mathcal{A}_{[-n,n]}$. We can define a shift automorphism θ_λ on $\mathcal{A}$ by $\theta_\lambda(e_i) = e_{i+1}$, $i \in \mathbf{Z}$. Moreover, the so-called *λ-Markov trace* ϕ_λ is defined on $\mathcal{A}$, which satisfies

$$\phi_\lambda(ae_{j+1}) = \lambda\phi_\lambda(a), \qquad a \in \mathcal{A}_{[i,j]}.$$

Then (I)–(V) are satisfied. Relations (1.1) first appeared in [51] in some model of statistical physics; so the C^*-algebras $\mathcal{A}_{[i,j]}$ as well as $\mathcal{A}$ are sometimes called the *Temperley-Lieb algebra* (see also [17, 37]).

When $\lambda^{-1} = 4\cos^2 \pi/(m+1)$, the Bratteli diagrams of

$$\mathbf{C} \subset \mathcal{A}_{[1,1]} \subset \mathcal{A}_{[1,2]} \subset \cdots \tag{1.2}$$

are determined by the graph A_m ([24]), so that ϕ_λ is the unique tracial state on $\mathcal{A}$ and (VI) is satisfied (see Example 1.4 and Proposition 1.5 below). When $\lambda \leq 1/4$, the Bratteli diagrams of (1.2) have the graph A_∞. In this case, $\mathcal{A}_{[i,j]}$ and hence $\mathcal{A}$ are independent of $0 < \lambda \leq 1/4$. Furthermore, it is easy to check using the string algebra description of $\{\mathcal{A}_{[i,j]}\}$ ([36]) that θ_λ is also independent of λ up to unitary equivalence on each $\mathcal{A}_{[i,j]}$. Hence all tracial states ϕ_λ $(0 < \lambda \leq 1/4)$ are invariant for $\gamma = \theta_{1/4}$. In fact, the set of all extremal points of $\mathcal{T}(\mathcal{A})$ is $\{\phi_\lambda : 0 \leq \lambda \leq 1/4\}$, where ϕ_0 is a degenerate trace such that $\phi_0(e_i) = 0$, $i \in \mathbf{Z}$. So $\mathcal{T}(\mathcal{A}) = \mathcal{T}_\gamma(\mathcal{A})$. Note that $\phi_{1/4}$ satisfies (VI), because the $\phi_{1/4}$-trace vectors (i.e. the $\phi_{1/4}$-values of minimal projections) of $\mathcal{A}_{[1,2n-1]}$ and $\mathcal{A}_{[1,2n]}$ are respectively $2^{-(2n-1)}(2, 4, \dots, 2n)$ and $2^{-2n}(1, 3, \dots, 2n-1)$ ([24]). See Examples 1.4 and 4.6 for somewhat different formulations of this example.

Example 1.4. Let $N \subset M$ be an inclusion of type II_1 factors with the *Jones index* $[M : N] < +\infty$. Let

$$\cdots \subset M_{-3} \subset M_{-2} \subset M_{-1} = N \subset M_0 = M \subset M_1 \subset M_2 \subset \cdots$$

be the *Jones tower* of tunnel and basic constructions. Set $\mathcal{A}_{[i,j]} = M'_{i-1} \cap M_j$ for $i \leq j$, which are finite-dimensional algebras. The *derived tower* of $N \subset M$ is given as $\mathbf{C} \subset \mathcal{A}_{[1,1]} \subset \mathcal{A}_{[1,2]} \subset \cdots$, whose inclusions are described as the *principal graph.* Sometimes the principal graph comes from $\mathbf{C} \subset \mathcal{A}_{[0,0]} \subset \mathcal{A}_{[0,1]} \subset \cdots$; then the above one is called the *dual principal graph.* (See [24, 17] on the index theory for type II_1 subfactors.) Let $\mathcal{A}$ be the C^*-completion of $\bigcup_{n=1}^{\infty} \mathcal{A}_{[-n,n]}$, and set a normalized trace τ on $\mathcal{A}$ as the restriction of the unique trace on $\bigcup_n M_n$ (called the *λ-Markov trace* with $\lambda = [M : N]^{-1}$). The mirrorings and the canonical shift on the derived tower were introduced in [36]. The *mirroring* γ_n of $\mathcal{A}_{[1,2n]}$ is defined by

$$\gamma_n(x) = J_n x^* J_n, \qquad x \in \mathcal{A}_{[1,2n]},$$

where J_n is the modular conjugation on $L^2(M_n, \tau)$. Since $\gamma_{n+1} \circ \gamma_n = \gamma_n \circ \gamma_{n-1}$ on $\mathcal{A}_{[1,2n-2]}$ (see [13] for details), we can define the *canonical shift* γ on $\bigcup_n \mathcal{A}_{[1,n]}$ by

$$\gamma(x) = \gamma_{n+1}(\gamma_n(x)), \qquad x \in \mathcal{A}_{[1,2n]}. \tag{1.3}$$

Indeed, the canonical shift γ can be defined from each $M_{-n} \subset M_{-n+1}$, $n \geq 0$, so that γ extends to an automorphism on $\mathcal{A}$ preserving τ. Then (I)–(V) are satisfied with the localization $\{\mathcal{A}_{[2i,2j]}\}_{i \leq j}$, because γ is a 2-shift on $\{\mathcal{A}_{[i,j]}\}$. Concerning (V), the following a bit stronger condition is satisfied: for any $i \leq j \leq k$

$$\begin{array}{ccc} \mathcal{A}_{[i,j]} & \subset & \mathcal{A}_{[i,k]} \\ \cup & & \cup \\ \mathcal{A}_{[i+1,j]} & \subset & \mathcal{A}_{[i+1,k]} \end{array}$$

is a commuting square in the sense that

$$E_{[i,j]}E_{[i+1,k]} = E_{[i+1,k]}E_{[i,j]} = E_{[i+1,j]}; \text{ equivalently } E_{[i,j]}(\mathcal{A}_{[i+1,k]}) = \mathcal{A}_{[i+1,j]},$$

where $E_{[i,j]}$ is the conditional expectation onto $\mathcal{A}_{[i,j]}$ with respect to τ.

The canonical shift γ sometimes has a natural square root [12]. For instance, when $N \subset M$ is Jones' subfactor $R_\lambda \subset R$ ([24]) with index $\lambda^{-1} \leq 4$ or Popa's subfactor $N^s \subset M^s$ ([41]) with index $\lambda^{-1} = s > 4$, $\{\mathcal{A}_{[i,j]}\}$ coincides with Example 1.3 and $\gamma = \theta_\lambda^2$.

The *standard matrix* (or the principal graph) $\Gamma = [a_{kl}]_{k \in K, l \in L}$ and the *standard eigenvector* $\vec{s} = (s_k)_{k \in K}$ are attached to $N \subset M$, so that $\Gamma\Gamma^t$ describes the inclusions of $\mathcal{A}_{[1,2n]}$, $n \geq 0$, and $\Gamma\Gamma^t\vec{s} = \lambda^{-1}\vec{s}$. Furthermore, the trace vector of $\mathcal{A}_{[1,2n]}$ is given as $(\lambda^n s_k)_{k \in K_n}$, where K_n is the set of vertices corresponding to the direct summands of $\mathcal{A}_{[1,2n]}$ and so $K_n \subset K_{n+1}$, $\bigcup_n K_n = K$. Hence, denoting the minimal central projections of $\mathcal{A}_{[1,2n]}$ by $(f_{n,k})_{k \in K_n}$, we have

$$\frac{d\tau_n}{d\mathrm{Tr}_n} = \lambda^n \sum_{k \in K_n} s_k f_{n,k}.$$

If $N \subset M$ is *extremal* in the sense of [42], then we have $s_k = [f_{n,k}M_{2n}f_{n,k} : Mf_{n,k}]^{1/2}$, $k \in K_n$. (See [39, 42] for details on standard invariants.) Note that (VI) holds if and only if $\vec{s}$ has *subexponential growth*, i.e.

$$\lim_{n \to \infty} \frac{1}{n} \log\Big(\max_{k \in K_n} s_k\Big) = 0,$$

and in this case $N \subset M$ is extremal [20]. In particular, if $N \subset M$ has *finite depth* (this is the case when $[M : N] < 4$), i.e. $|K| < +\infty$ ($|K|$ denotes the cardinality of K), then (VI) holds and furthermore τ is a unique tracial state on $\mathcal{A}$ as is shown below for completeness. Also, note that τ is an extremal tracial state if and only if $N \subset M$ has the *ergodic core* [42], that is, $M^{st} = (\bigcup_n M'_{-n} \cap M)''$ is a factor. So, a tracial state on $\mathcal{A}$ is not extremal in general even though it satisfies (IV)–(VI). (The last fact will be more explicitly known in Remark 5.2.)

Proposition 1.5 If $N \subset M$ has finite depth, then τ is a unique tracial state on $\mathcal{A}$.

Proof. Choose n_0 such that $K_{n_0} = K$ and hence the inclusion matrix of $\mathcal{A}_{[1,2n]} \subset \mathcal{A}_{[1,2n+2k]}$ is $\Gamma\Gamma^t$ when $n \geq n_0$ and $k \geq 1$. Let ϕ be any tracial state on $\mathcal{A}$ and $\vec{v}_n$ the ϕ-trace vector of $\mathcal{A}_{[1,2n]}$ for $n \geq n_0$. Then we get $\vec{v}_n = (\Gamma\Gamma^t)^k\vec{v}_{n+k}$ for $k \geq 1$, so that

$$\vec{v}_n \in \bigcap_{k=1}^{\infty} (\Gamma\Gamma^t)^k \mathbf{R}_+^d, \qquad n \geq n_0,$$

where $d = |K|$. Hence $\vec{v}_n$ is proportional to the Perron-Frobenius eigenvector of $\Gamma\Gamma^t$, which implies that $\phi|\mathcal{A}_{[1,2n]} = \tau|\mathcal{A}_{[1,2n]}$ for all $n \geq n_0$. Therefore $\phi = \tau$.

2. KMS condition and Gibbs condition

From now on let $(\mathcal{A}, \{\mathcal{A}_{[i,j]}\}, \gamma)$ be a C^*-system introduced in §1, which always satisfies conditions (I)–(III). The symbol X stands for a finite interval in $\mathbf{Z}$, and let $|X| = j - i + 1$ for $X = [i, j]$, $i \leq j$. We say that Φ is an *interaction* if a selfadjoint element $\Phi(X)$ in $\mathcal{A}_X$ is given for each X. An interaction Φ is said to be γ*-invariant* or *translation-invariant* if

$$\gamma(\Phi(X)) = \Phi(X + 1), \qquad X \subset \mathbf{Z},$$

where $X + 1 = \{k + 1 : k \in X\}$. Given an interaction Φ and a finite interval $\Lambda \subset \mathbf{Z}$, the *local Hamiltonian* H_Λ is defined by

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X).$$

Also the *surface energy* W_Λ is defined by

$$W_\Lambda = \sum \{\Phi(X) : X \cap \Lambda \neq \emptyset,\ X \cap \Lambda^c \neq \emptyset\},$$

whenever the sum in the right-hand side converges in norm.

We use the notion of the inner perturbation of a state on $\mathcal{A}$ to introduce the Gibbs condition. The theory of state perturbation was first developed in [1] (see [48] for its extension). The variational approach to state perturbation was exploited in [5, 16] by means of the relative entropy. Let ω, ψ be two state on $\mathcal{A}$. For a finite interval $\Lambda \subset \mathbf{Z}$, the *relative entropy* of $\psi_\Lambda = \psi|\mathcal{A}_\Lambda$ with respect to $\omega_\Lambda = \omega|\mathcal{A}_\Lambda$ is given by

$$S(\psi_\Lambda, \omega_\Lambda) = \mathrm{Tr}_\Lambda\Big(\frac{d\psi_\Lambda}{d\mathrm{Tr}_\Lambda}\Big(\log\frac{d\psi_\Lambda}{d\mathrm{Tr}_\Lambda} - \log\frac{d\omega_\Lambda}{d\mathrm{Tr}_\Lambda}\Big)\Big),$$

where Tr_Λ is the canonical trace of $\mathcal{A}_\Lambda$. Then the relative entropy $S(\psi, \omega)$ is defined as

$$S(\psi, \omega) = \sup_{\Lambda \subset \mathbf{Z}} S(\psi_\Lambda, \omega_\Lambda) = \lim_{n \to \infty} S(\psi_{[-n,n]}, \omega_{[-n,n]}).$$

(See [37] on the relative entropy for states on a C^*-algebra and for normal states on a von Neumann algebra.) For any state ω on $\mathcal{A}$ and $h = h^* \in \mathcal{A}$, since

$$\psi \mapsto S(\psi, \omega) + \psi(h)$$

is weakly* lower semicontinuous and strictly convex on the state space of $\mathcal{A}$, the perturbed state $[\omega^h]$ is defined as a unique minimizer of this functional ([16, 37]). Recall [5, 16] that for selfadjoint $h, k \in \mathcal{A}$ the chain rule

$$[[\omega^h]^k] = [\omega^{h+k}] \tag{2.1}$$

holds and

$$|S(\psi,\omega)-S(\psi,[\omega^h])|\le 2||h||.$$

In particular,

$$S(\omega,[\omega^h])\le 2||h||. \tag{2.2}$$

Let π_ω be the GNS representation of $\mathcal{A}$ associated with ω and Ω_ω the corresponding cyclic vector. When Ω_ω is separating for $\pi_\omega(\mathcal{A})''$ and Δ_ω is the modular operator for Ω_ω, the perturbed vector Ω_ω^h is defined by ([1, 2])

$$\begin{aligned}\Omega_\omega^h &= \sum_{n=0}^{\infty}(-1)^n\int_0^{1/2}dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n \\ &\qquad \Delta_\omega^{t_n}\pi_\omega(h)\Delta_\omega^{t_{n-1}-t_n}\pi_\omega(h)\cdots\Delta_\omega^{t_1-t_2}\pi_\omega(h)\Omega_\omega \\ &= \exp\Big(\frac{\log\Delta_\omega-h}{2}\Big)\Omega_\omega.\end{aligned}$$

(Note that Ω_ω^h stands for $\Omega_\omega(-h)$ in the notation of [1, 2].) Then another definition of $[\omega^h]$ is given as $[\omega^h]=\langle\pi_\omega(\cdot)\Omega_\omega^h,\Omega_\omega^h\rangle/||\Omega_\omega^h||^2$. If $\tilde\omega$ and $[\omega^h]\tilde{}$ are the normal extensions of ω and $[\omega^h]$ to $\pi_\omega(\mathcal{A})''$, i.e. $\tilde\omega(x)=\langle x\Omega_\omega,\Omega_\omega\rangle$ and $[\omega^h]\tilde{}(x)=\langle x\Omega_\omega^h,\Omega_\omega^h\rangle/||\Omega_\omega^h||^2$, $x\in\pi_\omega(\mathcal{A})''$, then we have $[\omega^h]\tilde{}=[\tilde\omega^{\pi_\omega(h)}]$ because for a state ψ on $\mathcal{A}$ (see e.g. [37, p. 93])

$$S(\psi,\omega)=\begin{cases} S(\tilde\psi,\tilde\omega) & \text{if } \psi \text{ has the normal extension } \tilde\psi \text{ to } \pi_\omega(\mathcal{A})'', \\ +\infty & \text{otherwise.}\end{cases} \tag{2.3}$$

When an interaction Φ, $\phi\in\mathcal{T}(\mathcal{A})$, and a finite interval $\Lambda\subset\mathbf{Z}$ are given, the *canonical state* or the *local Gibbs state* ϕ_Λ^c on $\mathcal{A}_\Lambda$ with respect to Φ and ϕ is defined by

$$\phi_\Lambda^c(a)=\frac{\phi(e^{-H_\Lambda}a)}{\phi(e^{-H_\Lambda})},\qquad a\in\mathcal{A}_\Lambda.$$

Under the above preparation let us introduce the notion of the Gibbs condition as follows.

Definition 2.1. Let ω be a state on $\mathcal{A}$, Φ an interaction, and $\phi\in\mathcal{T}(\mathcal{A})$. Assume that W_Λ is defined for any finite interval $\Lambda\subset\mathbf{Z}$.

