Entropy Densities for Gibbs States of Quantum Spin Systems

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Dedicated to Professor Huzihiro Araki on his 60th birthday

This paper is a contribution to the general theory of quantum spin systems. We deal with a general theory in the sense that no concrete interaction shows up but an arbitrary relatively short range interaction is chosen.

It is well-known that the mean entropy plays an important role in the thermodynamics of quantum spin systems, it is one of the ingredients of the Lanford-Ruelle-Robinson variational principle. We show that in the background of the existence of the mean entropy, there is an operator convergence which resembles the McMillan theorem from information theory. Asymptotic equipartition property and several entropy densities are investigated, in particular, the relation of the Gibbs condition to relative entropies.

Although this paper is not intended to be a review, we try to give an overview of the theory of quantum Gibbs states in the mathematical sense. The paper is organized as follows. Section 1 provides an introduction and the further sections contain our new results. Each section is closed with a short discussion on sources and references.

1. Introduction to thermodynamics of quantum spin systems. C*-algebras are known to provide a suitable mathematical formalism for quantum statistical mechanics of spin systems on an infinitely extended lattice. In fact, the rigorous and comprehensive treatment of quantum lattice systems was one of the early successes of the algebraic approach to quantum statistical thermodynamics. The subject is summarized in [7] and in [30], for example, the latter book contains more for physics. In this section we give a concise introduction to the general theory of translationally invariant interactions and we also fix the notations for the other sections.

Let an infinitely extended system of spins be considered in the simple cubic lattice $L = \mathbb{Z}^\nu$. The observables confined to a lattice site $x \in \mathbb{Z}^\nu$ form the selfadjoint part of a finite dimensional C*-algebra $\mathcal{A}_x$ which is a copy of $M_d(\mathbb{C})$. It is assumed that

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the local observables in any bounded region $\Lambda \subset \mathbb{Z}^\nu$ are those of the finite quantum system

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{A}_x .$$

It follows from the definition that for $\Lambda \subset \Lambda'$ we have $\mathcal{A}_{\Lambda'} = \mathcal{A}_\Lambda \otimes \mathcal{A}_{\Lambda' \setminus \Lambda}$, where $\Lambda' \setminus \Lambda$ is the complement of Λ in Λ'. The definition also implies that if Λ_1 and Λ_2 are disjoint then elements of $\mathcal{A}_{\Lambda_1}$ commute with those of $\mathcal{A}_{\Lambda_2}$. The quasilocal C*-algebra $\mathcal{A}$ is the norm completion of the normed algebra $\mathcal{A}_\infty = \cup_\Lambda \mathcal{A}_\Lambda$, the union of all local algebras $\mathcal{A}_\Lambda$ associated with bounded (finite) regions $\Lambda \subset \mathbb{Z}^\nu$.

We denote by a_x the element of $\mathcal{A}_x$ corresponding to $a \in \mathcal{A}_0$ ($x \in \mathbb{Z}^\nu$). It follows from the definition that the algebra $\mathcal{A}_\infty$ consists of linear combinations of terms $a^{(1)}_{x_1} \ldots a^{(k)}_{x_k}$, where $x_1, \ldots, x_k$ and $a^{(1)}, \ldots, a^{(k)}$ run through $\mathbb{Z}^\nu$ and $\mathcal{A}_0$, respectively. We define γ_x to be the linear transformation

$$a^{(1)}_{x_1} \ldots a^{(k)}_{x_k} \longmapsto a^{(1)}_{x_1+x} \ldots a^{(k)}_{x_k+x} .$$

γ_x corresponds to the space-translation by $x \in \mathbb{Z}^\nu$ and it extends to an automorphism of $\mathcal{A}$. Hence γ is a representation of the abelian group $\mathbb{Z}^\nu$ by automorphisms of the quasilocal algebra $\mathcal{A}$. Clearly, the covariance condition

$$\gamma_x(\mathcal{A}_\Lambda) = \mathcal{A}_{\Lambda+x}$$

holds, where $\Lambda + x$ is the space-translate of the region Λ by the displacement x. An important property of space-translations is that they are asymptotically abelian: If $|x|$ is large enough then

$$[\gamma_x(\mathcal{A}_{\Lambda_1}), \mathcal{A}_{\Lambda_2}] = 0 \tag{1.1}$$

for given finite regions $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^\nu$.

