Entropy Densities for Algebraic States

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Abstract. Algebraic (or finitely correlated) states are translation-invariant states on an infite tensor product C*-algebra, whose construction is rather general including quantum Markov chains. For a strongly mixing algebraic state, we obtain the relation between the mean entropy and another entropy density, which means the macroscopic uniformity under the relevant state in the statistical mechanical sense. For this purpose an outstanding property of approximately pruduct type is shown for strongly mixing algebraic states. We also obtain similar relations of the mean relative entropy to other entropy densities of two translation-invariant states when the reference one is a strongly mixing algebraic state.

Introduction

In this paper we discuss entropy densities and relative entropy densities for strongly mixing algebraic states. Algebraic states are stationary (or translation-invariant) states on a quantum spin C^*-algebra $\mathcal{A} = \bigotimes_{i\in\mathbf{Z}} \mathcal{A}_i$, the infinite C^*-tensor product of $\mathcal{A}_i = M_d(\mathbf{C})$, $i \in \mathbf{Z}$, which are constructed in a certain way from finite-dimensional algebraic objects. The prototype of these states is the class of classical Markov chains. The most general and complete exposition on "algebraic" states was presented in [12] under the term "finitely correlated" states, while similar notions were introduced in [1, 11, 19] with other terms such as "quantum Markov" states.

The ergodicity for algebraic states was already characterized in [12]. In Section 1 of this paper we characterize the strong mixing for algebraic states in several ways. In particular, it is proved that the strong mixing, the weak mixing, and the complete ergodicity are all equivalent for algebraic states as in the case of classical Markov states. Also we prove that strong mixing algebraic states have some property of approximately product type, which is quite useful in the subsequent analysis on entropy densities.

The mean entropy is a typical entropy density for states on quantum spin C^*-algebras. This can be defined for any stationary state φ on $\mathcal{A}$ as $s(\varphi) = \lim_{n\to\infty} \frac{1}{n} S(\varphi_n)$. Here $S(\varphi_n)$ is the von Neumann entropy of $\varphi_n = \varphi|\mathcal{A}_{[1,n]}$ where $\mathcal{A}_{[1,n]} = \bigotimes_{i=1}^{n} \mathcal{A}_i$. The mean entropy plays an important role in the theory of quantum statistical thermodynamics of spin systems, which is one of the ingredients in the variational principle for equilibrium states. For $0 < \varepsilon < 1$ another entropy density is defined by

$$\beta_\varepsilon(\varphi_n) = \inf\{\log \mathrm{Tr}_n(q) : q \in \mathcal{A}_{[1,n]} \text{ is a projection with } \varphi_n(q) \geq 1 - \varepsilon\},$$

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where Tr_n denotes the canonical trace on $\mathcal{A}_{[1,n]}$. Then it may be rather natural to conjecture that when φ is an ergodic state on $\mathcal{A}$, the relation

$$\lim_{n\to\infty}\frac{1}{n}\beta_\varepsilon(\varphi_n)=s(\varphi) \tag{0.1}$$

holds for every $0<\varepsilon<1$. This equality was obtained in [21] particularly when φ is a homogeneous product state that is a simplest example of algebraic states. Furthermore we proved in [16] that the equality (0.1) holds when φ is an ergodic Gibbs (or equilibrium) state in the setup of $\mathbf{Z}^\nu$-quantum lattice spin systems.

The relative entropy is an entropy quantity attached to two states of a system. Let φ and ω be states on a matrix algebra with the canonical trace Tr. Then the Umegaki relative entropy is given as

$$S(\omega,\varphi)=\mathrm{Tr}\,D_\omega(\log D_\omega-\log D_\varphi),$$

where D_φ and D_ω are the densities of φ and ω with respect to Tr. Also let us introduce the following relative entropy quantities:

$$S_{\mathrm{co}}(\omega,\varphi)=\sup\left\{\sum_i\omega(q_i)\log\frac{\omega(q_i)}{\varphi(q_i)}:q_i\text{ are projections with }\sum_i q_i=I\right\}$$

and for $0<\varepsilon<1$

$$\beta_\varepsilon(\omega,\varphi)=\inf\{\log\varphi(q):q\text{ is a projection with }\omega(q)\ge 1-\varepsilon\}.$$

The quantity S_{co} appeared in [9] and may be related to measurements described by projection-valued measures. On the other hand, the quantity $\beta_\varepsilon(\omega,\varphi)$ has a natural meaning from the viewpoint of quantum hypotesis testing (see [5, 8, 14]). Let H_0 and H_1 be the hypotheses so that the system has states ω and φ under H_0 and H_1, respectively. A projection q can be regarded as a (quantum) question whose outcomes are the eigenvalues 1 and 0. The decision rule is that H_0 [or H_1] is true if the outcome of q is 1 [or 0]. Then $\varphi(q)$ [or $\omega(I-q)$] means the error probability of accepting H_0 [or H_1] when H_1 [or H_0] actually is true. So the quantity $\exp\{\beta_\varepsilon(\omega,\varphi)\}$ can be considered as the lower bound of the first error probability over all decision rules whose second error probability does not exceed ε.

Now let φ and ω be stationary states on $\mathcal{A}$. The mean relative entropy is defined as $S_{\mathrm{M}}(\omega,\varphi)=\lim_{n\to\infty}\frac{1}{n}S(\omega_n,\varphi_n)$ whenever the limit exists. When φ is a product state, we proved in [15] that for every stationary state ω

$$\lim_{n\to\infty}\frac{1}{n}S_{\mathrm{co}}(\omega_n,\varphi_n)=S_{\mathrm{M}}(\omega,\varphi), \tag{0.2}$$

and that for every completely ergodic state ω and $0<\varepsilon<1$

$$\limsup_{n\to\infty}\frac{1}{n}\beta_\varepsilon(\omega_n,\varphi_n)\le -S_{\mathrm{M}}(\omega,\varphi), \tag{0.3}$$

$$\liminf_{n\to\infty}\frac{1}{n}\beta_\varepsilon(\omega_n,\varphi_n)\ge -\frac{1}{1-\varepsilon}S_{\mathrm{M}}(\omega,\varphi). \tag{0.4}$$

The equality (0.2) means that the relative entropy and its variant S_{co} give rise to the same asymptotics in the infinite tensor product system. The inequalities (0.3) and (0.4) show that we obtain $\exp\{\frac{1}{n}\beta_\varepsilon(\omega_n,\varphi_n)\}\approx\exp\{-S_{\mathrm{M}}(\omega,\varphi)\}$ for large n and small ε.

Our main aim here is to extend the above relations (0.1)–(0.4) to the case when φ is a strongly mixing algebriac state. These are proved in Sections 2 and 3.

1. Preliminaries on algebraic states

Let $M_d(\mathbf{C})$ be the $d \times d$ matrix algebra and $\mathcal{A} = \bigotimes_{i\in\mathbf{Z}} \mathcal{A}_i$ be the infinite C^*-tensor product of $\mathcal{A}_i = M_d(\mathbf{C})$, $i \in \mathbf{Z}$. Given a subset K of $\mathbf{Z}$ we denote by $\mathcal{A}_K$ the C^*-tensor product $\bigotimes_{i\in K} \mathcal{A}_i$ which is viewed as a C^*-subalgebra of $\mathcal{A}$. Let γ denote the automorphism of translation, i.e. the right shift automorphism on $\mathcal{A}$.