(1) We say that ω satisfies the *Gibbs condition in the strong sense* with respect to Φ and ϕ if Ω_ω is separating for $\pi_\omega(\mathcal{A})''$ and if for any $\Lambda\subset\mathbf{Z}$ the conditional expectation from $\pi_\omega(\mathcal{A})''$ onto $\pi_\omega(\mathcal{A}_\Lambda)\vee\pi_\omega(\mathcal{A}_{\Lambda^c})''$ with respect to $[\omega^{-W_\Lambda}]\tilde{}$ exists and the following holds:

$$[\omega^{-W_\Lambda}](ab)=\phi_\Lambda^c(a)[\omega^{-W_\Lambda}](b),\qquad a\in\mathcal{A}_\Lambda,\ b\in\mathcal{A}_{\Lambda^c}.$$

(2) We say that ω satisfies the *Gibbs condition in the weak sense* with respect to Φ and ϕ if $[\omega^{-W_\Lambda}]|\mathcal{A}_\Lambda=\phi_\Lambda^c$ for any $\Lambda\subset\mathbf{Z}$.

Obviously (1) implies (2). When ω satisfies the Gibbs condition in the sense of (2), $\phi \in \mathcal{T}(\mathcal{A})$ is unique because $\phi_\Lambda^c = \phi_\Lambda'^c$ gives $\phi = \phi'$ on $\mathcal{A}_\Lambda$ for $\phi, \phi' \in \mathcal{T}(\mathcal{A})$. In the usual quantum spin case, the Gibbs condition (1) coincides with [9, 6.2.16], because $\mathcal{A}$ has the unique tracial state and ϕ_Λ^c is the unique local Gibbs state on $\mathcal{A}_\Lambda$.

In this section we assume that an interaction Φ is not necessarily γ-invariant but satisfies the following:

(a) $\sup_{i\in\mathbf{Z}} \sum_{X\ni i} ||\Phi(X)|| < +\infty$,

(b) $\sup_{\Lambda\subset\mathbf{Z}} ||W_\Lambda|| < +\infty$.

Note that (a) and (b) are satisfied if Φ is γ-invariant and $\sum_{X\ni 0} |X|\,||\Phi(X)|| < +\infty$.

Under assumptions (a) and (b), [27, Theorem 8] (also [9, 6.2.6]) says that Φ generates a one-parameter dynamics α^Φ on $\mathcal{A}$. More precisely, there exists a strongly continuous one-parameter automorphism group α_t^Φ ($t \in \mathbf{R}$) on $\mathcal{A}$ such that

$$\lim_{\Lambda\to\mathbf{Z}} ||\alpha_t^\Phi(a) - e^{itH_\Lambda} a e^{-itH_\Lambda}|| = 0$$

for all $a \in \mathcal{A}$ and uniformly for t in finite intervals. Furthermore, the generator of α^Φ is the closure of the derivation δ_0 with $D(\delta_0) = \bigcup_\Lambda \mathcal{A}_\Lambda$ given by

$$\delta_0(a) = i \sum_{X\cap\Lambda\neq\emptyset} [\Phi(X), a], \qquad a \in \mathcal{A}_\Lambda.$$

So we can consider the KMS condition (at $\beta = 1$) for a state on $\mathcal{A}$ with respect to α^Φ. Note that $\alpha_t^\Phi \circ \gamma = \gamma \circ \alpha_t^\Phi$ if Φ is γ-invariant. (A recent development on the existence of one-parameter dynamics in the $\mathbf{Z}^\nu$-lattice spin and CAR cases is found in [34, 35].)

Our main result in this section is presented as follows.

Theorem 2.2 If ω is a state on $\mathcal{A}$ and Φ is an interaction as above, then the following conditions are equivalent:

(i) ω satisfies the KMS condition with respect to α^Φ;

(ii) ω satisfies the Gibbs condition in the strong sense with respect to Φ and ϕ for some $\phi \in \mathcal{T}(\mathcal{A})$.

Proof. (ii) $\Rightarrow$ (i). Assume that ω satisfies (ii) for some $\phi \in \mathcal{T}(\mathcal{A})$. Fix $\Lambda \subset \mathbf{Z}$ arbitrarily and write $\mathcal{M} = \pi_\omega(\mathcal{A})''$, $\mathcal{M}_\Lambda = \pi_\omega(\mathcal{A}_\Lambda)$, and $\mathcal{M}_{\Lambda^c} = \pi_\omega(\mathcal{A}_{\Lambda^c})''$. Let $\tilde{\omega}$ be the normal extension of ω, and let $\hat{\omega} = [\tilde{\omega}^{\pi_\omega(-W_\Lambda)}]$ ($= [\omega^{-W_\Lambda}]\tilde{}$, the normal extension of $[\omega^{-W_\Lambda}]$ to $\mathcal{M}$). Since $\hat{\omega}$ as well as $\tilde{\omega}$ is faithful by [1, Corollary 4.4], we have the modular automorphism groups σ_t and $\hat{\sigma}_t$ ($t \in \mathbf{R}$) for $\tilde{\omega}$ and $\hat{\omega}$, respectively. Since the conditional expectation from $\mathcal{M}$ onto $\mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}$ with respect to $\hat{\omega}$ exists, it follows by [50] that $\hat{\sigma}_t(\mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}) = \mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}$, $t \in \mathbf{R}$, and hence $\hat{\sigma}|\mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}$ is the modular automorphism group for $\hat{\omega}|\mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}$. Let σ^Λ and σ^{Λ^c} be the modular automorphism groups for $\hat{\omega}|\mathcal{M}_\Lambda$ and $\hat{\omega}|\mathcal{M}_{\Lambda^c}$, respectively. Since $[\mathcal{M}_\Lambda, \mathcal{M}_{\Lambda^c}] = 0$ with finite-dimensional $\mathcal{M}_\Lambda$ and $\hat{\omega}(xy) = \hat{\omega}(x)\hat{\omega}(y)$, $x \in \mathcal{M}_\Lambda$, $y \in \mathcal{M}_{\Lambda^c}$, we see that $(\mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}, \hat{\omega})$ is naturally isomorphic to $(\mathcal{M}_\Lambda \otimes \mathcal{M}_{\Lambda^c}, \hat{\omega}|\mathcal{M}_\Lambda \otimes \hat{\omega}|\mathcal{M}_{\Lambda^c})$. Hence $\hat{\sigma}|\mathcal{M}_\Lambda = \sigma^\Lambda$. But since

$$\hat{\omega}(\pi_\omega(a)) = [\omega^{-W_\Lambda}](a) = \phi_\Lambda^c(a), \qquad a \in \mathcal{A}_\Lambda,$$

it is easy to see that

$$\hat{\sigma}(x) = e^{it\pi_\omega(H_\Lambda)} x e^{-it\pi_\omega(H_\Lambda)}, \qquad x \in \mathcal{M}_\Lambda. \tag{2.4}$$

Now let δ_σ and $\delta_{\hat{\sigma}}$ be the generators of σ and $\hat{\sigma}$. Then the perturbation theory for the modular automorphism group says (see [9, §5.4.1]) that

$$\delta_\sigma(x) = \delta_{\hat{\sigma}}(x) + i[\pi_\omega(W_\Lambda), x], \qquad x \in D(\delta_\sigma) = D(\delta_{\hat{\sigma}}).$$

By (2.4) we have $\mathcal{M}_\Lambda \subset D(\delta_{\hat{\sigma}})$ and for every $a \in \mathcal{A}_\Lambda$

$$\begin{aligned}\delta_\sigma(\pi_\omega(a)) &= i[\pi_\omega(H_\Lambda), \pi_\omega(a)] + i[\pi_\omega(W_\Lambda), \pi_\omega(a)] \\ &= i\pi_\omega\Bigg(\sum_{X\cap\Lambda\neq\emptyset} [\Phi(X), a]\Bigg) \\ &= \pi_\omega(\delta(a)),\end{aligned}$$

where δ is the generator of α^Φ. Thus we have shown that

$$\pi_\omega(\delta(a)) = \delta_\sigma(\pi_\omega(a)), \qquad a \in \bigcup_{\Lambda\subset\mathbf{Z}} \mathcal{A}_\Lambda.$$

Since $\bigcup_\Lambda \mathcal{A}_\Lambda$ is a core of δ, we obtain $\pi_\omega \circ \delta \subset \delta_\sigma \circ \pi_\omega$ and hence $\pi_\omega \circ \alpha^\Phi = \sigma \circ \pi_\omega$. This implies that ω satisfies the KMS condition with respect to α^Φ.

To prove the converse, the next lemma is useful.

Lemma 2.3 Let $\mathcal{M}$ be a von Neumann algebra and φ a faithful normal state on $\mathcal{M}$. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be von Neumann subalgebras of $\mathcal{M}$ and assume that the conditional expectation $E_1 : \mathcal{M} \to \mathcal{M}_1$ with respect to φ exists. If $h \in \mathcal{M}_1$ is selfadjoint, then

(1) $[\varphi^h] = [(\varphi|\mathcal{M}_1)^h] \circ E_1$,

(2) $[\varphi^h]|\mathcal{M}_0 = \varphi|\mathcal{M}_0$ whenever $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x \in \mathcal{M}_0$ and $y \in \mathcal{M}_1$.

Proof. (1) Put $\varphi_1 = \varphi|\mathcal{M}_1$ so that $\varphi = \varphi_1 \circ E_1$. When $h \in \mathcal{M}_1$ is selfadjoint, we have for every state ψ on $\mathcal{M}$

$$\begin{aligned}S(\psi, \varphi) + \psi(h) &\geq S(\psi|\mathcal{M}_1, \varphi_1) + (\psi|\mathcal{M}_1)(h) \\ &\geq S([\varphi_1^h], \varphi_1) + [\varphi_1^h](h) \\ &= S([\varphi_1^h] \circ E_1, \varphi) + [\varphi_1^h] \circ E_1(h).\end{aligned}$$

This implies that $[\varphi^h] = [\varphi_1^h] \circ E_1$.

(2) The assumption means that $E_1(x) = \varphi(x)1$, $x \in \mathcal{M}_0$. Hence for every $x \in \mathcal{M}_0$ we get by (1)

$$[\varphi^h](x) = [\varphi_1^h](E_1(x)) = \varphi(x).$$

Proof of (i) $\Rightarrow$ (ii) of Theorem 2.2: Assume that ω is an α^Φ-KMS state. Then the cyclic vector Ω_ω is separating for $\mathcal{M} = \pi_\omega(\mathcal{A})''$ ([9, 5.3.9]) and the normal extension

σ of α^Φ to $\mathcal{M}$ is the modular automorphism group for $\tilde{\omega}$. For any fixed $\Lambda = [l, m]$, put $Q = -(H_\Lambda + W_\Lambda)$ and let $\alpha^{\Phi,Q}$ be the perturbation of α^Φ by Q; namely the generator δ^Q of $\alpha^{\Phi,Q}$ is given as

$$\delta^Q(a) = \delta(a) + i[Q, a], \qquad a \in D(\delta) = D(\delta^Q).$$

Furthermore define

$$\alpha_t^{(n)}(a) = e^{it(H_{[-n,l-1]}+H_{[m+1,n]})} a e^{-it(H_{[-n,l-1]}+H_{[m+1,n]})}, \qquad a \in \mathcal{A}.$$

Now let us prove that

$$\alpha_t^{\Phi,Q}(a) = \lim_{n\to\infty} \alpha_t^{(n)}(a), \qquad a \in \mathcal{A},\ t \in \mathbf{R}. \tag{2.5}$$

To do so, it suffices (see [48, 4.1.2]) to show that $(1 \pm \delta^{(n)})^{-1} \to (1 \pm \delta^Q)^{-1}$ strongly where $\delta^{(n)}$ is the generator of $\alpha^{(n)}$. For any $\Lambda' \subset \mathbf{Z}$ and $a \in \mathcal{A}_{\Lambda'}$ we have

$$\begin{aligned} &||\{(1 \pm \delta^{(n)})^{-1} - (1 \pm \delta^Q)^{-1}\}(1 \pm \delta^Q)(a)|| \\ &\quad = ||(1 \pm \delta^{(n)})^{-1}\{(1 \pm \delta^Q)(a) - (1 \pm \delta^{(n)})(a)\}|| \\ &\quad \le ||\delta^Q(a) - \delta^{(n)}(a)||. \end{aligned}$$

Let

$$\Xi_n = \sum\{\Phi(X) : X \cap \Lambda \ne \emptyset,\ X \cap [-n, n]^c \ne \emptyset\},$$

$$\Xi'_n = \sum\{\Phi(X) : X \cap \Lambda' \ne \emptyset,\ X \cap [-n, n]^c \ne \emptyset\}.$$

Then $||\Xi_n||, ||\Xi'_n|| \to 0$ $(n \to \infty)$. When $\Lambda \subset [-n, n]$, since

$$H_{[-n,n]} + Q = H_{[-n,l-1]} + H_{[m+1,n]} - \Xi_n,$$

we get

$$\begin{aligned} \delta^Q(a) - \delta^{(n)}(a) &= i\Big[\sum_{X\cap\Lambda'\ne\emptyset} \Phi(X) + Q, a\Big] - i[H_{[-n,l-1]} + H_{[m+1,n]}, a] \\ &= -i[\Xi_n, a] + i\Big[\sum_{X\cap\Lambda'\ne\emptyset} \Phi(X) - \sum_{X\subset[-n,n]} \Phi(X), a\Big] \\ &= -i[\Xi_n, a] + i[\Xi'_n, a], \end{aligned}$$

so that $||\delta^Q(a) - \delta^{(n)}(a)|| \to 0$. Therefore (2.5) is shown. Since $\alpha_t^{(n)}(a) = a$ for $a \in \mathcal{A}_\Lambda$ and $\alpha_t^{(n)}(a) \in \mathcal{A}_{\Lambda^c}$ for $a \in \mathcal{A}_{\Lambda^c}$, it follows from (2.5) that

$$\alpha_t^{\Phi,Q}|\mathcal{A}_\Lambda = \mathrm{id}_{\mathcal{A}_\Lambda}, \quad \alpha_t^{\Phi,Q}(\mathcal{A}_{\Lambda^c}) = \mathcal{A}_{\Lambda^c}. \tag{2.6}$$

Moreover let $\alpha^{\Phi,-W_\Lambda}$ be the perturbation of α^Φ by $-W_\Lambda$, which is the perturbation of $\alpha^{\Phi,Q}$ by H_Λ. Since $\alpha_t^{\Phi,-W_\Lambda}(a) = e^{itH_\Lambda}\alpha_t^{\Phi,Q}(a)e^{-itH_\Lambda}$, we get by (2.6)

$$\alpha_t^{\Phi,-W_\Lambda}(\mathcal{A}_\Lambda) = \mathcal{A}_\Lambda, \quad \alpha_t^{\Phi,-W_\Lambda}(\mathcal{A}_{\Lambda^c}) = \mathcal{A}_{\Lambda^c}. \tag{2.7}$$

The normal extensions of $\alpha^{\Phi,Q}$ and $\alpha^{\Phi,-W_\Lambda}$ are the perturbations of σ by $\pi_\omega(Q)$ and $\pi_\omega(-W_\Lambda)$, respectively, which are denoted by σ^Q and σ^{-W_Λ}. Recall [1, Proposition 4.3] (also [9, 5.4.4]) that σ^Q and σ^{-W_Λ} are the modular automorphism groups for the normal extensions $[\omega^Q]\tilde{}$ and $[\omega^{-W_\Lambda}]\tilde{}$ of $[\omega^Q]$ and $[\omega^{-W_\Lambda}]$ to $\mathcal{M}$, respectively. Let $\mathcal{M}_\Lambda$ and $\mathcal{M}_{\Lambda^c}$ be as in the proof of (ii) $\Rightarrow$ (i). Since $\sigma_t^{-W_\Lambda}(\mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}) = \mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}$ by (2.7), the conditional expectation from $\mathcal{M}$ onto $\mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}$ with respect to $[\omega^{-W_\Lambda}]\tilde{}$ exists by [50]. It follows from (2.6) that $\tilde{\phi}_\Lambda = [\omega^Q]\tilde{}|\mathcal{M}_\Lambda$ is a trace on $\mathcal{M}_\Lambda$ and the conditional expectation $E_1 : \mathcal{M} \to \mathcal{M}_{\Lambda^c}$ with respect to $[\omega^Q]\tilde{}$ exists. Furthermore, as is readily seen because $(\mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}, \sigma^Q)$ is naturally isomorphic to $(\mathcal{M}_\Lambda \otimes \mathcal{M}_{\Lambda^c}, \mathrm{id}_{\mathcal{M}_\Lambda} \otimes (\sigma^Q|\mathcal{M}_{\Lambda^c}))$ by (2.6), we have

$$\begin{aligned} [\omega^Q]\tilde{}(xy) &= [\omega^Q]\tilde{}(x)[\omega^Q]\tilde{}(y) \\ &= \tilde{\phi}_\Lambda(x)[\omega^Q]\tilde{}(y), \qquad x \in \mathcal{M}_\Lambda,\ y \in \mathcal{M}_{\Lambda^c}. \end{aligned} \tag{2.8}$$