Having described the kinematical structure of lattice systems we turn to the dynamics. The local Hamiltonian $H(\Lambda)$ is taken to be the total potential energy among the spins confined to Λ. This energy may come from many-body interactions of various orders. Most generally, we assume that there exists a global function Φ such that for any finite subsystem Λ the local Hamiltonian takes the form

$$H(\Lambda) = \sum_{X\subset\Lambda} \Phi(X) . \tag{1.2}$$

Each $\Phi(X)$ represents the interaction energy of the particles in X. Mathematically, $\Phi(X)$ is a selfadjoint element of $\mathcal{A}_X$ and $H(\Lambda)$ will be a selfadjoint operator in $\mathcal{A}_\Lambda$. We restrict our discussion to translationally invariant interactions which satisfy the additional requirement

$$\gamma_x(\Phi(X)) = \Phi(X + x) \tag{1.3}$$

for every $x \in \mathbb{Z}^\nu$ and every region $X \subset \mathbb{Z}^\nu$. For a finite subset $\Lambda \subset \mathbb{Z}^\nu$ let $|\Lambda|$ denote the volume of Λ (or the number of points in Λ). Furthermore, let $d(\Lambda)$ denote the largest distance between two points in Λ. $d(\Lambda)$ is called the diameter of Λ. An interaction Φ is said to be of finite range if there is a number $d_0 > 0$ such that $\Phi(\Lambda) = 0$ whenever $d(\Lambda) > d_0$. The infimum of such numbers is termed the range of

Φ. The interaction Φ is called finite body if there exist $N \in \mathbb{N}$ such that $\Phi(\Lambda) = 0$ whenever $|\Lambda| > N$. Any finite-range interaction is of finite body.

If ω is a state of the quasilocal algebra $\mathcal{A}$ then it will induce a state ω_Λ on $\mathcal{A}_\Lambda$, the finite system of the region Λ of $\mathbb{Z}^\nu$. The energy, entropy and free energy of this finite system are given by the following formulas.

$$\begin{aligned} E_\Lambda(\omega) &= \mathrm{Tr}_\Lambda D_\Lambda H(\Lambda)\,, \\ S_\Lambda(\omega) &= -\mathrm{Tr}_\Lambda D_\Lambda \log D_\Lambda\,, \\ F_\Lambda^\beta(\omega) &= E_\Lambda(\omega) - \frac{1}{\beta} S_\Lambda(\omega)\,. \end{aligned} \tag{1.4}$$

Here D_Λ denotes the density of ω_Λ with respect to the trace Tr_Λ of $\mathcal{A}_\Lambda$ and $\beta > 0$ denotes the inverse temperature. The functionals E_Λ, S_Λ and F_Λ^β are termed local. It is rather obvious that all three local functionals are continuous if the weak* topology is considered on the state space of the quasilocal algebra. The energy is affine, the entropy is concave and consequently, the free energy is a convex functional.

The thermodynamic limit, "Λ tends to infinity" may be taken along lattice parallelepipeds. Let $a \in \mathbb{Z}^\nu$ with positive coordinates and define

$$\Lambda(a) = \{x \in \mathbb{Z}^\nu : 0 \le x_i < a_i, \quad i = 1, 2, \ldots, \nu\}. \tag{1.5}$$

The volume of the parallelepiped $\Lambda(a)$ is $V(a) = \prod_{i=1}^{\nu} a_i$ and we write $a \to \infty$ if $a_i \to \infty$ for all $1 \le i \le \nu$. When $a \to \infty$, $\Lambda(a)$ tends to infinity in a manner suitable for the study of thermodynamic limit: The boundary of the parallelepipeds is getting more and more negligible compared with the volume. The notion of limit in the sense of van Hove makes this idea more precise and physically more satisfactory (see [28]). For the sake of simplicity we restrict ourselves to thermodynamic limit along parallelepipeds. (Although most results in the sequel hold more generally.)

We may define the global energy, entropy and free energy functionals of translationally invariant states to be

$$e(\omega) = \lim_{\Lambda \to \infty} E_\Lambda(\omega)/|\Lambda|\,, \tag{1.6}$$

$$s(\omega) = \lim_{\Lambda \to \infty} S_\Lambda(\omega)/|\Lambda|\,, \tag{1.7}$$

$$f^\beta(\omega) = \lim_{\Lambda \to \infty} F_\Lambda^\beta(\omega)/|\Lambda|\,, \tag{1.8}$$

and assume that the global time evolution is given by the limit

$$\sigma_t(a) = \lim_\Lambda \sigma_t^\Lambda(a) \qquad (t \in \mathbb{R}, a \in \mathcal{A}_\infty)\,, \tag{1.9}$$

where $\sigma_t^\Lambda(a) = e^{\mathrm{i}tH(\Lambda)} a e^{-\mathrm{i}tH(\Lambda)}$. The existence of the limit in (1.7) is guarenteed by the (strong) subadditivity of entropy, while that of the limits in (1.6), (1.8) and (1.9) is assumed if the interaction is suitably tempered, as it certainly does if the interaction is of finite range.