First let us recall the definition of algebraic states. A linear map $\Phi : \mathcal{B} \to \mathcal{C}$ between C^*-algebras $\mathcal{B}$ and $\mathcal{C}$ is said to be completely positive if, for any $n \in \mathbf{N}$, $\Phi_n : \mathcal{B} \otimes M_n(\mathbf{C}) \to \mathcal{C} \otimes M_n(\mathbf{C})$ defined as $\Phi_n([x_{ij}]) = [\Phi(x_{ij})]$ for $[x_{ij}]_{i,j=1}^n \in \mathcal{B} \otimes M_n(\mathbf{C})$ $(= M_n(\mathcal{B}))$ is positive. Let a completely positive unital map $\mathcal{E} : M_d(\mathbf{C}) \otimes M_k(\mathbf{C}) \to M_k(\mathbf{C})$ and a state ω on $M_k(\mathbf{C})$ be given so that

$$\omega(\mathcal{E}(I \otimes x)) = \omega(x), \qquad x \in M_k(\mathbf{C}). \tag{1.1}$$

For each $a \in M_d(\mathbf{C})$ define $\mathcal{E}_a : M_k(\mathbf{C}) \to M_k(\mathbf{C})$ by $\mathcal{E}_a(x) = \mathcal{E}(a \otimes x)$, $x \in M_k(\mathbf{C})$. Then we can uniquely define a stationary (or γ-invariant) state φ on $\mathcal{A}$ as follows:

$$\varphi(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \omega(\mathcal{E}_{a_0} \circ \mathcal{E}_{a_1} \circ \cdots \circ \mathcal{E}_{a_n}(I))$$

for every $a_i \in \mathcal{A}_i$, $0 \le i \le n$. Here the complete positivity of $\mathcal{E}$ ensures the positivity of φ. We call this φ the algebraic state generated by $(\mathcal{E}, \omega)$ because of its algebraic manner of definition.

The notion of algebraic states was introduced in [12] under the name "C^*-finitely correlated states". This notion is closely related to that of quantum Markov states in [1]; in fact, algebraic states become (stationary) quantum Markov states in the sense of [1] when $M_k(\mathbf{C}) = M_d(\mathbf{C})$.

Thanks to [12, Lemma 2.5] we can assume that ω is faithful in the above definition of an algebraic state φ. In the sequel, the faithfulness of ω always is assumed when we refer to the algebraic state generated by $(\mathcal{E}, \omega)$.

The set of all stationary states on $\mathcal{A}$ forms a Choquet simplex. An extremal point of this set is called an ergodic state. It is known [12] that the set of algeraic states on $\mathcal{A}$ forms a face (in particular, a convex set) in the set of stationary states, so that an algebraic state is ergodic if and only if it is extremal in the set of algebraic states. A stationary state φ on $\mathcal{A}$ is said to be strongly mixing (or strongly clustering) if

$$\lim_{n\to\infty} \varphi(a\gamma^n(b)) = \varphi(a)\varphi(b), \qquad a, b \in \mathcal{A},$$

and weakly mixing if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\varphi(a\gamma^n(b)) - \varphi(a)\varphi(b)| = 0, \qquad a, b \in \mathcal{A}.$$

Moreover we say that φ is completely ergodic if it is ergodic for all γ^n, $n \ge 1$. For ergodicity in general C^*-dynamical systems, see [10, 25] for instance.

The next proposition characterizes the ergodicity and mixing properties for algebraic states.

Proposition 1.1. *Let φ be an algebraic state generated by $(\mathcal{E}, \omega)$.*

(1) φ is ergodic if and only if $\mathcal{E}_I$ is irreducible, i.e. I is the only eigenvector of $\mathcal{E}_I$ with respect to the eigenvalue 1.

(2) The following conditions are equivalent:

(i) φ *is strongly mixing;*

(ii) φ *is weakly mixing;*

(iii) φ *is completely ergodic;*

(iv) $\mathcal{E}_I$ *is primitive, i.e.* $\mathcal{E}_I^n$ *is irreducible for all* $n \in \mathbf{N}$*;*

(v) $\lim_{n\to\infty} \mathcal{E}_I^n(x) = \omega(x)I$ *for all* $x \in M_k(\mathbf{C})$*.*

Proof. (1) was given in [12, Proposition 3.1].

(2) (i)⇒(ii)⇒(iii) holds for a general stationary state. For any $n \in \mathbf{N}$ we can define a completely positive unital map $\mathcal{E}^{(n)} : \mathcal{A}_{[0,n-1]} \otimes M_k(\mathbf{C}) \to M_k(\mathbf{C})$ by

$$\mathcal{E}^{(n)}((a_0 \otimes \cdots \otimes a_{n-1}) \otimes x) = \mathcal{E}_{a_0} \circ \cdots \circ \mathcal{E}_{a_{n-1}}(x)$$

for $a_0 \otimes \cdots \otimes a_{n-1} \in \mathcal{A}_{[0,n-1]}$ and $x \in M_k(\mathbf{C})$. Then φ on $\mathcal{A} = \bigotimes_{i\in\mathbf{Z}} \mathcal{A}_{[ni,n(i+1)-1]}$ is the algebraic state generated by $(\mathcal{E}^{(n)}, \omega)$. Since $\mathcal{E}_I^{(n)} = \mathcal{E}_I^n$, (iii)⇒(iv) immediately follows from the first assertion (1). (iv)⇔(v) was given in [24, Theorem 6]. Finally let us show (v)⇒(i). For $a = a_0 \otimes \cdots \otimes a_m$ and $b = b_0 \otimes \cdots \otimes b_m$ in $\mathcal{A}_{[0,m]}$ we get for $n > m$

$$\varphi(a\gamma^n(b)) = \omega(\mathcal{E}_{a_0} \circ \cdots \circ \mathcal{E}_{a_m} \circ \mathcal{E}_I^{n-m-1} \circ \mathcal{E}_{b_0} \circ \cdots \circ \mathcal{E}_{b_m}(I)).$$

Hence (v) implies that

$$\begin{aligned}\lim_{n\to\infty} \varphi(a\gamma^n(b)) &= \omega(\mathcal{E}_{a_0} \circ \cdots \circ \mathcal{E}_{a_m}(I))\omega(\mathcal{E}_{b_0} \circ \cdots \circ \mathcal{E}_{b_m}(I)) \\ &= \varphi(a)\varphi(b).\end{aligned}$$

This shows that φ is strongly mixing. □

We have another characterization of the strong mixing for algebraic states as follows.