Now let $\Lambda_1 \supset \Lambda$ and $Q_1 = -(H_{\Lambda_1} + W_{\Lambda_1})$. Since by (2.1)

$$[\omega^{Q_1}]\tilde{} = [\tilde{\omega}^{\pi_\omega(Q_1)}] = [[\tilde{\omega}^{\pi_\omega(Q)}]^{\pi_\omega(Q_1-Q)}]$$

and $\pi_\omega(Q_1 - Q) \in \mathcal{M}_{\Lambda^c}$, Lemma 2.3(2) together with (2.8) implies that $[\omega^{Q_1}]\tilde{}|\mathcal{M}_\Lambda = [\omega^Q]\tilde{}|\mathcal{M}_\Lambda$, that is, $\tilde{\phi}_{\Lambda_1}|_{\mathcal{M}_\Lambda} = \tilde{\phi}_\Lambda$. Hence there exists a tracial state $\tilde{\phi}$ on $\pi_\omega(\bigcup_\Lambda \mathcal{A}_\Lambda)$ such that $\tilde{\phi}|\mathcal{M}_\Lambda = \tilde{\phi}_\Lambda$ for all $\Lambda \subset \mathbf{Z}$. Thus we obtain $\phi \in \mathcal{T}(\mathcal{A})$ which extends $\tilde{\phi} \circ \pi_\omega$ on $\bigcup_\Lambda \mathcal{A}_\Lambda$. Since $\sigma_t^Q(\mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}) = \mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}$, the conditional expectation $E_0 : \mathcal{M} \to \mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c}$ with respect to $[\omega^Q]\tilde{}$ exists. So we have by Lemma 2.3(1) and (2.8)

$$\begin{aligned} [\omega^{-W_\Lambda}]\tilde{} &= [([\omega^Q]\tilde{})^{\pi_\omega(H_\Lambda)}] \\ &= [([\omega^Q]\tilde{}|\mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c})^{\pi_\omega(H_\Lambda)}] \circ E_0 \\ &= [(\tilde{\phi}_\Lambda \otimes [\omega^Q]\tilde{}|\mathcal{M}_{\Lambda^c})^{\pi_\omega(H_\Lambda)}] \circ E_0 \\ &= ([\tilde{\phi}_\Lambda^{\pi_\omega(H_\Lambda)}] \otimes [\omega^Q]\tilde{}|\mathcal{M}_{\Lambda^c}) \circ E_0 \end{aligned} \tag{2.9}$$

under the natural isomorphism $\mathcal{M}_\Lambda \vee \mathcal{M}_{\Lambda^c} \cong \mathcal{M}_\Lambda \otimes \mathcal{M}_{\Lambda^c}$. Since

$$\begin{aligned} [\tilde{\phi}_\Lambda^{\pi_\omega(H_\Lambda)}](\pi_\omega(a)) &= \frac{\tilde{\phi}_\Lambda(e^{-\pi_\omega(H_\Lambda)}\pi_\omega(a))}{\tilde{\phi}_\Lambda(e^{-\pi_\omega(H_\Lambda)})} \\ &= \frac{\phi(e^{-H_\Lambda}a)}{\phi(e^{-H_\Lambda})} = \phi_\Lambda^c(a), \qquad a \in \mathcal{A}_\Lambda, \end{aligned}$$

it follows from (2.9) that

$$[\omega^{-W_\Lambda}](ab) = \phi_\Lambda^c(a)[\omega^Q](b), \qquad a \in \mathcal{A}_\Lambda,\ b \in \mathcal{A}_{\Lambda^c},$$

completing the proof.

Let $K(\Phi)$ denote the set of all KMS states on $\mathcal{A}$ with respect to α^{Φ}, which becomes a simplex (see [9, 5.3.30]). Then Theorem 2.2 asserts that a correspondence $\omega \in K(\Phi) \mapsto \phi \in \mathcal{T}(\mathcal{A})$ is determined in the way that ω is a Gibbs state in the strong sense for Φ and ϕ.

Proposition 2.4 If Φ is γ-invariant and $\omega \in K(\Phi)$ is γ-invariant, then so is the corresonnding $\phi \in \mathcal{T}(\mathcal{A})$.

Proof. By the γ-invariance of Φ and ω we have

$$[\omega^{-W_\Lambda}] = [\omega^{-W_{\Lambda+1}}] \circ \gamma, \qquad \Lambda \subset \mathbf{Z}.$$

Indeed, this is easily checked due to the definition of state perturbation through the relative entropy. So we get $\phi_\Lambda^c = \phi_{\Lambda+1}^c \circ \gamma = (\phi\circ\gamma)_\Lambda^c$ on $\mathcal{A}_\Lambda$ for every $\Lambda \subset \mathbf{Z}$, which implies $\phi = \phi\circ\gamma$.

We here want to pose the following problems, which may be considered as an abstract version of the chemical potential theory [6] (see §4).

Problems 2.5.

(1) Is the correspondence $\omega \mapsto \phi$ bijective between $K(\Phi)$ and $\mathcal{T}(\mathcal{A})$?

(2) Under a suitable assumption, is ω an extremal α^{Φ}-KMS state if and only if the corresponding ϕ is an extremal tracial state? Furthermore, when Φ is γ-invariant, is ω an extremal γ-invariant α^{Φ}-KMS state if and only if ϕ is extremal in $\mathcal{T}_\gamma(\mathcal{A})$?

The next proposition is a partial answer concerning the injectivity of $\omega \mapsto \phi$ in the above (1), whose proof is a slight modification of [4] (also [48, 4.7.3]); so we omit it.

Proposition 2.6 Let $\omega, \omega' \in K(\Phi)$ and assume that ω is extremal. If ω and ω' correspond to the same $\phi \in \mathcal{T}(\mathcal{A})$, then $\omega = \omega'$.

As a consequence, when $\mathcal{T}(\mathcal{A})$ is a singleton (as in the finite depth case of Example 1.4), we obtain the uniqueness of α^{Φ}-KMS state as a slight generalization of [4, 28, 47]. (The existence of α^{Φ}-KMS state is due to [44].)

3. Variational principle

In this section let $(\mathcal{A}, \{\mathcal{A}_{[i,j]}\}, \gamma)$ be as in §1 and ϕ be a fixed traicial state on $\mathcal{A}$ satisfying (IV) and (V). For $n \geq 1$ and $\omega \in \mathcal{S}_\gamma(\mathcal{A})$, let $S(\omega_n, \phi_n)$ be the relative entropy of $\omega_n = \omega|\mathcal{A}_{[1,n]}$ with respect to $\phi_n = \phi|\mathcal{A}_{[1,n]}$.

Lemma 3.1 If $\omega \in \mathcal{S}_\gamma(\mathcal{A})$, then $\lim_{n\to\infty} \frac{1}{n} S(\omega_n, \phi_n)$ exists and

$$\lim_{n\to\infty} \frac{1}{n} S(\omega_n, \phi_n) = \sup_{n\geq 1} \frac{1}{n} S(\omega_n, \phi_n), \tag{3.1}$$

Proof. It suffices to show the superadditivity:

$$S(\omega_{m+n}, \phi_{m+n}) \geq S(\omega_m, \phi_m) + S(\omega_n, \phi_n), \qquad m, n \geq 1.$$

Put $D_n = d\omega_n/d\phi_n$, which is a positive element of $\mathcal{A}_{[1,n]}$ with $\phi(D_n) = 1$. Then $S(\omega_n, \phi_n) = \omega(\log D_n)$. By (II)–(V), $D_m\gamma^m(D_n)$ is a positive element of $\mathcal{A}_{[1,m+n]}$ and $\phi(D_m\gamma^m(D_n)) = \phi(D_m)\phi(\gamma^m(D_n)) = 1$. So we have by the positivity of relative entropy

$$\begin{aligned}
0 &\le S(D_{m+n}, D_m\gamma^m(D_n)) \\
&= \phi(D_{m+n}(\log D_{m+n} - \log D_m - \log \gamma^m(D_n))) \\
&= \omega(\log D_{m+n}) - \omega(\log D_m) - \omega \circ \gamma^m(\log D_n) \\
&= S(\omega_{m+n}, \phi_{m+n}) - S(\omega_m, \phi_m) - S(\omega_n, \phi_n).
\end{aligned}$$

Hence $\{S(\omega_n, \phi_n)\}$ is a superadditive sequence.

For each $\omega \in \mathcal{S}_\gamma(\mathcal{A})$, the *mean relative entropy* of ω with respect to ϕ is defined by (3.1) above and is denoted by $S_{\mathrm{M}}(\omega, \phi)$ ($\in [0, +\infty]$).

Proposition 3.2 The function $\omega \mapsto S_{\mathrm{M}}(\omega, \phi)$ is affine and weakly* lower semicontinuous on $\mathcal{S}_\gamma(\mathcal{A})$. Moreover, if $\omega \in \mathcal{S}_\gamma(\mathcal{A})$, then $S_{\mathrm{M}}(\omega, \phi) = 0$ if and only if $\omega = \phi$.

Proof. Since $\omega \mapsto S(\omega_n, \phi_n)$ is weakly* continuous, the weak* lower semicontinuity of $\omega \mapsto S_{\mathrm{M}}(\omega, \phi)$ follows from (3.1). The affinity is immediate from the following: For any $\omega, \omega' \in \mathcal{S}_\gamma(\mathcal{A})$ and $0 \le \alpha \le 1$

$$\begin{aligned}
\alpha S(\omega_n, \phi_n) + (1-\alpha)S(\omega'_n, \phi_n) &\ge S(\alpha\omega_n + (1-\alpha)\omega'_n, \phi_n) \\
&\ge \alpha S(\omega_n, \phi_n) + (1-\alpha)S(\omega'_n, \phi_n) + \alpha\log\alpha + (1-\alpha)\log(1-\alpha).
\end{aligned}$$

Indeed, the first inequality is a convexity property of relative entropy and the second is seen from the operator monotonity of $\log t$. If $\omega \in \mathcal{S}_\gamma(\mathcal{A})$ and $S_{\mathrm{M}}(\omega, \phi) = 0$, then $S(\omega_n, \phi_n) = 0$ for all $n \ge 1$ by (3.1), which implies $\omega = \phi$.

Proposition 3.2 shows that ϕ is extremal in $\mathcal{S}_\gamma(\mathcal{A})$ or equivalently extremal in $\mathcal{T}_\gamma(\mathcal{A})$ by Proposition 1.1. However, ϕ is not necessarily extremal in $\mathcal{T}(\mathcal{A})$ (see the last sentence of Example 1.4 or Remark 5.2).

For $\omega \in \mathcal{S}_\gamma(\mathcal{A})$, the *von Neumann entropy* $S(\omega_n)$ of ω_n is given as

$$\begin{aligned}
S(\omega_n) &= \mathrm{Tr}_n\Big(-\frac{d\omega_n}{d\mathrm{Tr}_n}\log\frac{d\omega_n}{d\mathrm{Tr}_n}\Big) = -\omega\Big(\log\frac{d\omega_n}{d\mathrm{Tr}_n}\Big) \\
&= -S(\omega_n, \phi_n) - \omega\Big(\log\frac{d\phi_n}{d\mathrm{Tr}_n}\Big),
\end{aligned} \tag{3.2}$$

because $d\omega_n/d\mathrm{Tr}_n = (d\omega_n/d\phi_n)(d\phi_n/d\mathrm{Tr}_n)$ and $d\phi_n/d\mathrm{Tr}_n$ belongs to the center of $\mathcal{A}_{[1,n]}$. We define the *mean entropy* $s(\omega)$ by

$$s(\omega) = \lim_{n\to\infty}\frac{1}{n}S(\omega_n),$$

whenever the limit exists.

Proposition 3.3 Assume that τ is a tracial state on $\mathcal{A}$ satisfying (IV)–(VI) in §1. Then for every $\omega \in \mathcal{S}_\gamma(\mathcal{A})$ the mean entropy $s(\omega)$ exists and

$$s(\omega) = \log\lambda^{-1} - S_{\mathrm{M}}(\omega, \tau). \tag{3.3}$$

The function $s(\omega)$ is affine and weakly* upper semicontinuous on $\mathcal{S}_\gamma(\mathcal{A})$. Moreover, $s(\omega) \le \log\lambda^{-1} = s(\tau)$, and $s(\omega) = \log\lambda^{-1}$ if and only if $\omega = \tau$.

Proof. The first assertion is obvious from (3.2) and (VI). What remains is immediate from Proposition 3.2.

The above proposition shows that if a tracial state τ on $\mathcal{A}$ satisfying (IV)–(VI) exists, then it is uniquely determined as a γ-invariant state maximizing the mean entropy.

Now let Φ be an interaction. In this section we assume that Φ is γ-invariant and has *relatively short range*:

$$\sum_{X \ni 0} \frac{||\Phi(X)||}{|X|} < +\infty.$$

Let $\mathcal{B}$ denote the set of all γ-invariant interactions of relatively short range, which becomes a real Banach space with the obvious linear operations and the norm $|||\Phi||| = \sum_{X\ni 0} ||\Phi(X)||/|X|$.

Given $\Phi \in \mathcal{B}$ define $A_\Phi \in \mathcal{A}$ by

$$A_\Phi = \sum_{X \ni 0} \frac{\Phi(X)}{|X|}.$$

Obviously $||A_\Phi|| \le |||\Phi|||$. Moreover, for simplicity we write $H_n = H_{[1,n]}$, the local Hamiltonian, and $\phi_n^c = \phi_{[1,n]}^c$, the local Gibbs state on $\mathcal{A}_{[1,n]}$ with respect to Φ and ϕ, i.e. $\phi_n^c(a) = \phi(e^{-H_n}a)/\phi(e^{-H_n})$, $a \in \mathcal{A}_{[1,n]}$.

Lemma 3.4 If $\Phi \in \mathcal{B}$ and $\omega \in \mathcal{S}_\gamma(\mathcal{A})$, then $\lim_{n\to\infty} \frac{1}{n}\omega(H_n)$ exists and

$$\lim_{n\to\infty} \frac{1}{n}\omega(H_n) = \omega(A_\Phi).$$

Proof. Since

$$\begin{aligned}
\left|\frac{\omega(H_n)}{n} - \omega(A_\Phi)\right| &\le \left|\left|\frac{H_n}{n} - \frac{1}{n}\sum_{k=1}^n \gamma^k(A_\Phi)\right|\right| \\
&\le \frac{1}{n}\sum_{k=1}^n \sum \left\{ \frac{||\Phi(X)||}{|X|} : X \ni k,\ X \cap [1,n]^c \ne \emptyset \right\},
\end{aligned}$$

we have $\frac{1}{n}\omega(H_n) \to \omega(A_\Phi)$ in the same way as [9, 6.2.39].