Theorem 1.1. For any translationally invariant state ω of the quasilocal algebra $\mathcal{A}$, the limit (1.7) exists and

$$s(\omega) = \inf\{S_{\Lambda(a)}(\omega)/|\Lambda(a)| : a \in \mathbb{Z}_+^\nu\}\,. \tag{1.10}$$

Moreover, the mean entropy functional s is affine and upper semicontinuous when the state space is endowed with the weak* topology.

In the treatment of the quantum spin system the set $\mathfrak{S}_\gamma$ of all translationally invariant states is essential. The global entropy functional s is an upper semicontinuous affine function on $\mathfrak{S}_\gamma$ and physically it is a macroscopic quantity which does not have a microscopic (that is, local) counterpart. Indeed, the local entropy functional is not an observable because it is not affine on the (local) state space. The local internal energy $E_\Lambda(\omega)$ is microscopic observable and the energy density functional e of $\mathfrak{S}_\gamma$ is the corresponding global extensive quantity.

The translationally invariant interactions of finite range form a real vector space $\mathcal{F}$ which may be endowed with the norm

$$\|\!|\Phi|\!\| = \sum_{\Lambda \ni 0} \frac{\|\Phi(\Lambda)\|}{|\Lambda|}. \tag{1.11}$$

This normed space $\mathcal{F}$ is not complete. Its completion $\mathcal{F}^-$ consists of the translationally invariant interactions Φ such that

$$\sum_{\Lambda \ni 0} \frac{\|\Phi(\Lambda)\|}{|\Lambda|} < +\infty.$$

(Such Φ are sometimes called relatively short range interactions.) The next results show that $\|\!| \,.\, |\!\|$ is the natural norm for interactions.

Theorem 1.2. Let $\Phi \in \mathcal{F}^-$ and ω be a translationally invariant state of the quasilocal algebra $\mathcal{A}$. Then the thermodynamic limit (1.6) exists and the energy density is given by

$$e(\omega) = \omega(A_\Phi) \quad \text{and} \quad A_\Phi = \sum_{\Lambda \ni 0} \frac{\Phi(\Lambda)}{|\Lambda|}\,.$$

Furthermore, $e(\omega)$ is an affine weakly* continuous functional of ω.

Under the assumption of the previous theorem the free energy density $f^\beta(\omega) = e(\omega) - \beta^{-1}s(\omega)$ exists and it is an affine lower semicontinuous function of the translationally invariant state ω.

We consider now a finite local system $\mathcal{A}_\Lambda$. The free energy functional

$$\psi \mapsto \psi(H(\Lambda)) - \frac{1}{\beta} S(\psi)$$

defined on the state space of $\mathcal{A}_\Lambda$ is minimized by the canonical state

$$\varphi^c_\Lambda(a) = \frac{\mathrm{Tr}_\Lambda\, a\, e^{-\beta H(\Lambda)}}{\mathrm{Tr}_\Lambda\, e^{-\beta H(\Lambda)}} \qquad (a \in \mathcal{A}_\Lambda). \tag{1.12}$$

The minimal value is given by

$$-\frac{1}{\beta} \log \mathrm{Tr}_\Lambda\, e^{-\beta H(\Lambda)}. \tag{1.13}$$

We shall call it canonical local free energy. This quantity depends on the inverse temperature β and on the interaction Φ. Our next goal is to discuss the existence of the thermodynamic limit of the canonical local free energy. The corresponding global quantity is called canonical free energy (density) but it is more usual to introduce notation for its constant multiple which is named pressure. So the next theorem establishes the existence of the pressure p. In terms of the pressure p the canonical global free energy is $-\beta^{-1}p$.