Proposition 1.2. *Let* φ *be the algebraic state generated by* $(\mathcal{E}, \omega)$*. Then* φ *is strongly mixing if and only if for any* $\alpha > 1$ *there exists* $l \in \mathbf{N}$ *such that for all* $m \in \mathbf{N}$

$$\begin{aligned}\alpha^{-1}(\varphi|\mathcal{A}_{(-\infty,m]}) \otimes (\varphi|\mathcal{A}_{[m+l+1,\infty)}) &\le \varphi|\mathcal{A}_{(-\infty,m]\cup[m+l+1,\infty)} \\ &\le \alpha(\varphi|\mathcal{A}_{(-\infty,m]}) \otimes (\varphi|\mathcal{A}_{[m+l+1,\infty)}).\end{aligned} \tag{1.2}$$

Proof. Suppose that φ is strongly mixing. We show that there exists $l \in \mathbf{N}$ such that

$$\alpha^{-1}\omega(\cdot)I \le \mathcal{E}_I^l \le \alpha\omega(\cdot)I \tag{1.3}$$

in the order of complete positivity; namely $\mathcal{E}_I^l - \alpha^{-1}\omega(\cdot)I$ and $\alpha\omega(\cdot)I - \mathcal{E}_I^l$ are completely positive maps of $M_k(\mathbf{C})$ into itself. For this sake it suffices due to [7] (also [22, Theorem 3.12]) to prove that for sufficiently large l

$$[\alpha^{-1}\omega(e_{ij})]_{i,j=1}^k \le [\mathcal{E}_I^l(e_{ij})]_{i,j=1}^k \le [\alpha\omega(e_{ij})]_{i,j=1}^k \tag{1.4}$$

in $M_k(M_k(\mathbf{C})) = M_{k^2}(\mathbf{C})$ with some matrix units $\{e_{ij}\}$ of $M_k(\mathbf{C})$. Here we can choose $\{e_{ij}\}$ so that the density of ω is written as $\sum_{i=1}^k \lambda_i e_{ii}$ where $\lambda_i > 0$. Since by Proposition 1.1(2)

$$\begin{aligned}\lim_{l\to\infty}[\mathcal{E}_I^l(e_{ij}) - \alpha^{-1}\omega(e_{ij})I] &= (1-\alpha^{-1})[\omega(e_{ij})I] \\ &= (1-\alpha^{-1})\begin{bmatrix} \lambda_1 I & 0 & \cdots & 0 \\ 0 & \lambda_2 I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k I \end{bmatrix}\end{aligned}$$

is strictly positive, we get the first inequality in (1.4) for large l. The second in (1.4) is similar. Hence there exists $l \in \mathbf{N}$ for which (1.3) holds. Now let $m, r, s \in \mathbf{N}$ with $r \le m$ and $m + l < s$. When $a_i \in \mathcal{A}_i$ for $i \in [r, m] \cup [m + l + 1, s]$,

$$\begin{aligned}&\varphi(a_r \otimes \cdots \otimes a_m \otimes I \otimes \cdots \otimes I \otimes a_{m+l+1} \otimes \cdots \otimes a_s) \\ &\quad = \omega(\mathcal{E}_{a_r} \circ \cdots \circ \mathcal{E}_{a_m} \circ \mathcal{E}_I^l \circ \mathcal{E}_{a_{m+l+1}} \circ \cdots \circ \mathcal{E}_{a_s}(I)).\end{aligned}$$

So it follows from (1.3) that

$$\begin{aligned}\alpha^{-1}(\varphi|\mathcal{A}_{[r,m]}) \otimes (\varphi|\mathcal{A}_{[m+l+1,s]}) &\le \varphi|\mathcal{A}_{[r,m]\cup[m+l+1,s]} \\ &\le \alpha(\varphi|\mathcal{A}_{[r,m]}) \otimes (\varphi|\mathcal{A}_{[m+l+1,s]}).\end{aligned}$$

Passing to the limit as $r \to -\infty$ and $s \to \infty$ we obtain (1.2).

Conversely suppose that for any $\alpha > 1$ there exists $l \in \mathbf{N}$ for which (1.2) holds. Let $a = a_0 \otimes \cdots \otimes a_m$ and $b = b_0 \otimes \cdots \otimes b_m$ with $a_i, b_i \in (\mathcal{A}_i)_+$, $0 \le i \le m$. When $n > m + l$, since $a \in (\mathcal{A}_{[0,m]})_+$ and $\gamma^n(b) \in (\mathcal{A}_{[m+l+1,\infty)})_+$, the assumption implies that

$$\alpha^{-1}\varphi(a)\varphi(b) \le \varphi(a\gamma^n(b)) \le \alpha\varphi(a)\varphi(b).$$

Hence $\varphi(a\gamma^n(b)) \to \varphi(a)\varphi(b)$, which shows the strong mixing of φ because $\mathcal{A}_{[0,m]}$ is the linear span of such elements as a, b above. □

Translationally invariant product states on $\mathcal{A}$ are known to be algebraic states (see [12]). In particular, the product state $\varphi = \bigotimes_{\mathbf{Z}} \varphi_0$ of a state φ_0 on $M_d(\mathbf{C})$ is a simplest example of strongly mixing algebraic states. Proposition 1.2 says that strongly mixing algebraic states have a certain property of approximately product type. This property will be very useful in the subsequent sections. Also note that the 'if' part of Proposition 1.2 is valid when φ is a general stationary state.

We close this section with typical examples of algebraic states. See [12] for more examples related to quantum statistical mechanics.

Example 1.3. We denote by $\{e_{rs}\}_{r,s=1}^d$ and $\{f_{ij}\}_{i,j=1}^k$ the usual matrix units of $M_d(\mathbf{C})$ and $M_k(\mathbf{C})$, respectively. Let E_r, $1 \le r \le d$, be $k \times k$ matrices with non-negative entries such that $E = \sum_{r=1}^d E_r$ is a stochastic matrix, i.e. $\sum_{j=1}^k E(i,j) = 1$, $1 \le i \le k$. Further let $(p_1, \ldots, p_k)$ be a probability distribution such that $\sum_{i=1}^k p_i E(i,j) = p_j$, $1 \le j \le k$. Define $\mathcal{E} : M_d(\mathbf{C}) \otimes M_k(\mathbf{C}) \to M_k(\mathbf{C})$ by

$$\mathcal{E}(e_{rs} \otimes f_{ij}) = \delta_{rs}\delta_{ij}\sum_{l=1}^k E_r(l,i)f_{ll},$$

and a state ω on $M_k(\mathbf{C})$ by $\omega(f_{ij}) = \delta_{ij}p_i$. Then $\mathcal{E}$ is a completely positive unital map and (1.1) is satisfied, so that $(\mathcal{E}, \omega)$ generates the algebraic state φ. This φ is essentially a state on the commutative C^*-subalgebra $\bigotimes_{\mathbf{Z}}(\bigoplus_{r=1}^{d} \mathbf{C}e_{rr}) = C(\{1,\dots,d\}^{\mathbf{Z}})$, the tensor product of the diagonal parts, which becomes an algebraic probability measure in the sense of [11]. (The definition of algebraic probability measures in [11] is formally a bit more general.)

In particular, let $M_k(\mathbf{C}) = M_d(\mathbf{C})$ and $[p_{ij}]_{i,j=1}^{d}$ be a stochastic matrix. When we set $E_r = [\delta_{ri}p_{ij}]$, the generated algebraic state is essentially a Markov state (in the classical sense) with the transition matrix $E = [p_{ij}]$, and the characterizations of its ergodicity and strong mixing are well known (see [4]).