Theorem 3.5 If $\Phi \in \mathcal{B}$, then $\lim_{n\to\infty} \frac{1}{n}\log\phi(e^{-H_n})$ exists and

$$\lim_{n\to\infty} \frac{1}{n}\log\phi(e^{-H_n}) = \sup_{\omega\in\mathcal{S}_\gamma(\mathcal{A})} \{-S_{\mathrm{M}}(\omega,\phi) - \omega(A_\Phi)\}.$$

Proof. For every $\omega \in \mathcal{S}_\gamma(\mathcal{A})$ we get

$$0 \le S(\omega_n, \phi_n^c) = \omega\left(\log \frac{d\omega_n}{d\phi_n} - \log \frac{e^{-H_n}}{\phi(e^{-H_n})}\right) \tag{3.4}$$
$$= S(\omega_n, \phi_n) + \omega(H_n) + \log \phi(e^{-H_n}),$$

so that

$$\frac{1}{n} \log \phi(e^{-H_n}) \ge -\frac{1}{n} S(\omega_n, \phi_n) - \frac{\omega(H_n)}{n}.$$

By Lemmas 3.1 and 3.4 we have

$$\liminf_{n\to\infty} \frac{1}{n} \log \phi(e^{-H_n}) \ge -S_{\mathrm{M}}(\omega, \phi) - \omega(A_\Phi).$$

Therefore

$$\liminf_{n\to\infty} \frac{1}{n} \log \phi(e^{-H_n}) \ge \sup_{\omega \in \mathcal{S}_\gamma(\mathcal{A})} \{-S_{\mathrm{M}}(\omega, \phi) - \omega(A_\Phi)\}. \tag{3.5}$$

On the other hand, thanks to (V), for each $n \in \mathbf{N}$ a state $\varphi^{(n)}$ on $\mathcal{A}$ can be defined by

$$\varphi^{(n)}(x) = \phi\left(\frac{\prod_{m=i}^{j-1} \gamma^{mn}(e^{-H_n})}{\phi(e^{-H_n})^{j-i}} x\right), \qquad x \in \mathcal{A}_{[in+1, jn]},\ i \le j,$$

which is periodic, i.e. $\varphi^{(n)} \circ \gamma^n = \varphi^{(n)}$. Then we define $\omega^{(n)} \in \mathcal{S}_\gamma(\mathcal{A})$ by

$$\omega^{(n)} = \frac{1}{n} \sum_{k=1}^{n} \varphi^{(n)} \circ \gamma^k.$$

Using the convexity and the monotonicity of relative entropy, we have for $j \in \mathbf{N}$

$$\begin{aligned}
S(\omega_{jn}^{(n)}, \phi_{jn}) &\le \frac{1}{n} \sum_{k=1}^{n} S(\varphi^{(n)} \circ \gamma^k | \mathcal{A}_{[1,jn]}, \phi | \mathcal{A}_{[1,jn]}) \\
&\le \frac{1}{n} \sum_{k=1}^{n} S(\varphi^{(n)} \circ \gamma^k | \mathcal{A}_{[-k+1,(j+1)n-k]}, \phi \circ \gamma^k | \mathcal{A}_{[-k+1,(j+1)n-k]}) \\
&= S(\varphi^{(n)} | \mathcal{A}_{[1,(j+1)n]}, \phi | \mathcal{A}_{[1,(j+1)n]}) \\
&= \varphi^{(n)}\left(\log \frac{d(\varphi^{(n)} | \mathcal{A}_{[1,(j+1)n]})}{d(\phi | \mathcal{A}_{[1,(j+1)n]})}\right) \\
&= \sum_{m=0}^{j} \varphi^{(n)} \circ \gamma^{mn}(-H_n) - (j+1) \log \phi(e^{-H_n}) \\
&= -(j+1)\varphi^{(n)}(H_n) - (j+1) \log \phi(e^{-H_n}).
\end{aligned}$$

Since

$$\left|\frac{\varphi^{(n)}(H_n)}{n} - \omega^{(n)}(A_\Phi)\right| = \left|\frac{\varphi^{(n)}(H_n)}{n} - \frac{1}{n}\sum_{k=1}^{n} \varphi^{(n)}(\gamma^k(A_\Phi))\right|$$
$$\leq \left\|\frac{H_n}{n} - \frac{1}{n}\sum_{k=1}^{n}\gamma^k(A_\Phi)\right\| \to 0 \quad (n\to\infty)$$

as in the proof of Lemma 3.4, we get given $\varepsilon > 0$

$$\begin{aligned} S_{\mathrm{M}}(\omega^{(n)},\phi) &= \lim_{j\to\infty}\frac{1}{jn}S(\omega^{(n)}_{jn},\phi_{jn}) \\ &\leq -\frac{\varphi^{(n)}(H_n)}{n} - \frac{1}{n}\log\phi(e^{-H_n}) \\ &\leq -\omega^{(n)}(A_\Phi) - \frac{1}{n}\log\phi(e^{-H_n}) + \varepsilon \end{aligned}$$

for every n large enough. Therefore

$$\limsup_{n\to\infty}\frac{1}{n}\log\phi(e^{-H_n}) \leq \sup_{\omega\in\mathcal{S}_\gamma(\mathcal{A})}\{-S_{\mathrm{M}}(\omega,\phi) - \omega(A_\Phi)\}. \tag{3.6}$$

The result follows from (3.5) and (3.6).

Definition 3.6. Define the *thermodynamic free energy* or the *pressure* $p(\Phi,\phi)$ of $\Phi\in\mathcal{B}$ with respect to ϕ by

$$p(\Phi,\phi) = \lim_{n\to\infty}\frac{1}{n}\log\phi(e^{-H_n}).$$

The above theorem asserts the variational equality:

$$p(\Phi,\phi) = \sup_{\omega\in\mathcal{S}_\gamma(\mathcal{A})}\{-S_{\mathrm{M}}(\omega,\phi) - \omega(A_\Phi)\}. \tag{3.7}$$

Since $\omega\mapsto -S_{\mathrm{M}}(\omega,\phi) - \omega(A_\Phi)$ is affine and weakly* upper semicontinuous by Proposition 3.2, it follows that

$$S(\Phi,\phi) = \{\omega\in\mathcal{S}_\gamma(\mathcal{A}) : p(\Phi,\phi) = -S_{\mathrm{M}}(\omega,\phi) - \omega(A_\Phi)\}$$

is nonempty and forms a face of $\mathcal{S}_\gamma(\mathcal{A})$, so that $S(\Phi,\phi)$ is a simplex. When $\omega\in S(\Phi,\phi)$, we say that ω satisfies the *variational principle* (or it is *thermodynamically stable*) with respect to Φ and ϕ.

From the variational equality (3.7) the following can be easily shown as in [9, 6.2.40].

Proposition 3.7 $p(\Phi,\phi)$ is convex in $\Phi\in\mathcal{B}$ and

$$|p(\Phi,\phi) - p(\Psi,\phi)| \leq |||\Phi-\Psi|||, \qquad \Phi,\Psi\in\mathcal{B}.$$

The next proposition says that when a trace τ satisfies (VI) as well, the pressure $p(\Phi,\tau)$ is identical (up to an additive constant) to the usual one defined by using Tr_n instead of τ_n. So, our variational principle reduces to the usual one when $\mathcal{A}$ is a spin C^*-algebra with $\phi=\tau$ the unique tracial state.

Proposition 3.8 Assume that τ is a tracial state on $\mathcal{A}$ satisfying (IV)–(VI). Then for every $\Phi\in\mathcal{B}$, the limit $P(\Phi)=\lim_{n\to\infty}\frac{1}{n}\log\mathrm{Tr}_n(e^{-H_n})$ exists and $P(\Phi)=p(\Phi,\tau)+\log\lambda^{-1}$. Furthermore

$$P(\Phi)=\sup_{\omega\in\mathcal{S}_\gamma(\mathcal{A})}\{s(\omega)-\omega(A_\Phi)\}.$$

Proof. Condition (VI) means that we have with $\varepsilon_n\downarrow 0$ $(n\to\infty)$

$$\exp\{n(\log\lambda-\varepsilon_n)\}1\le\frac{d\tau_n}{d\mathrm{Tr}_n}\le\exp\{n(\log\lambda+\varepsilon_n)\}1.$$

Since $\tau(e^{-H_n})=\mathrm{Tr}_n((d\tau_n/d\mathrm{Tr}_n)e^{-H_n})$, we get

$$\exp\{n(\log\lambda-\varepsilon_n)\}\mathrm{Tr}_n(e^{-H_n})\le\tau(e^{-H_n})\le\exp\{n(\log\lambda+\varepsilon_n)\}\mathrm{Tr}_n(e^{-H_n}),$$

so that

$$\left|\frac{1}{n}\log\mathrm{Tr}_n(e^{-H_n})-\frac{1}{n}\log\tau(e^{-H_n})+\log\lambda\right|\le\varepsilon_n.$$

This implies that $P(\Phi)$ exists and $P(\Phi)=p(\Phi,\tau)+\log\lambda^{-1}$. The variational equality for $P(\Phi)$ follows from (3.7) and (3.3).

The next proposition asserts that when $\Phi\in\mathcal{B}$ satisfies $||W_{[1,n]}||=\mathrm{o}(n)$ (this is the case if $\sum_{X\ni 0}||\Phi(X)||<+\infty$), a Gibbs state in the weak sense satisfies the variational principle.

Proposition 3.9 Let $\Phi\in\mathcal{B}$ be such that the surface energy $W_n=W_{[1,n]}$ is defined for every $n\ge 1$ and $\frac{1}{n}||W_n||\to 0$. If $\omega\in\mathcal{S}_\gamma(\mathcal{A})$ satisfies the Gibbs condition in the weak sense with respect to Φ and ϕ, then it satisfies the variational principle with respect to Φ and ϕ.

Proof. Since by (3.4)

$$S(\omega_n,\phi_n^c)=S(\omega_n,\phi_n)+\omega(H_n)+\log\phi(e^{-H_n}),$$

we have

$$\lim_{n\to\infty}\frac{1}{n}S(\omega_n,\phi_n^c)=S_{\mathrm{M}}(\omega,\phi)+\omega(A_\Phi)+p(\Phi,\phi).$$

Hence it suffices to show that $\lim_{n\to\infty}\frac{1}{n}S(\omega_n,\phi_n^c)=0$. But we get

$$S(\omega_n,\phi_n^c)\le S(\omega,[\omega^{-W_n}])\le 2||W_n||$$

due to the monotonicity of relative entropy and (2.2). Hence the conclusion follows from the assumption of Φ.

Now let $\mathcal{B}_0$ denote the set of all interactions $\Phi \in \mathcal{B}$ such that

$$\sum_{X \ni 0} ||\Phi(X)|| < +\infty \quad \text{and} \quad \sup_{n \geq 1} ||W_{[1,n]}|| < +\infty. \tag{3.8}$$

Then $\Phi \in \mathcal{B}_0$ generates a strongly continuous one-parameter automorphism group α^Φ satisfying $\alpha_t^\Phi \circ \gamma = \gamma \circ \alpha_t^\Phi$. The next theorem means that $S(\Phi, \phi) \subset K(\Phi)$ for $\Phi \in \mathcal{B}_0$.

Theorem 3.10 Let $\Phi \in \mathcal{B}_0$ and $\omega \in \mathcal{S}_\gamma(\mathcal{A})$. If ω satisfies the variational principle with respect to Φ and ϕ, then it satisfies the KMS condition with respect to α^Φ.

The details of this proof may be omitted because it is a slight modification of that in [46, 32] (also [9, 6.2.42]). It can be performed by use of convex analysis on the real Banach space $\mathcal{B}_0$ with the norm

$$||\Phi||_0 = \sum_{X \ni 0} ||\Phi(X)|| + \sup_{n \geq 1} ||W_{[1,n]}|| \ (\geq |||\Phi|||).$$

At the end we use the following approximation property: If $||\Phi_n - \Phi||_0 \to 0$ in $\mathcal{B}_0$, then $\alpha_t^{\Phi_n}(a) \to \alpha_t^\Phi(a)$ strongly as $n \to \infty$ for all $a \in \mathcal{A}$ and $t \in \mathbf{R}$, which can be seen because $(1 \pm \delta_n)^{-1} \to (1 \pm \delta)^{-1}$ strongy where δ_n and δ are the generators of α^{Φ_n} and α^Φ ([27], [48, 4.1.2]).

The following is a consequence of Theorem 2.2, Propositions 2.6, 3.9, and Theorem 3.10 altogether, which is a complete analogue of the one-dimensional quantum spin case.

Corollary 3.11 Assume that ϕ is a unique tracial state on $\mathcal{A}$. If $\Phi \in \mathcal{B}_0$, then there exists a unique $\omega \in \mathcal{S}_\gamma(\mathcal{A})$ which satisfies one (hence all) of the following equivalent conditions:

(i) the KMS condition with respect to α^Φ;
(ii) the Gibbs condition in the strong sense with respect to Φ and ϕ;
(iii) the Gibbs condition in the weak sense with respect to Φ and ϕ;
(iv) the variational principle with respect to Φ and ϕ.

In view of Problems 2.5 we are interested in the following:

Problems 3.12. Let ϕ, ϕ' be tracial states on $\mathcal{A}$ satisfying (IV) and (V).

(1) If $\omega \in S(\Phi, \phi)$, then does ω satisfy the Gibbs condition in the strong or weak sense with respect to Φ and ϕ?
(2) Is $S(\Phi, \phi)$ a singleton?
(3) Are $S(\Phi, \phi)$ and $S(\Phi, \phi')$ disjoint if $\phi \neq \phi'$?

As a weak result in this direction we give:

Proposition 3.13 Let ϕ, ϕ' be different tracial states on $\mathcal{A}$ satisfying (IV) and (V). Then $S(\Phi, \phi)$ and $S(\Phi, \phi')$ are disjoint for $\Phi \in \mathcal{B}$ with sufficiently small $|||\Phi|||$.

Proof. Choose $m \in \mathbf{N}$ with $\phi_m \neq \phi'_m$. Suppose that $\omega \in S(\Phi, \phi) \cap S(\Phi, \phi')$. Since for $0 \leq t \leq 1$

$$\begin{aligned} -S_{\mathrm{M}}(\omega, \phi) - \omega(A_\Phi) &\geq -S_{\mathrm{M}}((1-t)\omega + t\phi, \phi) - ((1-t)\omega + t\phi)(A_\Phi) \\ &= -(1-t)S_{\mathrm{M}}(\omega, \phi) - ((1-t)\omega + t\phi)(A_\Phi) \end{aligned}$$

by Proposition 3.2, we get

$$S_{\mathrm{M}}(\omega,\phi) \le (\phi-\omega)(A_\Phi) \le 2||A_\Phi|| \le 2|||\Phi|||.$$

An inequality for the relative entropy (see [37, 1.15]) gives

$$||\omega_m - \phi_m|| \le \{2S(\omega_m,\phi_m)\}^{1/2} \le 2(m|||\Phi|||)^{1/2}$$

thanks to (3.1). Since the same inequality holds with ϕ' instead of ϕ, we have

$$||\phi_m - \phi'_m|| \le ||\omega_m - \phi_m|| + ||\omega_m - \phi'_m|| \le 4(m|||\Phi|||)^{1/2},$$

so that $|||\Phi||| \ge \frac{1}{m}(||\phi_m - \phi'_m||/4)^2$. This shows the conclusion.

4. Gauge actions and chemical potentials

In this section we consider a quantum system obtained as the fixed point subalgebra of a quantum spin C^*-algebra by a gauge action of a compact group. This kind of C^*-systems are typical examples of field systems discussed in the chemical potential theory [7, 6]. Such systems arise also from Wassermann's subfactors [53] in the subfactor theory.

Let $\mathcal{F}$ denote a one-dimensional quantum spin C^*-algebra $\bigotimes_{k\in\mathbf{Z}}\mathcal{F}_k$, $\mathcal{F}_k$ being copies of $M_d(\mathbf{C})$, and γ the automorphism of right shift. Let G be a compact group and σ a (possibly reducible) unitary representation of G on $V=\mathbf{C}^d$. So G acts on $\mathrm{End}\,V = M_d(\mathbf{C})$ as $\mathrm{Ad}\,\sigma$. We set a product action β of G on $\mathcal{F}$ by $\beta_g = \bigotimes_{\mathbf{Z}}\mathrm{Ad}\,\sigma_g$, $g\in G$. Let $\mathcal{A}$ be the fixed point subalgebra $\mathcal{F}^\beta$ of $\mathcal{F}$. Then $\mathcal{A}$ is called the *observable algebra*, while $\mathcal{F}$ is called the *field algebra*. The restriction of the shift γ on $\mathcal{A}$ is denoted by the same γ. Moreover for $i,j\in\mathbf{Z}$, $i\le j$, set $\mathcal{F}_{[i,j]} = \bigotimes_{k=i}^{j}\mathcal{F}_k$ and $\mathcal{A}_{[i,j]} = \mathcal{A}\cap\mathcal{F}_{[i,j]} = \mathcal{F}^\beta_{[i,j]}$, the fixed point algebra of $\beta|\mathcal{F}_{[i,j]}$. Then $\mathcal{A}$ is the AF C^*-algebra generated by $\{\mathcal{A}_{[i,j]}\}$ by [45, Proposition 2.1]. We call $(\mathcal{A},\{\mathcal{A}_{[i,j]}\},\gamma)$ the *C^*-system of gauge invariance* for (G,σ).

Concerning the tracial state space $\mathcal{T}(\mathcal{A})$ we have:

Proposition 4.1 In the above setting, $\mathcal{T}(\mathcal{A}) = \mathcal{T}_\gamma(\mathcal{A})$, and $\phi\in\mathcal{T}(\mathcal{A})$ is extremal if and only if ϕ satisfies (V) in §1.