Example 1.4. Let $(V_1, V_2, \dots, V_d)$ be an operational partition of unity in $M_k(\mathbf{C})$, i.e. $V_i \in M_k(\mathbf{C})$ and $\sum_{i=1}^{d} V_i^* V_i = I$, and let ω be a state on $M_k(\mathbf{C})$ such that

$$\omega\left(\sum_{i=1}^{d} V_i^* x V_i\right) = \omega(x), \qquad x \in M_k(\mathbf{C}).$$

(The notion of operational partitions of unity was used in [18].) Define $\mathcal{E} : M_d(\mathbf{C}) \otimes M_k(\mathbf{C}) \to M_k(\mathbf{C})$ by

$$\mathcal{E}(e_{ij} \otimes x) = V_i^* x V_j, \qquad 1 \le i, j \le d,\ x \in M_k(\mathbf{C}).$$

It is easy to check that $\mathcal{E}$ is a completely positive unital map satisfying (1.1). Then the algebraic state φ generated by $(\mathcal{E}, \omega)$ is given by

$$\varphi(e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes \cdots \otimes e_{i_n j_n}) = \omega(V_{i_1}^* \cdots V_{i_n}^* V_{j_n} \cdots V_{j_1}).$$

In the above setup we may assume that ω is faithful (if not, V_i can be replaced by $V_i e$ where $e = \operatorname{supp}\omega$). Then it is seen that $\mathcal{E}_I$ is irreducible if and only if

$$\operatorname{span}\{V_{i_1}^* \cdots V_{i_n}^* \xi : 1 \le i_1, \dots, i_n \le d,\ n \ge 0\} = \mathbf{C}^k \tag{1.5}$$

for every nonzero $\xi \in \mathbf{C}^k$. In fact, let p denote the projection onto the subspace in the left-hand side of (1.5). Since $pV_i^* p = V_i^* p$, we get $\mathcal{E}_I(p) = \sum_{i=1}^{d} V_i^* p V_i \le p$, which implies $\mathcal{E}_I(p) = p$ because $\omega(p - \mathcal{E}_I(p)) = 0$. Hence the irreducibility of $\mathcal{E}_I$ implies $p = I$. Conversely suppose that $\mathcal{E}_I$ is not irreducible. Then $\mathcal{E}_I(x) = x$ for some $x \in M_k(\mathbf{C})$ with $x \notin \mathbf{C}I$. Such x can be selfadjoint. Writing $x = \sum_i \lambda_i p_i$ where $\lambda_1 > \lambda_2 > \dots$, since

$$\begin{aligned} 0 \le p_1 \mathcal{E}_I(I - p_1) p_1 &\le (\lambda_1 - \lambda_2)^{-1} p_1 \mathcal{E}_I(\lambda_1 I - x) p_1 \\ &= (\lambda_1 - \lambda_2)^{-1} p_1 (\lambda_1 I - x) p_1 = 0, \end{aligned}$$

we get $p_1 = \mathcal{E}_I(p_1) p_1 \le \mathcal{E}_I(p_1)$ and so $\mathcal{E}_I(p_1) = p_1$. This implies that $(I - p_1) V_i^* p_1 = 0$ for $1 \le i \le d$. Hence (1.5) does not hold for $\xi \in p_1 \mathbf{C}^k$.

For instance, when (V_i) is (e_{21}, e_{12}) in $M_2(\mathbf{C})$ and ω is the tracial state, $\mathcal{E}_I$ is irreducible but $\mathcal{E}_I^2$ is not; so by Proposition 1.1 the generated algebraic state φ is ergodic and not strongly mixing. When (V_i) is $(e_{11}/\sqrt{2}, e_{21}/\sqrt{2}, e_{12})$ in $M_2(\mathbf{C})$ and ω has the density $\frac{2}{3}e_{11} + \frac{1}{3}e_{22}$, $\mathcal{E}_I^n$ is irreducible for all $n \in \mathbf{N}$ and hence φ is strongly mixing.

2. Relative entropy densities

The theory of quantum relative entropy was studied first in [27] for normal positive functionals of semifinite von Neumann algebras. Extension to more general setups were done in [2, 3, 26] and so on. See [20] for details on the relative entropy. We here need the relative entropy of states on finite-dimensional C^*-algebras. So we mention the definition just in this case.

Let $\mathcal{B}$ be a finite-dimensional C^*-algebra with the canonical trace $\mathrm{Tr}_{\mathcal{B}}$ such that $\mathrm{Tr}(e) = 1$ for all one-dimensional projections e in $\mathcal{B}$. For two states φ and ω on $\mathcal{B}$ let D_φ and D_ω be the densities of φ and ω, respectively, with respect to $\mathrm{Tr}_{\mathcal{B}}$, whose support projections are denoted by $\mathrm{supp}\, D_\varphi$ and $\mathrm{supp}\, D_\omega$. Then the (Umegaki) relative entropy of ω with respect to φ is defined by

$$S(\omega, \varphi) = \begin{cases} \mathrm{Tr}_{\mathcal{B}} D_\omega (\log D_\omega - \log D_\varphi), & \mathrm{supp}\, D_\omega \le \mathrm{supp}\, D_\varphi, \\ +\infty, & \text{otherwise.} \end{cases}$$

When $\mathcal{B}$ is commutative so that $\mathcal{B} = \bigoplus_{i=1}^n \mathbf{C} e_i$ with minimal projections $\{e_i\}$, $S(\omega, \varphi)$ is given as

$$S(\omega, \varphi) = \sum_{i=1}^n \omega(e_i) \log \frac{\omega(e_i)}{\varphi(e_i)},$$

which is called also the (Kullback-Leibler) information divergence. In this connection, besides $S(\omega, \varphi)$ we may define a variant of relative entropy as follows:

$$S_{\mathrm{co}}(\omega, \varphi) = \sup\left\{ \sum_i \omega(q_i) \log \frac{\omega(q_i)}{\varphi(q_i)} : q_1, \dots, q_n \text{ are projections in } \mathcal{B}, \sum_i q_i = I \right\},$$

or equivalently

$$S_{\mathrm{co}}(\omega, \varphi) = \sup\{ S(\omega|\mathcal{C}, \varphi|\mathcal{C}) : \mathcal{C} \text{ is a commutative } C^*\text{-subalgebra of } \mathcal{B} \}.$$

The inequality $S_{\mathrm{co}}(\omega, \varphi) \le S(\omega, \varphi)$ follows from the monotonicity of relative entropy. It is known [23] that D_ω must commute with D_φ when $S_{\mathrm{co}}(\omega, \varphi) = S(\omega, \varphi) < +\infty$.

For $0 < \varepsilon < 1$ we define another type of relative entropy quantity as follows:

$$\beta_\varepsilon(\omega, \varphi) = \inf\{ \log \varphi(q) : q \text{ is a projection in } \mathcal{B},\ \omega(q) \ge 1 - \varepsilon \}.$$

Now let $\mathcal{A} = \bigotimes_{i \in \mathbf{Z}} \mathcal{A}_i$, $\mathcal{A}_i = M_d(\mathbf{C})$, be as in Section 1. Let φ and ω be stationary states on $\mathcal{A}$. Put $\omega_n = \omega|\mathcal{A}_{[1,n]}$ and $\varphi_n = \varphi|\mathcal{A}_{[1,n]}$ for $n \in \mathbf{N}$. Then the mean relative entropy $S_{\mathrm{M}}(\omega, \varphi)$ is defined by

$$S_{\mathrm{M}}(\omega, \varphi) = \lim_{n\to\infty} \frac{1}{n} S(\omega_n, \varphi_n)$$

whenever the limit exists (including $+\infty$).

The relations (0.2)–(0.4) between $S_{\mathrm{M}}(\omega, \varphi)$ and the asymptotic limits of $S_{\mathrm{co}}(\omega_n, \varphi_n)$ and $\beta_\varepsilon(\omega_n, \varphi_n)$ were obtained in [15] when φ is a product state $\bigotimes_{\mathbf{Z}} \varphi_0$. In this section we prove that the same relations hold also when φ is a strongly mixing algebraic state.

Theorem 2.1. *Let φ be a strongly mixing algebraic state on $\mathcal{A}$. Then for every stationary state ω on $\mathcal{A}$, $S_{\mathrm{M}}(\omega,\varphi)$ exists and*

$$\lim_{n\to\infty}\frac{1}{n}S_{\mathrm{co}}(\omega_n,\varphi_n)=S_{\mathrm{M}}(\omega,\varphi).$$

Proof. Let $\alpha>1$ and choose l as in Proposition 1.2. For each $m,n\in\mathbf{N}$ with $n\ge m+l$ write $n=r(m+l)+s$ where $0\le s<m+l$. Repeated use of Proposition 1.2 implies that

$$\begin{aligned}\alpha^{-r}\bigotimes_{i=0}^{r-1}(\varphi|\mathcal{A}_{[i(m+l)+1,i(m+l)+m]})&\le\varphi|\mathcal{A}_{\bigcup_{i=0}^{r-1}[i(m+l)+1,i(m+l)+m]}\\&\le\alpha^{r}\bigotimes_{i=0}^{r-1}(\varphi|\mathcal{A}_{[i(m+l)+1,i(m+l)+m]}).\end{aligned}\tag{2.1}$$

Put $\mathcal{B}_i=\mathcal{A}_{[i(m+l)+1,i(m+l)+m]}$ for $i\in\mathbf{Z}$. By the monotonicity of relative entropy and the above inequality we have

$$\begin{aligned}S_{\mathrm{co}}(\omega_n,\varphi_n)&\ge S_{\mathrm{co}}\left(\omega\,\Big|\bigotimes_{i=0}^{r-1}\mathcal{B}_i,\ \varphi\,\Big|\bigotimes_{i=0}^{r-1}\mathcal{B}_i\right)\\&\ge S_{\mathrm{co}}\left(\omega\,\Big|\bigotimes_{i=0}^{r-1}\mathcal{B}_i,\ \bigotimes_{i=0}^{r-1}(\varphi|\mathcal{B}_i)\right)-r\log\alpha.\end{aligned}\tag{2.2}$$

Now let us consider the infinite C^*-tensor product $\tilde{\mathcal{A}}=\bigotimes_{i\in\mathbf{Z}}\mathcal{B}_i$ of $\mathcal{B}_i$ $(=\mathcal{A}_{[1,m]})$, which is a C^*-subalgebra of $\mathcal{A}$. Let $\tilde{\varphi}=\bigotimes_{i\in\mathbf{Z}}(\varphi|\mathcal{B}_i)$ and $\tilde{\omega}=\omega|\tilde{\mathcal{A}}$. Then $\tilde{\varphi}$ is the product state of $\varphi_m=\varphi|\mathcal{A}_{[1,m]}$ and $\tilde{\omega}$ is a stationary state on $\tilde{\mathcal{A}}$. Hence we can apply [15, Theorem 2.1] (or (0.2)) to $\tilde{\varphi}$ and $\tilde{\omega}$, so that

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{r}S_{\mathrm{co}}\left(\omega\,\Big|\bigotimes_{i=0}^{r-1}\mathcal{B}_i,\ \bigotimes_{i=0}^{r-1}(\varphi|\mathcal{B}_i)\right)&=\lim_{r\to\infty}\frac{1}{r}S_{\mathrm{co}}(\tilde{\omega}_r,\tilde{\varphi}_r)\\&=S_{\mathrm{M}}(\tilde{\omega},\tilde{\varphi})\\&\ge S(\omega_m,\varphi_m).\end{aligned}\tag{2.3}$$

The inequality in (2.3) is seen by [15, (2.1)]. Since $r/n\to 1/(m+l)$ as $n\to\infty$ (hence $r\to\infty$), (2.2) and (2.3) imply that

$$\liminf_{n\to\infty}\frac{1}{n}S_{\mathrm{co}}(\omega_n,\varphi_n)\ge\frac{1}{m+l}S(\omega_m,\varphi_m)-\frac{1}{m+l}\log\alpha.\tag{2.4}$$

Letting $m\to\infty$, l being fixed, we obtain

$$\liminf_{n\to\infty}\frac{1}{n}S_{\mathrm{co}}(\omega_n,\varphi_n)\ge\limsup_{n\to\infty}\frac{1}{n}S(\omega_n,\varphi_n).$$

This completes the proof because $S_{\mathrm{co}}(\omega_n,\varphi_n)\le S(\omega_n,\varphi_n)$. □

Theorem 2.2. *Let φ be a strongly mixing algebraic state and ω a completely ergodic state on $\mathcal{A}$. Then for every $0 < \varepsilon < 1$,*

$$\limsup_{n\to\infty} \frac{1}{n}\beta_\varepsilon(\omega_n, \varphi_n) \le -S_{\mathrm{M}}(\omega, \varphi), \tag{2.5}$$

$$\liminf_{n\to\infty} \frac{1}{n}\beta_\varepsilon(\omega_n, \varphi_n) \ge -\frac{1}{1-\varepsilon}S_{\mathrm{M}}(\omega, \varphi). \tag{2.6}$$

Proof. We use the same notations as in the proof of Theorem 2.1. Since the complete ergodicity of ω implies that of $\tilde{\omega}$ with respect to the right shift on $\tilde{\mathcal{A}} = \bigotimes_{i\in\mathbf{Z}} \mathcal{B}_i$, we can apply [15, (2.3)] (or (0.3)) to $\tilde{\varphi}$ and $\tilde{\omega}$, so that

$$\limsup_{r\to\infty} \frac{1}{r}\beta_\varepsilon(\tilde{\omega}_r, \tilde{\varphi}_r) \le -S_{\mathrm{M}}(\tilde{\omega}, \tilde{\varphi}) \le -S(\omega_m, \varphi_m). \tag{2.7}$$

Let q be a projection in $\bigotimes_{i=0}^{r-1} \mathcal{B}_i$ such that $\tilde{\omega}(q) \ge 1-\varepsilon$. Then since $q \in \mathcal{A}_{[1,n]}$ and $\omega(q) = \tilde{\omega}(q)$, we get by (2.1)

$$\beta_\varepsilon(\omega_n, \varphi_n) \le \log\varphi(q) \le \log\tilde{\varphi}(q) + r\log\alpha.$$

Therefore

$$\beta_\varepsilon(\omega_n, \varphi_n) \le \beta_\varepsilon(\tilde{\omega}_r, \tilde{\varphi}_r) + r\log\alpha.$$

This together with (2.7) implies that

$$\limsup_{n\to\infty} \frac{1}{n}\beta_\varepsilon(\omega_n, \varphi_n) \le -\frac{1}{m+l}S(\omega_m, \varphi_m) + \frac{1}{m+l}\log\alpha.$$

Letting $m \to \infty$ we obtain (2.5).