Proof. Let $\phi\in\mathcal{T}(\mathcal{A})$ be extremal. Then by [45, Theorem 3.2] ϕ is the restriction of a product state $\bigotimes_{\mathbf{Z}}\psi_0$ on $\mathcal{A}$ where ψ_0 is a state on $M_d(\mathbf{C})$. This implies that ϕ is γ-invariant and multiplicative in the sense of (V). Hence $\mathcal{T}(\mathcal{A}) = \mathcal{T}_\gamma(\mathcal{A})$. Now let $\mathcal{ET}(\mathcal{A})$ denote the set of all extremal elements of $\mathcal{T}(\mathcal{A})$. Each $\phi\in\mathcal{T}(\mathcal{A})$ has the integral decomposition $\phi = \int_{\mathcal{ET}(\mathcal{A})}\rho\, d\nu(\rho)$ where ν is a probability measure on $\mathcal{ET}(\mathcal{A})$. Assume that ϕ satisfies (V). If $a\in\bigcup_{n=1}^\infty\mathcal{A}_{[-n,n]}$ is selfadjoint, then we get for j large enough

$$\begin{aligned}\left\{\int_{\mathcal{ET}(\mathcal{A})}\rho(a)d\nu(\rho)\right\}^2 &= \phi(a)^2 = \phi(a\gamma^j(a)) \\ &= \int_{\mathcal{ET}(\mathcal{A})}\rho(a\gamma^j(a))d\nu(\rho) \\ &= \int_{\mathcal{ET}(\mathcal{A})}\rho(a)^2 d\nu(\rho),\end{aligned}$$

which implies that $\rho(a)$ is constant for ν-a.e. $\rho \in \mathcal{ET}(\mathcal{A})$. Thus ν is supported on a single point, that is, ϕ is extremal.

Associated with (G,σ) above, let us introduce the set $\Xi(G,\sigma)$ as follows: $\Xi(G,\sigma)$ is the set of all continuous one-parameter subgroups $t \mapsto \xi_t$ of G, where we identify $\xi, \xi' \in \Xi(G,\sigma)$ if $\mathrm{Ad}\,\sigma_{\xi'_t} = \mathrm{Ad}\,\sigma_{g^{-1}\xi_t g}$, $t \in \mathbf{R}$, for some $g \in G$. Then $\Xi(G,\sigma)$ can be regarded as the chemical potentials in the setting of this section. Also, the next proposition says that $\Xi(G,\sigma)$ gives a parametrization of the faithful extremal (γ-invariant) tracial states on $\mathcal{A}$.

We use the notation $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ to mean the set of all faithful and extremal $\phi \in \mathcal{T}(\mathcal{A})$. By Proposition 4.1, $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ coincides with the set of all $\phi \in \mathcal{T}(\mathcal{A})$ satisfying (IV) and (V) in §1.

Proposition 4.2 The bijective correspondence $\phi \leftrightarrow \xi$ between $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ and $\Xi(G,\sigma)$ is determined in the way that ϕ extends to a KMS state with respect to β_{ξ_t}.

Proof. For each $\xi \in \Xi(G,\sigma)$, since $t \mapsto \sigma_{\xi_t}$ is a continuous one-parameter unitary group, there exists a unique selfadjoint $h \in M_d(\mathbf{C})$ such that $\tau_0(e^{-h}) = 1$ and $\mathrm{Ad}\,\sigma_{\xi_t} = \mathrm{Ad}\,e^{ith}$, $t \in \mathbf{R}$, where τ_0 is the tracial state of $M_d(\mathbf{C})$. Noting that the product state $\bigotimes_{\mathbf{Z}} \tau_0(e^{-h}\,\cdot\,)$ is a unique KMS state with respect to $\beta_{\xi_t} = \bigotimes_{\mathbf{Z}} \mathrm{Ad}\,\sigma_{\xi_t}$, we define ϕ as the restriction of $\bigotimes_{\mathbf{Z}} \tau_0(e^{-h}\,\cdot\,)$ on $\mathcal{A}$. For any $a \in \mathcal{A}_{[-n,n]}$, since

$$\begin{aligned}\exp\Bigl(it \sum_{j=-n}^{n} \gamma^j(h)\Bigr) a \exp\Bigl(-it \sum_{j=-n}^{n} \gamma^j(h)\Bigr) \\ = \Bigl(\bigotimes_{-n}^{n} e^{ith}\Bigr) a \Bigl(\bigotimes_{-n}^{n} e^{-ith}\Bigr) = \beta_{\xi_t}(a) = a, \qquad t \in \mathbf{R},\end{aligned}$$

we get

$$\Bigl[a, \sum_{j=-n}^{n} \gamma^j(h)\Bigr] = 0. \tag{4.1}$$

Hence for any $a, b \in \mathcal{A}_{[-n,n]}$

$$\phi(ab) = \Bigl(\bigotimes_{-n}^{n} \tau_0\Bigr)\Bigl(\exp\Bigl(-\sum_{j=-n}^{n} \gamma^j(h)\Bigr) ab\Bigr) = \phi(ba),$$

which imples that ϕ is tracial on $\mathcal{A}$. Moreover ϕ is faithful and obviously satisfies (V). So $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ by Proposition 4.1.

Conversely let $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ be given. By [6] (also the final remark of [9, §5.4.3]), ϕ has an extremal γ-invariant extension ψ to $\mathcal{F}$ and ψ is a KMS state with respect to $\beta_{\xi_t} = \bigotimes_{\mathbf{Z}} \mathrm{Ad}\,\sigma_{\xi_t}$ for some $\xi \in \Xi(G,\sigma)$. Furthermore, if ψ' is another extremal γ-invariant extension of ϕ, then $\psi' = \psi \circ \beta_g$ for some $g \in G$ by [6, Theorem II.1] (also [9, 5.4.24]) and the corresponding $\xi' \in \Xi(G,\sigma)$ satisfies $\beta_{\xi'_t} = \beta_{g^{-1}}\beta_{\xi_t}\beta_g$ or $\mathrm{Ad}\,\sigma_{\xi'_t} = \mathrm{Ad}\,\sigma_{g^{-1}\xi_t g}$. Thus we obtain the conclusion.

In particular, assume that G is a compact connected Lie group with maximal torus T of dimension N. Let $\mathcal{F}^T$ denote the fixed point algebra of the restriction of $\beta = \bigotimes_{\mathbf{Z}} \mathrm{Ad}\,\sigma$ to T, so that $\mathcal{A} \subset \mathcal{F}^T \subset \mathcal{F}$. It is known [18] that every extremal tracial state on $\mathcal{A}$ extends to an extremal tracial state on $\mathcal{F}^T$. Moreover, every element of $\mathcal{ET}^{\mathrm{f}}(\mathcal{F}^T)$ is of the form $\bigotimes_{\mathbf{Z}} \mathrm{Tr}(D(r)\,\cdot\,)$ where the density matrix $D(r)$ is explicitly written with a vector $r \in (\mathbf{R}^N)_{++}$. Thus the set $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ in this case is parametrized by $(\mathbf{R}^N)_{++}$, more precisely, by a fundamental domain for the action of the Weyl group of G on $(\mathbf{R}^N)_{++}$ (see [18] for details).

From now on an interaction enters into our discussions. Let $\Phi \in \mathcal{B}_0$, that is, Φ is a γ-invariant interaction in the observable algebra $\mathcal{A}$ satisfying (3.8), which is also considered as an interaction in the field algebra $\mathcal{F}$. So Φ generates a one-parameter automorphism group α^Φ on $\mathcal{F}$, whose restriction on $\mathcal{A}$ is the dynamics generated on $\mathcal{A}$. Let $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ be fixed, which corresponds to $\xi \in \Xi(G,\sigma)$ by Proposition 4.2. Then, as in the proof of Proposition 4.2, we have a unique selfadjoint element h of $\mathcal{F}_0 = M_d(\mathbf{C})$ such that $\tau_0(e^{-h}) = 1$ and $\mathrm{Ad}\,\sigma_{\xi_t} = \mathrm{Ad}\,e^{ith}$, $t \in \mathbf{R}$.

Associated with Φ and ϕ, an interaction Φ^h in the field algebra $\mathcal{F}$ is defined as follows:

$$\Phi^h(X) = \begin{cases} \Phi(\{j\}) + \gamma^j(h) & \text{if } X = \{j\},\ j \in \mathbf{Z}, \\ \Phi(X) & \text{otherwise.} \end{cases} \tag{4.2}$$

Then Φ^h is γ-invariant and generates a one-parameter automorphism group α^{Φ^h} on $\mathcal{F}$. So there exists a unique α^{Φ^h}-KMS state on $\mathcal{F}$, which is automatically extremal in $\mathcal{S}_\gamma(\mathcal{F})$ and faithful (see [4, 28]).

Lemma 4.3 With the above notations,

$$\alpha_t^\Phi \beta_g = \beta_g \alpha_t^\Phi, \qquad t \in \mathbf{R},\ g \in G, \tag{4.3}$$

$$\alpha_t^{\Phi^h} = \alpha_t^\Phi \beta_{\xi_t}, \qquad t \in \mathbf{R}. \tag{4.4}$$

Hence $\alpha^\Phi = \alpha^{\Phi^h}|\mathcal{A}$.

Proof. Since $\beta_g(\Phi(X)) = \Phi(X)$ and hence $\beta_g(H_{[-n,n]}) = H_{[-n,n]}$, we get

$$e^{itH_{[-n,n]}}\beta_g(a)e^{-itH_{[-n,n]}} = \beta_g(e^{itH_{[-n,n]}}ae^{-itH_{[-n,n]}})$$

for all $a \in \mathcal{A}$, $t \in \mathbf{R}$, and $g \in G$. Letting $n \to \infty$ gives (4.3). The local Hamiltonian $H_{[-n,n]}(\Phi^h)$ of Φ^h inside $[-n,n]$ is given by

$$H_{[-n,n]}(\Phi^h) = H_{[-n,n]} + \sum_{j=-n}^{n} \gamma^j(h).$$

Since $[H_{[-n,n]}, \sum_{j=-n}^{n} \gamma^j(h)] = 0$ by (4.1), we have

$$\exp(itH_{[-n,n]}(\Phi^h)) = e^{itH_{[-n,n]}} \exp\biggl(it \sum_{j=-n}^{n} \gamma^j(h)\biggr).$$

Hence (4.4) follows.

The above (4.3) shows that $(\mathcal{F}, \mathcal{A}, G, \alpha^\Phi, \beta, \gamma)$ is a field system in the chemical potential theory.

For an α^Φ-KMS state ω on $\mathcal{A}$ and $\xi \in \Xi(G, \sigma)$, we say that ξ is the *chemical potential* of ω if there exists an extension φ of ω to $\mathcal{F}$ which satisfies the $\alpha_t^\Phi \beta_{\xi_t}$-KMS condition. By (4.4) and the fact mentioned before Lemma 4.3, we see that there exists a unique α^Φ-KMS state with the chemical potential ξ, which is automatically γ-invariant and faithful.

Theorem 4.4 Let $\Phi \in \mathcal{B}_0$, $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$, and $\xi \in \Xi(G, \sigma)$ with $\phi \leftrightarrow \xi$ in the sense of Proposition 4.2. Consider the following conditions for $\omega \in \mathcal{S}_\gamma(\mathcal{A})$:

(i) ω is an α^Φ-KMS state with the chemical potential ξ;

(ii) ω satisfies the Gibbs condition in the strong sense with respect to Φ and ϕ;

(iii) ω satisfies the Gibbs condition in the weak sense with respect to Φ and ϕ;

(iv) $\omega \in S(\Phi, \phi)$ or ω satisfies the variational principle with respect to Φ and ϕ.

Then:

(1) (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) for every $\omega \in \mathcal{S}_\gamma(\mathcal{A})$.

(2) Assume that $\omega \in \mathcal{S}_\gamma(\mathcal{A})$ is extremal and faithful. Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

(3) Assume that ξ is in the center of G. Then all (i)–(iv) are equivalent for $\omega \in \mathcal{S}_\gamma(\mathcal{A})$. Moreover there exists a unique $\omega \in \mathcal{S}_\gamma(\mathcal{A})$ satisfying (i)–(iv), which is extremal and faithful.

Proof. (1) (ii) $\Rightarrow$ (iii) is trivial and (iii) $\Rightarrow$ (iv) is due to Proposition 3.9. To show (i) $\Rightarrow$ (ii), assume that ω extends to an $\alpha_t^\Phi \beta_{\xi_t}$-KMS state φ on $\mathcal{F}$. By (4.4) and [9, 6.2.42 and its remark], φ satisfies the Gibbs condition with respect to Φ^h. In the following let us work in the von Neumann algebra $\mathcal{M} = \pi_\varphi(\mathcal{F})''$ via the GNS representation π_φ. Since $\alpha_t^\Phi \beta_{\xi_t}(\mathcal{A}) = \alpha_t^\Phi(\mathcal{A}) = \mathcal{A}$ and the normal extension of $\alpha_t^\Phi \beta_{\xi_t}$ to $\mathcal{M}$ is the modular automorphism group for the normal extension $\tilde{\varphi}$ of φ, the conditional expectation $E : \mathcal{M} \to \pi_\varphi(\mathcal{A})''$ with respect to $\tilde{\varphi}$ exists. Set $\tilde{\omega} = \tilde{\varphi}|\pi_\varphi(\mathcal{A})''$ so that $\omega(a) = \tilde{\omega}(\pi_\varphi(a))$, $a \in \mathcal{A}$. For any finite interval $\Lambda \subset \mathbf{Z}$, since $\pi_\varphi(-W_\Lambda) \in \pi_\varphi(\mathcal{A})''$, we have by Lemma 2.3

$$(4.5) \qquad [\varphi^{-W_\Lambda}]^{\sim} = [\tilde{\varphi}^{\pi_\varphi(-W_\Lambda)}] = [\tilde{\omega}^{\pi_\varphi(-W_\Lambda)}] \circ E.$$

But it is seen that $[\tilde{\omega}^{\pi_\varphi(-W_\Lambda)}]$ is the normal extension of $[\omega^{-W_\Lambda}]$ via π_φ, i.e.

$$(4.6) \qquad [\omega^{-W_\Lambda}](a) = [\tilde{\omega}^{\pi_\varphi(-W_\Lambda)}](\pi_\varphi(a)), \qquad a \in \mathcal{A},$$

because (2.3) holds with $\pi_\varphi(\mathcal{A})''$ instead of $\pi_\omega(\mathcal{A})''$. By (4.5) and (4.6) we have $[\omega^{-W_\Lambda}] = [\varphi^{-W_\Lambda}]|\mathcal{A}$. Hence, if $a \in \mathcal{A}_\Lambda$ and $b \in \mathcal{A}_{\Lambda^c}$, then

$$[\omega^{-W_\Lambda}](ab) = [\varphi^{-W_\Lambda}](ab) = \frac{\tau(e^{-H_\Lambda(\Phi^h)}a)}{\tau(e^{-H_\Lambda(\Phi^h)})}[\omega^{-W_\Lambda}](b),$$

where τ is the tracial state of $\mathcal{F}$ and $H_\Lambda(\Phi^h)$ is the local Hamiltonian of Φ^h inside Λ. Since

$$H_\Lambda(\Phi^h) = H_\Lambda + \sum_{j \in \Lambda} \gamma^j(h)$$

and $[H_\Lambda, \sum_{j\in\Lambda}\gamma^j(h)] = 0$ as (4.1), we get

$$\tau(e^{-H_\Lambda(\Phi^h)}a) = \tau\Big(\exp\Big(-\sum_{j\in\Lambda}\gamma^j(h)\Big)e^{-H_\Lambda}a\Big) = \phi(e^{-H_\Lambda}a),$$

so that

$$[\omega^{-W_\Lambda}](ab) = \phi_\Lambda^c(a)[\omega^{-W_\Lambda}](b).$$

This together with (i) $\Rightarrow$ (ii) of Theorem 2.2 implies (ii).

(2) Let $\omega \in \mathcal{S}_\gamma(\mathcal{A})$ be extremal and faithful, and assume that ω satisfies (iii) and hence (iv). Then it follows from Theorem 3.10 and [6, Theorem II.4] that ω is an α^Φ-KMS state with some chemical potential ξ'. Hence by (1) above, ω satisfies (iii) for $\phi' \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ as well as ϕ where $\phi' \leftrightarrow \xi'$. This gives $\phi' = \phi$ as was mentioned after Definition 2.1, so that (i) follows.

(3) Let $\psi = \bigotimes_{\mathbf{Z}}\tau_0(e^{-h}\,\cdot\,)$ so that $\phi = \psi|\mathcal{A}$. The assumption of ξ being in the center of G implies that $\mathrm{Ad}\,\sigma_g(\sigma_{\xi_t}) = \sigma_{\xi_t}$, $g \in G$. Hence for any $\Lambda \subset \mathbf{Z}$, we have $\bigotimes_\Lambda e^{ith} \in \mathcal{A}_\Lambda$ and so $\bigotimes_\Lambda e^{-h} \in \mathcal{A}_\Lambda$. Define $E_\mathcal{A} : \mathcal{F} \to \mathcal{A}$ by $E_\mathcal{A}(a) = \int_G \beta_g(a)dg$ (dg is the normalized Haar measure on G), which is a τ-preserving conditional expectation and satisfies $E_\mathcal{A}(\mathcal{F}_\Lambda) = \mathcal{A}_\Lambda$ for all $\Lambda \subset \mathbf{Z}$. Define $\varphi = \omega \circ E_\mathcal{A}$ and put $\omega_n = \omega|\mathcal{A}_{[1,n]}$, $\varphi_n = \varphi|\mathcal{F}_{[1,n]}$, $E_n = E_\mathcal{A}|\mathcal{F}_{[1,n]}$, etc. Since $\omega_n \circ E_n = \varphi_n$ and

$$\begin{aligned}\phi_n(E_n(a)) &= \tau\Big(\Big(\bigotimes_1^n e^{-h}\Big)E_\mathcal{A}(a)\Big)\\ &= \tau\Big(\Big(\bigotimes_1^n e^{-h}\Big)a\Big) = \psi_n(a), \qquad a \in \mathcal{F}_{[1,n]},\end{aligned}$$

we get by [37, 5.15]

$$\begin{aligned}S(\omega_n, \phi_n) &= S(\omega_n \circ E_n, \phi_n \circ E_n) = S(\varphi_n, \psi_n)\\ &= \mathrm{Tr}_n\Big(\frac{d\varphi_n}{d\mathrm{Tr}_n}\Big(\log\frac{d\varphi_n}{d\mathrm{Tr}_n} - \log\frac{d\psi_n}{d\mathrm{Tr}_n}\Big)\Big)\\ &= -S(\varphi_n) - \varphi_n\Big(-\sum_{j=1}^n\gamma^j(h) - \log d^n\Big)\\ &= -S(\varphi_n) + n\varphi(h) + n\log d.\end{aligned}$$

Therefore

$$S_\mathrm{M}(\omega,\phi) = -s(\varphi) + \varphi(h) + \log d. \tag{4.7}$$

On the other hand, we get

$$\begin{aligned}p(\Phi,\phi) &= \lim_{n\to\infty}\frac{1}{n}\log\phi(e^{-H_{[1,n]}})\\ &= \lim_{n\to\infty}\frac{1}{n}\log\tau(\exp(-H_{[1,n]}(\Phi^h)))\\ &= P(\Phi^h) - \log d,\end{aligned} \tag{4.8}$$

where $P(\Phi^h)$ is the pressure of Φ^h in the field algebra $\mathcal{F}$. Furthermore

$$A_{\Phi^h} = A_\Phi + h. \tag{4.9}$$

By (4.7)–(4.9) altogether we see that $\omega \in S(\Phi, \phi)$ if and only if φ satisfies the variational principle with respect to Φ^h; equivalently φ is an $\alpha_t^\Phi \beta_{\xi_t}$-KMS state by (4.4). Thus (iv) $\Rightarrow$ (i) is shown. For the last statement, the unique existence of ω satisfying (i) was already shown before Theorem 4.4. The extremality of ω is due to (iv) because $S(\Phi, \phi)$ is a face of $\mathcal{S}_\gamma(\mathcal{A})$.

It is desirable to show the equivalence of (i)–(iv) of the theorem for a general $\xi \in \Xi(G, \sigma)$. In the particular case when G is abelian, the following is a consequence of Theorem 4.4(3); so Problems 2.5 and 3.12 are to some extent solved.

Corollary 4.5 Assume that G is abelian. Then for each $\phi \in \mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ (or $\xi \in \Xi(G, \sigma)$) there exists a unique $\omega \in \mathcal{S}_\gamma(\mathcal{A})$ satisfying the equivalent conditions (i)–(iv) of Theorem 4.4. Moreover, the correspondence $\phi \mapsto \omega$ is bijective from $\mathcal{ET}^{\mathrm{f}}(\mathcal{A})$ onto the set of all faithful extremal γ-invariant α^Φ-KMS states on $\mathcal{A}$.

In the rest of this section we briefly explain that C^*-systems treated in this section are derived from a certain model of Example 1.4. Let $\sigma : G \to \mathrm{End}\, V$ be a finite-dimensional unitary representation of a compact group G and set $\sigma_k = \sigma$ $(k \in 2\mathbf{Z})$, $\sigma_k = \bar\sigma$ $(k \in 2\mathbf{Z}+1)$ where $\bar\sigma$ is the representation conjugate to σ. For $j \in \mathbf{Z}$ let R_j be the type II_1 factor generated by $\bigotimes_{-\infty}^j \mathrm{End}\, V$ via the GNS representation for the tracial state, and M_j be the fixed point subalgebra R_j^β under the product action $\beta_j = \bigotimes_{k=-\infty}^j \mathrm{Ad}\,\sigma_k$. Since $(R_j^\beta)' \cap R_j = \mathbf{C}$ as is well known, M_j is a type II_1 factor. Then so-called Wassermann's subfactor [53] arising from σ is defined as $M_{-1} \subset M_0$, i.e. $R^\beta \subset (R \otimes \mathrm{End}\, V)^{\beta \otimes \mathrm{Ad}\,\sigma}$ with $R = R_{-1}$, which is an extremal inclusion of type II_1 factors with the index $[M_0 : M_{-1}] = (\dim V)^2$. In the construction above we may take as β any minimal action on R. (In fact, minimal actions on R are unique [43].)

The Jones tower of $M_{-1} \subset M_0$ is $\{M_j\}_{j=-\infty}^\infty$ defined above and the relative commutants $\mathcal{A}_{[i,j]} = M_{i-1}' \cap M_j$ are given by $\mathcal{A}_{[i,j]} = (\bigotimes_i^j \mathrm{End}\, V)^\beta$, the fixed point algebra of $\bigotimes_{k=i}^j \mathrm{Ad}\,\sigma_k$. In particular, the derived tower $\{\mathcal{A}_{[1,n]}\}_{n\ge 0}$ is

$$\begin{aligned}\mathbf{C} \subset (\mathrm{End}\, V)^{\mathrm{Ad}\,\bar\sigma} &\subset (\mathrm{End}\, V \otimes \mathrm{End}\, V)^{\mathrm{Ad}\,\bar\sigma \otimes \mathrm{Ad}\,\sigma} \\ &\subset (\mathrm{End}\, V \otimes \mathrm{End}\, V \otimes \mathrm{End}\, V)^{\mathrm{Ad}\,\bar\sigma \otimes \mathrm{Ad}\,\sigma \otimes \mathrm{Ad}\,\bar\sigma} \\ &\subset \cdots\end{aligned}$$

and the standard invariant of $M_{-1} \subset M_0$ is determined by the way of irreducible decompositions of $\bar\sigma \otimes \sigma \otimes \cdots \otimes \bar\sigma \otimes \sigma$ (like the Clebsch-Gordan rules). For example, the vertices of the principal graph corresponding to the direct summands of $\mathcal{A}_{[1,2n]}$ is given by

$$K_n = \{\rho \in \widehat{G} : \rho \prec \bar\sigma \otimes \sigma \otimes \cdots \otimes \bar\sigma \otimes \sigma \ (n \text{ factors of } \bar\sigma \otimes \sigma)\}$$

and the standard eigenvector $\vec{s} = (s_\rho)_{\rho \in K}$, $K = \bigcup_n K_n$, is given by $s_\rho = \dim \rho$. Recall [43, 56] that $M_{-1} \subset M_0$ is strongly amenable (see [42] on strong amenability).

The C^*-algebra generated by $\{\mathcal{A}_{[i,j]}\}$ is $\mathcal{A} = \mathcal{F}^{\beta}$ where $\mathcal{F} = \bigotimes_{\mathbf{Z}} \mathrm{End}\, V$. Note that the canonical shift (1.3) on $\mathcal{A}$ coincides with $\gamma^2|\mathcal{A}$ where γ is the right shift on $\mathcal{F}$. In this way, we observe that the C^*-system derived from Wassermann's subfactor $M_{-1} \subset M_0$ is nothing but the field system obtained by the gauge action $\bar{\sigma} \otimes \sigma : G \to \mathrm{End}\, V \otimes \mathrm{End}\, V$. In particular, when σ is self-conjugate, γ restricts on $\mathcal{A}$ and we arrive at the same setting as was stated at the beginning of this section.

Example 4.6. According to Popa's classification [40, 42] of hyperfinite type II_1 subfactors of index 4, there are three cases having infinite depth. All of these are realized as Wassermann's subfactors (see [17, 4.7.d]). Consider the following three unitary representations on $V = \mathbf{C}^2$, all of which are self-conjugate:

(1) $G = SU(2)$ and $\sigma(g) = g$, $g \in SU(2)$.

(2) $G = D_\infty = \left\{ \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}, \begin{bmatrix} 0 & -w^{-1} \\ w & 0 \end{bmatrix} : z, w \in \mathbf{T} \right\}$, the infinite dihedral group, and $\sigma(g) = g$, $g \in D_\infty$.

(3) $G = \mathbf{T}$ and $\sigma(z) = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$, $z \in \mathbf{T}$.

Then Wassermann's subfactors for representations (1)–(3) have the principal graphs A_∞, D_∞, and $A_{\infty,\infty}$, respectively. (See [19, 56] for principal graphs of Wassermann's subfactors from other irreducible representations of $SU(2)$). Wassermann's subfactor for (1) coincides with Jones' subfactor $R_{1/4} \subset R$. The derived C^*-algebras are the fixed point algebras $\mathcal{F}^{SU(2)} \subset \mathcal{F}^{D_\infty} \subset \mathcal{F}^{\mathbf{T}}$ of the CAR algebra $\mathcal{F} = \bigotimes_{\mathbf{Z}} M_2(\mathbf{C})$ under the respective product actions. ($\mathcal{F}^{\mathbf{T}}$ is sometimes called the GICAR algebra.) For $0 \le r \le 1$ let ψ_r denote the product state $\bigotimes_{\mathbf{Z}} \mathrm{Tr}\left(\begin{bmatrix} r & 0 \\ 0 & 1-r \end{bmatrix} \cdot \right)$ on $\mathcal{F}$. Then the set $\mathcal{ET}(\mathcal{A})$ of extremal tracial states on $\mathcal{A} = \mathcal{F}^G$ is given as $\mathcal{ET}(\mathcal{A}) = \{\psi_r|\mathcal{A} : 0 \le r \le 1\}$ for each G. For cases (1) and (2), since $\psi_r|\mathcal{A} = \psi_{1-r}|\mathcal{A}$, we write $\mathcal{ET}(\mathcal{A}) = \{\phi_\lambda : 0 \le \lambda \le 1/4\}$ with parameter $\lambda = r(1-r)$. Also, for case (3) we write $\mathcal{ET}(\mathcal{A}) = \{\phi_r : 0 \le r \le 1\}$. The trace vectors of $\mathcal{A}_{[1,n]}$ for these extremal traces can be recursively computed (we omit details). Note that ϕ_0 in cases (1), (2) and ϕ_0, ϕ_1 in case (3) are trivial traces whose GNS representations are one-dimensional. The chemical potential $\xi \in \Xi(G, \sigma)$ corresponding to ϕ_λ or ϕ_r where $0 < r < 1$ is given by $\xi_t = \begin{bmatrix} e^{it\mu} & 0 \\ 0 & e^{-it\mu} \end{bmatrix}$, $t \in \mathbf{R}$, where $\mu \in \mathbf{R}$ is determined by $r = e^{-\mu}/(e^{-\mu} + e^{\mu})$ or $\mu = \log \sqrt{r^{-1} - 1}$. In cases (1) and (2) this ξ_t is identified with ξ_{-t}; note $\xi_{-t} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \xi_t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $-\mu = \log \sqrt{(1-r)^{-1} - 1}$.

5. Quantum random walks on groups

There is another important model of subfactors obtained by finitely generated discrete groups of automorphisms ([8, 42, 49]), whose derived C^*-systems are imbedded in quantum spin C^*-algebras. Taking account of the description of derived towers in [8] and [42, 5.1.5], we introduce a C^*-system associated with a finitely generated discrete group.

Let G be a discrete group with a finite number of generators $g_1, \ldots, g_d$ in the sense that the semigroup generated by $g_1, \ldots, g_d$ is all of G. Here, $g_r = g_s$ for different r, s is allowed and the identity e of G is not necessarily contained in $\{g_r\}$. Let $\mathcal{F}_k = M_d(\mathbf{C})$, $k \in \mathbf{Z}$, and $\{e^k_{rs}\}^d_{r,s=1}$ be the matrix units of $\mathcal{F}_k$. For $i, j \in \mathbf{Z}$, $i \le j$, define $\mathcal{A}_{[i,j]} \subset \bigotimes_{k=i}^j \mathcal{F}_k$ by

$$\mathcal{A}_{[i,j]} = \mathrm{Alg}\left\{ \bigotimes_{k=i}^{j} e^k_{r_k s_k} : g_{r_j} \cdots g_{r_i} = g_{s_j} \cdots g_{s_i} \right\},$$

whose direct summands are indexed by $g \in K_{j-i+1}$ where

$$K_n = \{g \in G : g = g_{r_n} \cdots g_{r_1} \text{ for some } 1 \le r_1, \dots, r_n \le d\}, \qquad n \in \mathbf{N}. \tag{5.1}$$

So $\mathcal{A}_{[1,n]} \cong \bigoplus_{g \in K_n} M_{d_{g,n}}(\mathbf{C})$ where

$$d_{g,n} = \#\{(r_1, \dots, r_n) : 1 \le r_1, \dots, r_n \le d,\ g = g_{r_n} \cdots g_{r_1}\}. \tag{5.2}$$

The C^*-algebra generated by $\{\mathcal{A}_{[i,j]}\}$ is included in $\mathcal{F} = \bigotimes_{\mathbf{Z}} \mathcal{F}_k$ and the right shift γ restricts on $\mathcal{A}$. Then we call $(\mathcal{A}, \{\mathcal{A}_{[i,j]}\}, \gamma)$ the *C^*-system of quantum random walk* on G.

Let G^* denote the set of all homomorphisms from G into the multiplicative group $\mathbf{R}_{++} = (0, \infty)$. We use the notation $\mathcal{T}^{\mathrm{m,f}}_\gamma(\mathcal{A})$ to mean the set of all $\phi \in \mathcal{T}(\mathcal{A})$ satisfying (IV) and (V) in §1. For each $\chi \in G^*$, set $W = \sum_{r=1}^d \chi(g_r)$ and define $D \in M_d(\mathbf{C})$ by

$$D = \frac{1}{W} \sum_{r=1}^{d} \chi(g_r) e_{rr}. \tag{5.3}$$

Furthermore, define a product state $\psi = \bigotimes_{\mathbf{Z}} \mathrm{Tr}(D\,\cdot\,)$ on $\mathcal{F}$ where Tr is the canonical trace of $M_d(\mathbf{C})$.

Proposition 5.1 For every $\chi \in G^*$ define ψ as above and $\phi = \psi|\mathcal{A}$. Then the correspondnece $\chi \mapsto \phi$ is bijective from G^* onto $\mathcal{T}^{\mathrm{m,f}}_\gamma(\mathcal{A})$.

Proof. Let ϕ be defined from $\chi \in G^*$ as above. Let $i, j \in \mathbf{Z}$, $i \le j$, and $r_k, s_k \in \{1, \dots, d\}$ for $i \le k \le j$. If $(r_i, \dots, r_j) \ne (s_i, \dots, s_j)$, then

$$\phi\left(\bigotimes_{k=i}^{j} e^k_{r_k s_k}\right) = \prod_{k=i}^{j} \mathrm{Tr}(D e^k_{r_k s_k}) = 0.$$

Also we get

$$\phi\left(\bigotimes_{k=i}^{j} e^k_{r_k r_k}\right) = \prod_{k=i}^{j} \frac{\chi(g_{r_k})}{W} = \frac{\chi(g)}{W^{j-i+1}}$$

with $g = g_{r_j} \cdots g_{r_i}$. By definition of $\mathcal{A}_{[i,j]}$ these imply that ϕ is tracial on $\mathcal{A}_{[i,j]}$. Hence ϕ is a trace. It is immediate that ϕ satisfies (IV) and (V).