To prove (2.6), first suppose $S_{\mathrm{M}}(\omega, \varphi) = 0$. Then $S(\omega_m, \varphi_m) \le \log\alpha$ is seen by (2.4) and Theorem 2.1. Here we can let $\alpha \to 1$ and $m \to \infty$ independently, so that $\omega = \varphi$ follows. Therefore (2.6) is obvious. Next suppose $S_{\mathrm{M}}(\omega, \varphi) > 0$. Then the proof of (2.6) is the same as that of [15, (2.4)] (or (0.4)), where we did not use the assumption of φ being a product state. □

As was explained in Introduction, the quantity $\exp\{\beta_\varepsilon(\omega_n, \varphi_n)\}$ has a meaning as the lower bound of an error probability in the quantum hypothesis test for $\{\omega_n, \varphi_n\}$. So Theorem 2.2 says that $\exp\{-S_{\mathrm{M}}(\omega, \varphi)\}$ is a certain kind of asymptotic error bound in the quantum hypothesis test for $\{\omega, \varphi\}$.

It may be conjectured that $\lim_{n\to\infty} \frac{1}{n}\beta_\varepsilon(\omega_n, \varphi_n) = -S_{\mathrm{M}}(\omega, \varphi)$ holds under suitable ergodicity assumptions of φ and ω. However this equality is not known even when both φ and ω are product states.

3. Entropy densities

For $n \in \mathbf{N}$ let Tr_n denote the usual trace on $\mathcal{A}_{[1,n]}$, i.e. $\mathrm{Tr}_n = \bigotimes_1^n \mathrm{Tr}$, Tr being the usual trace on $M_d(\mathbf{C})$. Let φ be a stationary state on $\mathcal{A}$ and $\varphi_n = \varphi|\mathcal{A}_{[1,n]}$. The (von Neumann) entropy $S(\varphi_n)$ of φ_n is given as

$$S(\varphi_n) = -\mathrm{Tr}_n\, D_n \log D_n,$$

where D_n is the density of φ_n with respect to Tr_n. Then the limit

$$s(\varphi) = \lim_{n\to\infty} \frac{1}{n} S(\varphi_n)$$

exists, which is called the mean entropy of φ. Indeed the mean entropy can be defined for translation-invariant states on C^*-algebras of $\mathbf{Z}^\nu$-lattice spin systems, and it plays an important role in the theory of quantum statistical mechanics of spin systems (see [6, 17] for instance).

For $0 < \varepsilon < 1$ let us define

$$\beta_\varepsilon(\varphi_n) = \inf\{\log \mathrm{Tr}_n(q) : q \in \mathcal{A}_{[1,n]} \text{ is a projection with } \varphi_n(q) \ge 1-\varepsilon\}.$$

We can give a more concrete formula of $\beta_\varepsilon(\varphi_n)$ as follows. Let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{d^n}$ be the eigenvalue list of D_n in decreasing order (counting multiplicities). Define

$$N_\varepsilon(\varphi_n) = \min\left\{N : \sum_{i=1}^N \lambda_i \ge 1-\varepsilon\right\}.$$

Then it is obvious that $\beta_\varepsilon(\varphi_n) = \log N_\varepsilon(\varphi_n)$.

It was shown in [21] that when φ is a product state, the following holds for every $0 < \varepsilon < 1$:

$$\lim_{n\to\infty} \frac{1}{n} \beta_\varepsilon(\varphi_n) = s(\varphi). \tag{3.1}$$

Moreover we proved in [16] that the relation (3.1) extends to the case when φ is an ergodic Gibbs state in a $\mathbf{Z}^\nu$-quantum lattice system.

In this section we prove that (3.1) holds also when φ is a strongly mixing algebraic state. First let us give a result which is weaker than (3.1) but is valid for any completely ergodic state.

Proposition 3.1. *If φ is a completely ergodic state on $\mathcal{A}$, then for every $0 < \varepsilon < 1$,*

$$\limsup_{n\to\infty} \frac{1}{n} \beta_\varepsilon(\varphi_n) \le s(\varphi), \tag{3.2}$$

$$\liminf_{n\to\infty} \frac{1}{n} \beta_\varepsilon(\varphi_n) \ge \frac{1}{1-\varepsilon} s(\varphi) - \frac{\varepsilon}{1-\varepsilon} \log d.$$

Proof. Let τ denote the tracial state on $\mathcal{A}$, i.e. the product state of the normalized trace on $M_d(\mathbf{C})$. Then [15, Theorem 2.2] (also Theorem 2.2) says that

$$\limsup_{n\to\infty} \frac{1}{n} \beta_\varepsilon(\varphi_n, \tau_n) \le -S_{\mathrm{M}}(\varphi, \tau),$$

$$\liminf_{n\to\infty}\frac{1}{n}\beta_\varepsilon(\varphi_n,\tau_n)\ge-\frac{1}{1-\varepsilon}S_{\mathrm{M}}(\varphi,\tau).$$

Since $\tau_n=d^{-n}\mathrm{Tr}_n$, we get

$$\frac{1}{n}\beta_\varepsilon(\varphi_n,\tau_n)=\frac{1}{n}\beta_\varepsilon(\varphi_n)-\log d.$$

Furthermore

$$-S_{\mathrm{M}}(\varphi,\tau)=\lim_{n\to\infty}\frac{1}{n}\{S(\varphi_n)+\log d^{-n}\}=s(\varphi)-\log d.$$

These yield the desired inequalities. □

Theorem 3.2. *Let φ be a strongly mixing algebraic state on $\mathcal{A}$. Then for every $0<\varepsilon<1$,*

$$\lim_{n\to\infty}\frac{1}{n}\beta_\varepsilon(\varphi_n)=s(\varphi).$$

We divide the proof of the theorem into several lemmas. It suffices by (3.2) to prove that

$$\liminf_{n\to\infty}\frac{1}{n}\beta_\varepsilon(\varphi_n)\ge s(\varphi). \tag{3.3}$$

Assume that φ is strongly mixing algebraic state. Let $\alpha>1$ and choose l as in Proposition 1.2, so that (2.1) holds for all $r\in\mathbf{N}$. For each $m,n\in\mathbf{N}$ write $n=r(m+l)+s$ where $0\le s<m+l$ and put

$$\mathcal{B}_i=\mathcal{A}_{[i(m+l)+1,i(m+l)+m]},\qquad 0\le i\le r-1,$$

$$\mathcal{C}_n=\mathcal{A}_{[1,n]\setminus\bigcup_{i=0}^{r-1}[i(m+l)+1,i(m+l)+m]}.$$

Then $\mathcal{A}_{[1,n]}=(\bigotimes_{i=0}^{r-1}\mathcal{B}_i)\otimes\mathcal{C}_n$. Besides φ_n and D_n, define states φ'_n and $\hat\varphi_n$ on $\mathcal{A}_{[1,n]}$ by

$$\varphi'_n=\left(\varphi\,\middle|\,\bigotimes_{i=0}^{r-1}\mathcal{B}_i\right)\otimes(\tau|\mathcal{C}_n),$$

$$\hat\varphi_n=\left(\bigotimes_{i=0}^{r-1}(\varphi|\mathcal{B}_i)\right)\otimes(\tau|\mathcal{C}_n),$$

and let D'_n and $\hat D_n$ be the densities of φ'_n and $\hat\varphi_n$, respectively, with respect to Tr_n. Of course, D'_n and $\hat D_n$ depend on l,m too, while for simplicity we omit them from the notations.