Conversely let $\phi \in \mathcal{T}^{\mathrm{m,f}}_\gamma(\mathcal{A})$ be given. Assume that $e = g_{p_l} \cdots g_{p_1} = g_{p'_{l'}} \cdots g_{p'_1}$ where $p_k, p'_{k'} \in \{1, \dots, d\}$. Since $e = (g_{p_l} \cdots g_{p_1})^{l'} = (g_{p'_{l'}} \cdots g_{p'_1})^l$, it follows that

$$\bigotimes_{j=1}^{l'} \left(\bigotimes_{k=1}^{l} e^{(j-1)l'+k}_{p_k p_k} \right) \quad \text{and} \quad \bigotimes_{j=1}^{l} \left(\bigotimes_{k=1}^{l'} e^{(j-1)l+k}_{p'_k p'_k} \right)$$

are in the same direct summand of $\mathcal{A}_{[1,ll']}$. Hence by the γ-invariance and the multiplicativity property of ϕ we get

$$\phi\bigg(\bigotimes_{k=1}^{l} e^k_{p_k p_k}\bigg)^{l'} = \phi\bigg(\bigotimes_{k=1}^{l'} e^k_{p'_k p'_k}\bigg)^{l}.$$

So we can define $W = \phi(\bigotimes_{k=1}^{l} e^k_{p_k p_k})^{-1/l}$ independently of the expression $e = g_{p_l} \cdots g_{p_1}$. Now let $g = g_{r_n} \cdots g_{r_1} = g_{r'_{n'}} \cdots g_{r'_1}$ and $g^{-1} = g_{r''_{n''}} \cdots g_{r''_1}$. Since

$$e = g_{r''_{n''}} \cdots g_{r''_1} g_{r_n} \cdots g_{r_1} = g_{r''_{n''}} \cdots g_{r''_1} g_{r'_{n'}} \cdots g_{r'_1},$$

we have

$$W^{-(n+n'')} = \phi\bigg(\bigotimes_{k=1}^{n} e^k_{r_k r_k}\bigg)\phi\bigg(\bigotimes_{k=1}^{n''} e^k_{r''_k r''_k}\bigg),$$

$$W^{-(n'+n'')} = \phi\bigg(\bigotimes_{k=1}^{n'} e^k_{r'_k r'_k}\bigg)\phi\bigg(\bigotimes_{k=1}^{n''} e^k_{r''_k r''_k}\bigg),$$

so that $W^n \phi(\bigotimes_{k=1}^{n} e^k_{r_k r_k}) = W^{n'} \phi(\bigotimes_{k=1}^{n'} e^k_{r'_k r'_k})$. Hence $\chi(g) \in \mathbf{R}^*_+$ is well defined by

$$\chi(g) = W^n \phi\bigg(\bigotimes_{k=1}^{n} e^k_{r_k r_k}\bigg) \quad \text{if} \quad g = g_{r_n} \cdots g_{r_1}.$$

If $g = g_{r_n} \cdots g_{r_1}$ and $h = g_{s_m} \cdots g_{s_1}$, then

$$\chi(gh) = W^{n+m} \phi\bigg(\bigotimes_{k=1}^{n} e^k_{r_k r_k}\bigg)\phi\bigg(\bigotimes_{k=1}^{m} e^k_{s_k s_k}\bigg) = \chi(g)\chi(h).$$

Therefore $\chi \in G^*$. For this χ we have $\sum_{r=1}^{d} \chi(g_r) = W \sum_{r=1}^{d} \phi(e^1_{rr}) = W$. Thus the conclusion follows.

Remark 5.2. Let $\chi \in G^*$ and $\phi \in \mathcal{T}^{\mathrm{m,f}}_\gamma(\mathcal{A})$ with $\chi \leftrightarrow \phi$ in the sense of Proposition 5.1. Set a finitely supported probability measure μ on G by $\mu(g) = W^{-1} \sum_{g_r = g} \chi(g_r)$ for $g \in G$. Then ϕ can be described in terms of the random walk (or the Markov chain) on G with the initial distribution δ_e and the transition probabilities $p(g|h) = \mu(gh^{-1})$, $g, h \in G$. Indeed, the ϕ-value of a minimal central projection of $\mathcal{A}_{[1,n]}$ corresponding to $g \in K_n$ is given by $\mu^n(g)$, where μ^n is the nth convolution of μ. Hence we know by [25, 26] that ϕ is an extremal tracial state if and only if the random walk (G, μ) has the trivial Poisson boundary (i.e. the μ-harmonic bounded functions on G are trivial), equivalently the entropy $h(G, \mu) = \lim_{n\to\infty} \frac{1}{n} H(\mu^n)$ ([26]) is equal to 0. In particular, if G has subexponential growth, then $h(G, \mu) = 0$ for any finitely supported measure μ, so that all $\phi \in \mathcal{T}^{\mathrm{m,f}}_\gamma(\mathcal{A})$ are extremal in $\mathcal{T}(\mathcal{A})$. By the way, it is worth noting [26] that there is a solvable (hence amenable) group G for which $h(G, \mu) > 0$ for any finitely supported nondegenerate measure μ on G. So, in this case, every $\phi \in \mathcal{T}_\gamma^{\mathrm{m,f}}(\mathcal{A})$ is not extremal in $\mathcal{T}(\mathcal{A})$, though extremal in $\mathcal{T}_\gamma(\mathcal{A})$ by the remark after Proposition 3.2.

Remark 5.3. Assume that G is abelian. Then for any $i, j \in \mathbf{Z}$, $i < j$, it is obvious that $U(i,j) = \sum_{r,s=1}^d e_{rs}^i \otimes e_{sr}^j$ belongs to $\mathcal{A}_{[i,j]}$. Since $\mathrm{Ad}\, U(i,j)$ acts on $\mathcal{F}$ as the transposition interchanging the ith and jth components, the permutation group $S(\infty)$ on $\mathbf{Z}$ appears in the unitary group of $\mathcal{A}$. So, applying Størmer's results in the same way as in [45] we can show that any extremal tracial state on $\mathcal{A}$ is the restriction of a symmetric product state on $\mathcal{F}$. Here we need a conditional expectation $E_\mathcal{A} : \mathcal{F} \to \mathcal{A}$, whose existence is seen from the fact that the trace-preserving conditional expectations $E_\Lambda : \mathcal{F}_\Lambda \to \mathcal{A}_\Lambda$ satisfy the commuting square property for $\begin{matrix} \mathcal{A}_{\Lambda'} & \subset & \mathcal{F}_{\Lambda'} \\ \cup & & \cup \\ \mathcal{A}_\Lambda & \subset & \mathcal{F}_\Lambda \end{matrix}$, $\Lambda \subset \Lambda'$. Hence Proposition 4.1 holds true and we have $\mathcal{E}\mathcal{T}^{\mathrm{f}}(\mathcal{A}) = \mathcal{T}_\gamma^{\mathrm{m,f}}(\mathcal{A})$. (See also Example 5.6(2) below.)

Lemma 5.4 For any $\chi \in G^*$ set $D \in M_d(\mathbf{C})$ by (5.3). Then $\bigotimes_\Lambda D$ belongs to the center $Z(\mathcal{A}_\Lambda)$ of $\mathcal{A}_\Lambda$ for every finite interval $\Lambda \subset \mathbf{Z}$.

Proof. The case $\Lambda = [1,n]$ is enough. We get

$$\begin{aligned} \bigotimes_1^n D &= \frac{1}{W^n} \sum_{r_1,\dots,r_n} \chi(g_{r_n} \cdots g_{r_1}) e_{r_1 r_1}^1 \otimes \cdots \otimes e_{r_n r_n}^n \\ &= \frac{1}{W^n} \sum_{g \in K_n} \chi(g) \sum_{g_{r_n} \cdots g_{r_1} = g} e_{r_1 r_1}^1 \otimes \cdots \otimes e_{r_n r_n}^n. \end{aligned}$$

Since $\sum_{g_{r_n} \cdots g_{r_1} = g} e_{r_1 r_1}^1 \otimes \cdots \otimes e_{r_n r_n}^n$, $g \in K_n$, are central projections of $\mathcal{A}_{[1,n]}$, we have the result.

Now let $\Phi \in \mathcal{B}_0$, $\phi \in \mathcal{T}_\gamma^{\mathrm{m,f}}(\mathcal{A})$, and $\chi \in G^*$ with $\phi \leftrightarrow \chi$ in the sense of Proposition 5.1. Set

$$h = -\log D = -\sum_{r=1}^d \Big(\log \frac{\chi(g_r)}{W}\Big) e_{rr}.$$

Define a one-parameter automorphism group β_t on $\mathcal{F}$ by $\beta_t = \bigotimes_\mathbf{Z} \mathrm{Ad}\, e^{ith}$ and an interaction Φ^h in $\mathcal{F}$ by (4.2). Since by Lemma 5.4

$$-\sum_{j \in \Lambda} \gamma^j(h) = \log\Big(\bigotimes_\Lambda D\Big) \in Z(\mathcal{A}_\Lambda), \qquad \Lambda \subset \mathbf{Z},$$

we have as Lemma 4.3

$$\beta_t(a) = a, \qquad a \in \mathcal{A},\ t \in \mathbf{R},$$

$$\alpha_t^\Phi \beta_t = \beta_t \alpha_t^\Phi = \alpha_t^{\Phi^h}, \qquad t \in \mathbf{R}.$$

Thus, the next theorem can be proved in the same way as Theorem 4.4 and Corollary 4.5. In the proof of (iv) $\Rightarrow$ (i), we use a conditional expectation $E_\mathcal{A} : \mathcal{F} \to \mathcal{A}$ preserving the tracial state τ of $\mathcal{F}$ mentioned in Remark 5.3.

Theorem 5.5 With the above assumptions and notations, the following conditions for $\omega \in \mathcal{S}_\gamma(\mathcal{A})$ are equivalent:

(i) ω is an α^Φ-KMS state extending to an $\alpha_t^\Phi \beta_t$-KMS state on $\mathcal{F}$;

(ii) ω satisfies the Gibbs condition in the strong sense with respect to Φ and ϕ;

(iii) ω satisfies the Gibbs condition in the weak sense with respect to Φ and ϕ;

(iv) $\omega \in S(\Phi, \phi)$.

Moreover, for each $\phi \in \mathcal{T}_\gamma^{\mathrm{m,f}}(\mathcal{A})$ there exists a unique $\omega \in \mathcal{S}_\gamma(\mathcal{A})$ satisfying the above conditions, and the correspondence $\phi \mapsto \omega$ is injective from $\mathcal{T}_\gamma^{\mathrm{m,f}}(\mathcal{A})$ into the set of faithful extremal γ-invariant α^Φ-KMS states on $\mathcal{A}$.

Examples 5.6. (1) Assume that G is a finite group and fix a generating finite set $\{g_1, \dots, g_d\}$. Then $G^* = \{1\}$ and $\phi = \tau|\mathcal{A}$ is a unique trace on $\mathcal{A}$ as Proposition 1.5. Hence Corollary 3.11 holds in this case.

(2) Let $G = \mathbf{Z}^N$ and $\{w_1, \dots, w_d\}$ be any generating set. Define a unitary representation σ of $\widehat{G} = \mathbf{T}^N$ on $\mathbf{C}^d$ by

$$\sigma_z = \mathrm{diag}(z^{w_1}, \dots, z^{w_d}),$$

where $z^w = z_1^{w(1)} \cdots z_N^{w(N)}$ for $z = (z_1, \dots, z_N) \in \mathbf{T}^N$ and $w = (w(1), \dots, w(N)) \in \mathbf{Z}^N$. Since

$$\bigotimes_1^n \sigma_z = \sum_{r_1,\dots,r_n} z^{w_{r_1}+\dots+w_{r_n}} e^1_{r_1 r_1} \otimes \dots \otimes e^n_{r_n r_n},$$

it follows that

$$\mathcal{A}_{[1,n]} = \mathrm{Alg}\left\{ \bigotimes_{k=1}^n e^k_{r_k s_k} : w_{r_1} + \dots + w_{r_n} = w_{s_1} + \dots + w_{s_n} \right\}$$

is equal to $\mathcal{F}^\beta_{[1,n]}$ where β is the product action of $\mathrm{Ad}\,\sigma_z$, $z \in \mathbf{T}^N$. Hence the C^*-system of quantum random walk on $\mathbf{Z}^N$ coincides with the C^*-system of gauge invariance for $(\mathbf{T}^N, \sigma)$. Thus $(\mathbf{Z}^N)^*$ is isomorphic to $\Xi(\mathbf{T}^N, \sigma)$ and by Corollary 4.5 it parametrizes, given $\Phi \in \mathcal{B}_0$, the faithful extremal γ-invariant α^Φ-KMS states on $\mathcal{A}$. Here the isomorphisms $\chi \in (\mathbf{Z}^N)^* \leftrightarrow r \in (\mathbf{R}^N)_{++} \leftrightarrow \xi \in \Xi(\mathbf{T}^N, \sigma)$ are given by

$$r = (r_1, r_2, \dots, r_N) = (\chi(1, 0, \dots, 0), \chi(0, 1, 0, \dots, 0), \dots, \chi(0, \dots, 0, 1)),$$

$$\xi_t = (e^{-itr_1}, e^{-itr_2}, \dots, e^{-itr_N}), \qquad t \in \mathbf{R}.$$

Note that the above arguments remain valid when G is an arbitrary abelian discrete group with a generating finite set. So Theorem 5.5 for abelian G is equivalent to Corollary 4.5.

(3) Let $G = F_N$, the free group with N generators $g_1, \dots, g_N$. Set $g_{N+k} = g_k^{-1}$, $1 \le k \le N$. Then the C^*-system $(\mathcal{A}, \{\mathcal{A}_{[i,j]}\}, \gamma)$ of quantum random walk on F_N is a subsystem of $\mathcal{F} = \bigotimes_{\mathbf{Z}} M_{2N}(\mathbf{C})$. Since $(F_N)^* \cong (\mathbf{R}^N)_{++}$, by Theorem 5.5 there are, given $\Phi \in \mathcal{B}_0$, faithful extremal γ-invariant α^Φ-KMS states parametrized by $(\mathbf{R}^N)_{++}$, while it is not known whether these exhaust such α^Φ-KMS states on $\mathcal{A}$.

6. Dynamical entropies

In this section let us discuss dynamical entropy and topological entropy (and their relation) in C^*-systems introduced in §1. Noncommutative dynamical entropy was first studied in [15] for an automorphism of a finite von Neumann algebra with respect to a normal tracial state. This Connes-Størmer dynamical entropy was later on extended in [14] to the so-called CNT-dynamical entropy in a general C^*-algebra setup. As was clarified in [14], the CNT-dynamical entropy nicely behaves for automorphisms (or endomorphisms) of nuclear C^*-algebras. So we consider this entropy as the most appropriate one in our AF C^*-systems $(\mathcal{A}, \{\mathcal{A}_{[i,j]}\}, \gamma)$. Let $h_\omega(\gamma)$ denote the *CNT-dynamical entropy* of γ with respect to $\omega \in \mathcal{S}_\gamma(\mathcal{A})$. Since $\mathcal{A} = \overline{\bigcup_n \mathcal{A}_{[-n,n]}}$, we have by [14, Corollary V.4]

$$h_\omega(\gamma) = \lim_{n\to\infty} h_{\omega,\gamma}(\mathcal{A}_{[-n,n]}),$$

where

$$h_{\omega,\gamma}(\mathcal{A}_{[-n,n]}) = \lim_{m\to\infty} \frac{1}{m} H_\omega(\mathcal{A}_{[-n,n]}, \gamma(\mathcal{A}_{[-n,n]}), \dots, \gamma^{m-1}(\mathcal{A}_{[-n,n]})).$$

(See [14] or [37, Chap. 10] for the definition of H_ω above.)

On the other hand, noncommutative versions of topological entropy were recently studied in [21, 22, 52] in AF C^*-algebras (or rather "local" C^*-algebras). Although the definitions of topological entropy in [21] and [52] are a bit different, they are identical for our γ. In fact, the *topological entropy* $\hbar(\gamma)$ of γ on the local C^*-algebra $\bigcup_n \mathcal{A}_{[-n,n]}$ (also $\bigcup_n \mathcal{A}_{[1,n]}$) is given as ([21, Theorem 3.32], [52, Lemma 3.2])

$$\hbar(\gamma) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Tr}_n(1),$$

where $\mathrm{Tr}_n(1)$ means the tracial dimension (i.e. the number of orthogonal minimal projections) of $\mathcal{A}_{[1,n]}$.

As in the classical probabilistic case we first have:

Proposition 6.1 $h_\omega(\gamma) \le \hbar(\gamma)$ for every $\omega \in \mathcal{S}_\gamma(\mathcal{A})$.

Proof. For any $\omega \in \mathcal{S}_\gamma(\mathcal{A})$, since by [14, Proposition III.6]

$$\begin{aligned} &H_\omega(\mathcal{A}_{[-n,n]}, \gamma(\mathcal{A}_{[-n,n]}), \dots, \gamma^{m-1}(\mathcal{A}_{[-n,n]})) \\ &\quad = H_\omega(\mathcal{A}_{[1,2n+1]}, \mathcal{A}_{[2,2n+2]}, \dots, \mathcal{A}_{[m,2n+m]}) \\ &\quad \le H_\omega(\mathcal{A}_{[1,2n+m]}) \le S(\omega_{2n+m}), \end{aligned} \tag{6.1}$$

and $S(\omega_{2n+m}) \le \log \mathrm{Tr}_{2n+m}(1)$, we have

$$h_{\omega,\gamma}(\mathcal{A}_{[-n,n]}) \le \lim_{m\to\infty} \frac{1}{m} \log \mathrm{Tr}_{2n+m}(1) = \hbar(\gamma).$$

Hence $h_\omega(\gamma) \le \hbar(\gamma)$.