Lemma 3.3. *In the above notations,*

$$\alpha^{-r}\hat D_n\le D'_n\le\alpha^r\hat D_n, \tag{3.4}$$

$$D'_n=d^{-(rl+s)}\frac{d(\varphi|\bigotimes_{i=0}^{r-1}\mathcal{B}_i)}{d\mathrm{Tr}}, \tag{3.5}$$

$$\hat D_n=d^{-(rl+s)}\bigotimes_{i=0}^{r-1}\gamma^{i(m+l)}(D_m), \tag{3.6}$$

$$\operatorname{supp} D_n' = \operatorname{supp} \hat{D}_n = \bigotimes_{i=0}^{r-1} \gamma^{i(m+l)}(\operatorname{supp} D_m), \tag{3.7}$$

where Tr *in (3.5) is the canonical trace on* $\bigotimes_{i=0}^{r-1} \mathcal{B}_i$.

Proof. We have $\alpha^{-r}\hat{\varphi}_n \le \varphi_n' \le \alpha^r \hat{\varphi}_n$ by (2.1), so that (3.4) holds. Since

$$\frac{d(\tau|\mathcal{C}_n)}{d\mathrm{Tr}_{\mathcal{C}_n}} = d^{-(n-rm)} I_{\mathcal{C}_n} = d^{-(rl+s)} I_{\mathcal{C}_n},$$

(3.5) and (3.6) are immediate. (3.7) follows from (3.4) and (3.6). □

Let π_φ be the GNS representation of $\mathcal{A}$ induced by φ with the cyclic vector Ω_φ and the implementing unitary V_φ; namely $\varphi(a) = \langle \pi_\varphi(a)\Omega_\varphi, \Omega_\varphi\rangle$ and $\pi_\varphi(\gamma(a)) = V_\varphi \pi_\varphi(a) V_\varphi^*$ for $a \in \mathcal{A}$.

Lemma 3.4. *As* $n \to \infty$ *(hence* $r \to \infty$*),*

$$\left\| \pi_\varphi\left(\frac{1}{r}\sum_{i=0}^{r-1} \gamma^{i(m+l)}(-(\operatorname{supp} D_m)\log D_m)\right)\Omega_\varphi - S(\varphi_m)\Omega_\varphi \right\| \to 0.$$

Proof. Since φ is ergodic with respect to γ^{m+l}, the mean ergodic theorem says that

$$\begin{aligned} &\pi_\varphi\left(\frac{1}{r}\sum_{i=0}^{r-1} \gamma^{i(m+l)}(-(\operatorname{supp} D_m)\log D_m)\right)\Omega_\varphi \\ &\qquad = \frac{1}{r}\sum_{i=0}^{r-1} V_\varphi^{i(m+l)} \pi_\varphi(-(\operatorname{supp} D_m)\log D_m)\Omega_\varphi \end{aligned}$$

converges to

$$\varphi(-(\operatorname{supp} D_m)\log D_m)\Omega_\varphi = S(\varphi_m)\Omega_\varphi,$$

as desired. □

Lemma 3.5. *For any* $\delta > 0$ *let* p_n *be the spectral projection of* D_n' *corresponding to the interval* $[0, \exp\{-n(s(\varphi)-\delta)\})$*. If* m *is sufficiently large, then* $\varphi(p_n) \to 1$ *as* $n \to \infty$.

Proof. We can choose m such that

$$s(\varphi) - \delta < \frac{1}{m+l} S(\varphi_m) - \frac{1}{m+l}\log\alpha.$$

Let

$$a_n = \frac{1}{n}\sum_{i=0}^{r-1} \gamma^{i(m+l)}(-(\operatorname{supp} D_m)\log D_m) - \frac{r}{n}\log\alpha,$$

$$\xi = \frac{1}{m+l} S(\varphi_m) - \frac{1}{m+l}\log\alpha.$$

Then since $r/n \to 1/(m+l)$ as $n \to \infty$, Lemma 3.4 implies that $||(\pi_\varphi(a_n)-\xi)\Omega_\varphi|| \to 0$. Moreover

$$\begin{aligned} a_n &= (\operatorname{supp} \hat{D}_n)\Big(-\frac{1}{n}\log \hat{D}_n\Big) - \frac{r}{n}\log\alpha \\ &\le (\operatorname{supp} D'_n)\Big(-\frac{1}{n}\log D'_n\Big) \end{aligned}$$

by (3.7), (3.6), and (3.4) due to the operator monotonicity of $\log t$, $t > 0$. Since $I - p_n \le \operatorname{supp} D'_n$ and $I - p_n$ is the spectral projection of $(\operatorname{supp} D'_n)(-\frac{1}{n}\log D'_n)$ corresponding to $(-\infty, s(\varphi) - \delta]$, we have

$$\begin{aligned} (I-p_n)(a_n-\xi)(I-p_n) &\le (I-p_n)\Big(-\frac{1}{n}\log D'_n - \xi\Big) \\ &\le (s(\varphi)-\delta-\xi)(I-p_n). \end{aligned}$$

Therefore

$$(s(\varphi)-\delta-\xi)\varphi(I-p_n) \ge \varphi((I-p_n)(a_n-\xi)), \tag{3.8}$$

because p_n belongs to the centralizer of $\varphi|\bigotimes_{i=0}^{r-1}\mathcal{B}_i$ by (3.5). Since $s(\varphi)-\delta-\xi < 0$ and

$$\begin{aligned} \varphi((I-p_n)(a_n-\xi)) &= \langle(\pi_\varphi(a_n)-\xi)\Omega_\varphi, (I-\pi_\varphi(p_n))\Omega_\varphi\rangle \\ &\le ||(\pi_\varphi(a_n)-\xi)\Omega_\varphi|| \to 0, \end{aligned}$$

(3.8) implies taht $\varphi(I-p_n) \to 0$ and hence $\varphi(p_n) \to 1$. □

Lemma 3.6. *For any $\eta > 0$, if m is large enough, then*

$$\limsup_{n\to\infty} \frac{1}{n} S(\varphi_n, \varphi'_n) < \eta.$$

Proof. Noting that $\operatorname{supp} D_n \le \operatorname{supp} D'_n$, we have by (3.5)

$$\begin{aligned} S(\varphi_n, \varphi'_n) &= \operatorname{Tr}_n(D_n \log D_n - D_n \log D'_n) \\ &= -S(\varphi_n) - \operatorname{Tr}_n\Big(D_n \log \frac{d(\varphi|\bigotimes_{i=0}^{r-1}\mathcal{B}_i)}{d\mathrm{Tr}}\Big) + (rl+s)\log d \\ &= -S(\varphi_n) + S\Bigg(\varphi\,\Bigg|\bigotimes_{i=0}^{r-1}\mathcal{B}_i\Bigg) + (rl+s)\log d \\ &\le -S(\varphi_n) + \sum_{i=0}^{r-1} S(\varphi|\mathcal{B}_i) + (rl+s)\log d \\ &= -S(\varphi_n) + rS(\varphi_m) + (rl+s)\log d. \end{aligned}$$

In the above estimate, the inequality follows from the subadditivity of entropy. Hence

$$\limsup_{n\to\infty} \frac{1}{n} S(\varphi_n, \varphi'_n) \le -s(\varphi) + \frac{1}{m+l} S(\varphi_m) + \frac{l}{m+l}\log d.$$

This yields the conclusion because the right-hand side tends to 0 as $m \to \infty$. □

Lemma 3.7. *Let* $a_n \in \mathcal{A}_n$*,* $0 \le a_n \le I$*, and* $0 < \varepsilon' < 1$*. If*

$$\liminf_{n\to\infty} \varphi(a_n) \ge 1 - \varepsilon',$$

then for any $\delta > 0$

$$\liminf_{n\to\infty} \frac{1}{n} \log \varphi'_n(a_n) \ge -\delta$$

whenever m *is sufficiently large.*

Proof. By Lemma 3.6 we get

$$\limsup_{n\to\infty} \frac{1}{n} S(\varphi_n, \varphi'_n) < \delta(1 - \varepsilon') \tag{3.9}$$

whenever m is sufficiently large. Suppose

$$\liminf_{n\to\infty} \frac{1}{n} \log \varphi'_n(a_n) < -\delta.$$

Then choosing a subsequence we may assume that $\varphi'_n(a_n) < e^{-n\delta}$ for all n. As in the proof of [15, (2.4)], letting

$$F(s,t) = s \log \frac{s}{t} + (1-s) \log \frac{1-s}{1-t}, \qquad 0 \le s, t \le 1,$$

we can estimate for n large enough

$$\begin{aligned} S(\varphi_n, \varphi'_n) &\ge F(\varphi_n(a_n), \varphi'_n(a_n)) \\ &\ge F(\varphi_n(a_n), e^{-n\delta}) \\ &\ge n\delta\varphi(a_n) - \log 2. \end{aligned}$$

Therefore

$$\liminf_{n\to\infty} \frac{1}{n} S(\varphi_n, \varphi'_n) \ge \delta \liminf_{n\to\infty} \varphi(a_n) \ge \delta(1 - \varepsilon'),$$

contradicting (3.9). □

Proof of (3.3). For each n choose a projection q_n in $\mathcal{A}_{[1,n]}$ such that $\varphi(q_n) \ge 1 - \varepsilon$ and

$$\log \mathrm{Tr}_n(q_n) < \beta_\varepsilon(\varphi_n) + 1. \tag{3.10}$$

For any $\delta > 0$ let p_n be as in Lemma 3.5 with m being large enough. Then $\varphi(p_n) \to 1$ by Lemma 3.5. Since

$$p_n \ge \exp\{n(s(\varphi) - \delta)\} D'_n p_n,$$

we get

$$\mathrm{Tr}_n(q_n) \ge \mathrm{Tr}_n(p_n q_n) \ge \exp\{n(s(\varphi) - \delta)\} \varphi'_n(p_n q_n),$$

so that

$$\frac{1}{n} \log \mathrm{Tr}_n(q_n) \ge s(\varphi) - \delta + \frac{1}{n} \log \varphi'_n(p_n q_n p_n). \tag{3.11}$$

On the other hand, since

$$\begin{aligned}\varphi(I-p_nq_np_n) &= \varphi(I-p_n)+\varphi(p_n(I-q_n))+\varphi(p_nq_n(I-p_n)) \\ &\le \varphi(I-p_n)+\varphi(p_n)^{1/2}\varphi(I-q_n)^{1/2}+\varphi(p_nq_np_n)^{1/2}\varphi(I-p_n)^{1/2} \\ &\le \varphi(I-p_n)+\varepsilon^{1/2}+\varphi(I-p_n)^{1/2},\end{aligned}$$

we get

$$\limsup_{n\to\infty}\varphi(I-p_nq_np_n)\le\varepsilon^{1/2},$$

that is,

$$\liminf_{n\to\infty}\varphi(p_nq_np_n)\ge 1-\varepsilon^{1/2}.$$

Hence Lemma 3.7 implies that

$$\liminf_{n\to\infty}\frac{1}{n}\log\varphi_n'(p_nq_np_n)\ge-\delta. \tag{3.12}$$

By (3.10)–(3.12) we obtain

$$\liminf_{n\to\infty}\frac{1}{n}\beta_\varepsilon(\varphi_n)\ge s(\varphi)-\delta,$$

showing (3.3). □

The conclusion of Theorem 3.2 says that for every $0<\varepsilon<1$ and $\delta>0$ we obtain

$$\exp\{n(s(\varphi)-\delta)\}\le N_\varepsilon(\varphi_n)\le\exp\{n(s(\varphi)+\delta)\}$$

when n is large. In particular, this means that the probability distribution consisting of the eigenvalues of D_n is concentrated on a smaller and smaller proportion of eigenprojections as $n\to\infty$ whenever $s(\varphi)<\log d$. The limit property of this type resembles the asymptotic equipartition property in information theory (see [4] for instance) and expresses the macroscopic uniformity from the viewpoint of statistical mechanics (see [13, p. 76–77]).

4. Concluding remarks

(1) Let τ be the tracial state and φ be any stationary state on $\mathcal{A}$. Then for $0<\varepsilon<1/2$ we have $S_{\mathrm{M}}(2\varepsilon\varphi+(1-2\varepsilon)\tau,\tau)>0$ but

$$\lim_{n\to\infty}\frac{1}{n}\beta_\varepsilon(2\varepsilon\varphi_n+(1-2\varepsilon)\tau_n,\tau_n)=0$$

according to [15, Example 2.4(1)]. Since

$$\frac{1}{n}\beta_\varepsilon(2\varepsilon\varphi_n+(1-2\varepsilon)\tau_n,\tau_n)=\frac{1}{n}\beta_\varepsilon(2\varepsilon\varphi_n+(1-2\varepsilon)\tau_n)-\log d$$

as in the proof of Proposition 3.1, we get

$$\lim_{n\to\infty}\frac{1}{n}\beta_\varepsilon(2\varepsilon\varphi_n+(1-2\varepsilon)\tau_n)=\log d.$$

On the other hand, the affinity of mean entropy yields

$$s(2\varepsilon\varphi+(1-2\varepsilon)\tau)=2\varepsilon s(\varphi)+(1-2\varepsilon)\log d.$$

These estimates show that Theorem 2.2 and Proposition 3.1 (also Theorem 3.2) cannot hold without the ergodicity assumptions of ω and φ. (See also [16, the remark after Theorem 3.5].) It may be possible that Theorem 3.2 holds true for any ergodic algebraic state (or more generally for any ergodic state).

(2) In Sections 2 and 3 we only used the property of strongly mixing algebraic states described in Proposition 1.2. Moreover we needed to let $\alpha\to 1$ just once in the proof of (2.6). So all the results of Sections 2 and 3 except (2.6) are valid for a stationary state φ on $\mathcal{A}$ such that the relation (1.2) is satisfied for some $\alpha>1$ and some $l\in\mathbf{N}$.

(3) In the definition of algebraic states, the finite-dimensionality of $M_k(\mathbf{C})$ is essential but that of $M_d(\mathbf{C})$ is not. Indeed, algebraic states can be formulated on the infinite C^*-tensor product $\mathcal{A}=\bigotimes_{i\in\mathbf{Z}}\mathcal{A}_i$ of $\mathcal{A}_i=\mathcal{A}_0$ where $\mathcal{A}_0$ is an arbitrary unital C^*-algebra (see [12]). In this extended setting, the characterizations of the strong mixing for algebraic states in Section 1 are valid without any change. Furthermore the results in [15] extend to this setting as was remarked in [15, §4]. So the arguments in Section 2 show that Theorems 2.1 and 2.2 hold true also when $\mathcal{A}$ is the C^*-tensor product of a general C^*-algebra with the use of the relative entropy for C^*-algebra states in defining $S_{\mathrm{M}}(\omega,\varphi)$. However the arguments in Section 3 heavily depend on the finite-dimensionality of $M_d(\mathbf{C})$.

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