The next result is on the lines of [11, Theorem 1] and [13, Proposition 4.2].

Proposition 6.2 If $\omega \in \mathcal{S}_\gamma(\mathcal{A})$ is multiplicative in the sense of (V) in §1, then the mean entropy $s(\omega)$ exists and

$$h_\omega(\gamma) = h_\omega(\gamma|\mathcal{A}_{[1,\infty)}) = s(\omega).$$

Proof. The first equality is immediate because $h_{\omega,\gamma}(\mathcal{A}_{[-n,n]}) = h_{\omega,\gamma}(\mathcal{A}_{[1,2n+1]})$ by the proof of Proposition 6.1. For each $n, m \in \mathbf{N}$, let $\mathcal{B} = \bigvee_{k=0}^{m-1} \gamma^{nk}(\mathcal{A}_{[1,n]})$ and $\{q_j : 1 \le j \le l\}$ be a set of minimal projections in the centralizer of ω_n such that $\sum_{j=1}^l q_j = \operatorname{supp}\omega_n$, the support projection of ω_n. Let $q_j^k = \gamma^{n(k-1)}(q_j)$ for $1 \le j \le l$ and $1 \le k \le m$. Then by the multiplicativity of ω, $\{q_{j_1}^1 \cdots q_{j_m}^m : 1 \le j_1, \dots, j_m \le l\}$ is a set of minimal projections in the centralizer of $\omega|\mathcal{B}$ such that $\sum_{j_1,\dots,j_m} q_{j_1}^1 \cdots q_{j_m}^m = \operatorname{supp}(\omega|\mathcal{B})$ and $\omega(q_{j_1}^1 \cdots q_{j_m}^m) = \omega(q_{j_1}^1) \cdots \omega(q_{j_m}^m)$. Hence we get by [14, Corollary VIII.8]

$$H_\omega(\mathcal{A}_{[1,n]}, \gamma^n(\mathcal{A}_{[1,n]}), \cdots, \gamma^{n(m-1)}(\mathcal{A}_{[1,n]})) = S(\omega|\mathcal{B}) = mS(\omega_n),$$

so that by [14, VII.5 ii)]

$$\begin{aligned} h_\omega(\gamma) &= \frac{1}{n} h_\omega(\gamma^n) \\ &\ge \frac{1}{n} \lim_{m\to\infty} \frac{1}{m} H_\omega(\mathcal{A}_{[1,n]}, \gamma^n(\mathcal{A}_{[1,n]}), \dots, \gamma^{n(m-1)}(\mathcal{A}_{[1,n]})) \\ &= \frac{1}{n} S(\omega_n). \end{aligned}$$

Therefore

$$h_\omega(\gamma) \ge \limsup_{n\to\infty} \frac{1}{n} S(\omega_n).$$

On the other hand, since (6.1) gives

$$h_{\omega,\gamma}(\mathcal{A}_{[-n,n]}) \le \liminf_{m\to\infty} \frac{1}{m} S(\omega_m),$$

it follows that $s(\omega)$ exists and $h_\omega(\gamma) = s(\omega)$.

Theorem 6.3 Assume that τ is a tracial state on $\mathcal{A}$ satisfying (IV)–(VI). Then:

$$h_\tau(\gamma) = \hbar(\gamma) = \log\lambda^{-1} = \sup_{\omega\in\mathcal{S}_\gamma(\mathcal{A})} h_\omega(\gamma).$$

Moreover, if $\omega \in \mathcal{S}_\gamma(\mathcal{A})$, then $h_\omega(\gamma) = \hbar(\gamma)$ if and only if $\omega = \tau$.

Proof. The equality $h_\tau(\gamma) = \log\lambda^{-1}$ follows from Propositions 3.3 and 6.2. We get $\hbar(\gamma) = \log\lambda^{-1}$ by letting $\Phi = 0$ (so $H_n = 0$) in the proof of Proposition 3.8. Furthermore, for any $\omega \in \mathcal{S}_\gamma(\mathcal{A})$ we have

$$h_\omega(\gamma) \le s(\omega) \le \log\lambda^{-1} \tag{6.2}$$

by Proposition 3.3 and the proof of Proposition 6.1. Hence the first part is shown. The second is immediate from (6.2) together with the last part of Proposition 3.3.

For instance, let γ be the canonical shift and τ the λ-Markov trace in the C^*-system of Example 1.4 arising from an inclusion $N \subset M$ of type II_1 factors. If $N \subset M$ has the standard eigenvector $\vec{s}$ of subexponential growth, then τ satisfies (IV)–(VI) and so we have Theorem 6.3 with $\lambda^{-1} = [M : N]$. Furthermore, in this setting, it can be seen by [20, Corollary 4.9] that Theorem 6.3 remains true when $N \subset M$ is extremal and strongly amenable.

In what remains we give a bit more detailed discussions on entropies specializing to the cases of C^*-systems in §4 and §5. First let σ be a unitary representation of a compact group G on $V = \mathbf{C}^d$ and $(\mathcal{A}, \{\mathcal{A}_{[i,j]}\}, \gamma)$ the C^*-system of gauge invariance for (G, σ), that is, $\mathcal{A}$ is the fixed point algebra of the product action $\beta_g = \bigotimes_{\mathbf{Z}} \mathrm{Ad}\,\sigma_g$, $g \in G$, on $\mathcal{F} = \bigotimes_{\mathbf{Z}} M_d(\mathbf{C})$. Let τ be the restriction on $\mathcal{A}$ of the tracial state of $\mathcal{F}$, which of course satisfies (IV) and (V). The direct summands of $\mathcal{A}_{[1,n]}$ are indexed by $K_n = \{\rho \in \widehat{G} : \rho \prec \bigotimes_1^n \sigma\}$ and the τ-trace vector of $\mathcal{A}_{[1,n]}$ is given by $(\dim \rho / d^n)_{\rho \in K_n}$. Hence, if $(\dim \rho)_{\rho \in \bigcup_n K_n}$ has subexponential growth in the sense that

$$\lim_{n\to\infty} \frac{1}{n} \log\Bigl(\max_{\rho \in K_n} \dim \rho\Bigr) = 0, \tag{6.3}$$

then τ satisfies (VI) as well with $\lambda^{-1} = d = \dim V$; so Theorem 6.3 holds.

In the next theorem, under assumption (6.3), we exactly compute $h_\omega(\gamma)$ when ω is the restriction on $\mathcal{A}$ of a symmetric product state on $\mathcal{F}$.

Theorem 6.4 In the above setting assume (6.3). If $\varphi = \bigotimes_{\mathbf{Z}} \varphi_0$ on $\mathcal{F}$ and $\omega = \varphi|\mathcal{A}$ where φ_0 is a state on $M_d(\mathbf{C})$, then

$$h_\omega(\gamma) = s(\omega) = h_\varphi(\gamma) = S(\varphi_0).$$

Proof. The first equality follows from Proposition 6.2 and the last is well known. By [6, Corollary III.2.3(iii)] we have $\pi_\omega(\mathcal{F}^{\beta|G_\varphi})'' = \pi_\omega(\mathcal{A})''$ for the GNS representation π_ω and the subgroup $G_\varphi = \{g \in G : \varphi \circ \beta_g = \varphi\}$ of G. Recall [14, Theorem VII.2] that $h_\omega(\gamma) = h_{\tilde{\omega}}(\tilde{\gamma})$ where $\tilde{\omega}$ and $\tilde{\gamma}$ are the normal extensions of ω and γ via π_ω. Also, (6.3) is satisfied for $\sigma|G_\varphi$ as well, because the irreducible decomposition of $(\bigotimes_1^n \sigma)|G_\varphi$ is a refinement of that of $\bigotimes_1^n \sigma$. Thus, to prove the theorem, we may assume that $\varphi \circ \beta_g = \varphi$ for all $g \in G$. Now let $K_n = \{\rho_1, \ldots, \rho_{l_n}\}$, that is, $\bigotimes_1^n \sigma = \sum_{k=1}^{l_n} m_{n,k} \rho_k$ with multiplicities $m_{n,k}$. Then $\mathcal{A}_{[1,n]}$ is decomposed as $\bigoplus_{k=1}^{l_n} \mathcal{A}_{[1,n]} f_{n,k}$, where $f_{n,k}$ are the minimal central projections of $\mathcal{A}_{[1,n]}$ and $\mathcal{A}_{[1,n]} f_{n,k} \cong M_{m_{n,k}}(\mathbf{C})$. Let $\mathrm{Tr}_{\mathcal{F}_{[1,n]}}$ be the canonical trace of $\mathcal{F}_{[1,n]}$ and set $D_n = d\varphi_n / d\mathrm{Tr}_{\mathcal{F}_{[1,n]}}$. The β-invariance of φ implies that $D_n \in \mathcal{A}_{[1,n]}$ and hence D_n is written as

$$D_n = \sum_{k=1}^{l_n} \sum_{j=1}^{m_{n,k}} \mu_{n,k}(j) p_{n,k}(j),$$

where $p_{n,k}(j)$, $1 \le j \le m_{n,k}$, are minimal projections in $\mathcal{A}_{[1,n]} f_{n,k}$ so that they are of rank $\dim \rho_k$ as projections in $\mathcal{F}_{[1,n]}$. Therefore $\sum_{k=1}^{l_n} \sum_{j=1}^{m_{n,k}} \mu_{n,k}(j) \dim \rho_k = 1$ and

$$\begin{aligned} S(\omega_n) &= -\sum_{k=1}^{l_n} \sum_{j=1}^{m_{n,k}} \mu_{n,k}(j) \dim \rho_k \log(\mu_{n,k}(j) \dim \rho_k) \\ &= S(\varphi_n) - \sum_{k=1}^{l_n} \sum_{j=1}^{m_{n,k}} \mu_{n,k}(j) \dim \rho_k \log \dim \rho_k. \end{aligned}$$

Since

$$0 \le \sum_{k=1}^{l_n} \sum_{j=1}^{m_{n,k}} \mu_{n,k}(j) \dim \rho_k \log \dim \rho_k \le \log\Bigl(\max_{1 \le k \le l_n} \dim \rho_k \Bigr),$$

it follows from (6.3) that $s(\omega) = s(\varphi) = S(\varphi_0)$, completing the proof.

The above theorem can be applied to extremal tracial states on $\mathcal{A}$ due to [45, Theorem 3.2] (also Proposition 4.1).

Example 6.5. It is immediate that the unitary representations (1)–(3) of Example 4.6 satisfy (6.3). Hence Theorems 6.3 and 6.4 show that in cases (1) and (2)

$$\begin{aligned} h_{\phi_\lambda}(\gamma) = s(\phi_\lambda) &= -r \log r - (1-r) \log(1-r) \\ &\le h_{\phi_{1/4}}(\gamma) = \hbar(\gamma) = \log 2 \end{aligned}$$

for $\lambda = r(1-r)$, $0 \le r \le 1$. The results in case (3) are analogous. Example 1.3 for $\lambda \le 1/4$ is nothing but case (1) above. The above formula $h_{\phi_\lambda}(\gamma) = -r \log r - (1-r)\log(1-r)$ as well as $h_{\phi_\lambda}(\gamma) = \frac{1}{2} \log \lambda^{-1}$ for $\lambda > 1/4$ of Example 1.3 was computed in [38, 10, 55].

Finally let $(\mathcal{A}, \{\mathcal{A}_{[i,j]}\}, \gamma)$ be the C^*-system of quantum random walk on a discrete group G with generators $g_1, \dots, g_d$. In this case, the restriction τ on $\mathcal{A}$ of the tracial state of $\mathcal{F}$ satisfies (IV)–(VI), because $n^{-1} \log d\tau_n / d\mathrm{Tr}_n = (\log d^{-1})1$ for all n. Hence Theorem 6.3 holds with $\lambda^{-1} = d$.

Proposition 6.6 Assume that G has subexponential growth. If $\varphi \in \mathcal{S}_\gamma(\mathcal{F})$ and $\omega = \varphi | \mathcal{A}$, then $s(\omega) = s(\varphi)$.

Proof. Let $E_n : \mathcal{F}_{[1,n]} \to \mathcal{A}_{[1,n]}$ be the conditional expectation with respect to $\mathrm{Tr}_{\mathcal{F}_{[1,n]}}$, which is written as

$$E_n(a) = \sum_{k=1}^{l_n} f_{n,k} a f_{n,k}, \qquad a \in \mathcal{F}_{[1,n]},$$

where $f_{n,k}$ are the minimal central projections of $\mathcal{A}_{[1,n]}$. Set $D_n = d\varphi_n / d\mathrm{Tr}_{\mathcal{F}_{[1,n]}}$. Then, since $\mathrm{Tr}_n = \mathrm{Tr}_{\mathcal{F}_{[1,n]}} | \mathcal{A}_{[1,n]}$, we have $d\omega_n / d\mathrm{Tr}_n = E_n(D_n)$ and

$$\begin{aligned} S(\omega_n) - S(\varphi_n) &= \mathrm{Tr}_n(D_n(\log D_n - \log E_n(D_n))) \\ &= S(D_n, E_n(D_n)). \end{aligned}$$

Now write $D_n = \sum_{j=1}^N \mu_j p_j$ where $\mu_j > 0$, $\sum_{j=1}^N \mu_j = 1$, and p_j are projections of rank one. By the joint convexity of relative entropy we get

$$S(D_n, E_n(D_n)) \le \sum_{j=1}^N \mu_j S(p_j, E_n(p_j)) = \sum_{j=1}^N \mu_j S(E_n(p_j)).$$

Since the rank of $E_n(p_j) = \sum_{k=1}^{l_n} f_{n,k} p_j f_{n,k}$ is at most l_n, we have $S(E_n(p_j)) \le \log l_n$ and hence $0 \le S(D_n, E_n(D_n)) \le \log l_n$. Therefore

$$0 \le \frac{1}{n} S(\omega_n) - \frac{1}{n} S(\varphi_n) \le \frac{1}{n} \log l_n \to 0 \quad (n \to \infty)$$

thanks to the assumption of subexponential growth. This gives the conclusion.

By Propositions 5.1, 6.2, and 6.6 we have:

Corollary 6.7 Assume that G has subexponential growth. If $\phi \in \mathcal{T}_\gamma^{\mathrm{m,f}}(\mathcal{A})$ and $\chi \in G^*$ with $\phi \leftrightarrow \chi$ as in Proposition 5.1, then

$$h_\phi(\gamma) = s(\phi) = -\sum_{r=1}^d \frac{\chi(g_r)}{W} \log \frac{\chi(g_r)}{W}.$$

Example 6.8. Let $D = \sum_{r,s=1}^d d^{-1} e_{rs}$, which is a rank one projection in $M_d(\mathbf{C})$. Define $\varphi = \bigotimes_{\mathbf{Z}} \mathrm{Tr}(D\,\cdot\,)$ on $\mathcal{F}$ and $\omega = \varphi|\mathcal{A}$. Let K_n and $d_{g,n}$ be given by (5.1) and (5.2). Then it is easy to check that $d\omega_n/d\mathrm{Tr}_n = \sum_{g\in K_n} (d_{g,n}/d^n) p_{g,n}$ where $p_{g,n}$ is a minimal projection in the direct summand of $\mathcal{A}_{[1,n]}$ corresponding to $g \in K_n$. Hence, if μ is the distribution on G given by $\mu(g) = \#\{r : g_r = g\}/d$ for $g \in G$, then

$$S(\omega_n) = -\sum_{g\in K_n} \frac{d_{g,n}}{d^n} \log \frac{d_{g,n}}{d^n} = H(\mu^n),$$

so that by Proposition 6.2 we have $h_\omega(\gamma) = s(\omega) = h(G,\mu)$. On the other hand, $h_\varphi(\gamma) = s(\varphi) = 0$. So, in view of Remark 5.2, this example shows that the assumption of subexponential growth is essential in Proposition 6.6. When the generating set is $\{g_1, \dots, g_N, g_1^{-1}, \dots, g_N^{-1}\}$ $(d = 2N)$, it is known [8] that $h(G,\mu) \le \frac{2N-2}{2N} \log(2N-1)$ for the above μ and the equality occurs only when $G = F_N$, the free group on N generators.

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