Theorem 1.3. For a relatively short range interaction $\Phi \in \mathcal{F}^-$ and $0 < \beta < \infty$ the thermodynamic limit

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \log \mathrm{Tr}_\Lambda e^{-\beta H(\Lambda)} \equiv p(\beta, \Phi)$$

exists and satisfies the inequality

$$|p(\beta,\Phi) - p(\beta,\Psi)| \le \beta \|\Phi - \Psi\| \qquad (\Phi, \Psi \in \mathcal{F}^-).$$

Moreover, p is a convex functional of the interaction.

As an analogue of the variational principle for finite quantum systems, the global free energy functional f^β attains an absolute minimum at a translationally invariant state, and the minimum value of f^β is equal to the canonical global free energy. In the next theorem this global variational principle will be formulated in a slightly different but equivalent way.

Theorem 1.4. For a relatively short range interaction $\Phi \in \mathcal{F}^-$

$$p(\beta,\Phi) = \sup\{s(\omega) - \beta\, e(\omega) : \omega \text{ is a translationally invariant state on } \mathcal{A}\}.$$

Let ω be a translationally invariant state of the quasilocal algebra $\mathcal{A}$. It is said that ω is globally thermodynamically stable (at the inverse temperature β and with respect to the interaction Φ) if $p(\beta,\Phi) = s(\omega) - \beta\, e(\omega)$, that is, ω minimizes the global free energy functional. (Some people say that ω satisfies the variational principle when it is thermodynamically stable.)

The thermodynamic stability condition may be reformulated in terms of relative entropy. Let ψ_1 and ψ_2 be states of a matrix algebra with densities D_1 and D_2. The relative entropy of ψ_1 with respect to ψ_2 is defined by

$$S(\psi_1,\psi_2) = \begin{cases} \mathrm{Tr}\, D_1(\log D_1 - \log D_2) & \text{if } \operatorname{supp}\psi_1 \le \operatorname{supp}\psi_2 \\ +\infty & \text{otherwise .} \end{cases} \tag{1.14}$$

In particular, $S(\psi_1,\psi_2)$ is always finite if the density D_2 has strictly positive eigenvalues. The relative entropy may be defined to states of an arbitrary C*-algebra and it is a jointly convex functional. The general definition is not treated here. (Something more on relative entropy is found in Section 4.) For states φ and ω of the quasilocal algebra $\mathcal{A}$, we have

$$S(\omega,\varphi) = \sup\{S(\omega_\Lambda,\varphi_\Lambda) : \Lambda \subset \mathbb{Z}^\nu\} = \lim_{\Lambda\to\infty} S(\omega_\Lambda,\varphi_\Lambda)\,, \tag{1.15}$$

where the relative entropy of finite dimensional restrictions is given by (1.14).

For a translationally invariant state ω we have

$$S(\omega_\Lambda, \varphi^c_\Lambda) = -S(\omega_\Lambda) + \beta E_\Lambda(\omega) + \log \mathrm{Tr}_\Lambda e^{-\beta H(\Lambda)}$$

and in the thermodynamic limit

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} S(\omega_\Lambda, \varphi^c_\Lambda) \equiv h(\omega|\beta, \Phi) = -s(\omega) + \beta e(\omega) + p(\beta, \Phi)\,. \tag{1.16}$$

Theorem 1.4 tells us that $h(\omega|\beta,\Phi) \geq 0$ and this vanishes only for thermodynamically stable states.

Let us use the notation $\mathfrak{S}^\beta_\Phi$ for the set of all thermodynamically stable translationally invariant states (with respect to the interaction Φ and inverse temperature β). This set is convex and is viewed as the set of equilibrium states of the infinite lattice system at inverse temperature β.

Theorem 1.5. For a relatively short range interaction $\Phi \in \mathcal{F}^-$ the convex set $\mathfrak{S}^\beta_\Phi$ is a Choquet simplex and its extremal boundary consists of extremal translationally invariant states.

By an ergodic state we shall understand an extremal translationally invariant state.

The existence of the global time evolution (1.9) is usually proved for a class of interactions which is between finite range and relative short range. Set for $\lambda > 0$

$$\|\Phi\|_\lambda = \sum_{\Lambda\ni 0} \|\Phi(\Lambda)\| e^{\lambda|\Lambda|}. \tag{1.17}$$

When $\|\Phi\|_\lambda$ is finite, then the global time evolution exists and thermodynamic stability is equivalent to the KMS condition for translationally invariant states. This is the case if $\Phi \in \mathcal{F}^-$ is of finite body. The set of all those interactions will be denoted by $\mathcal{F}^-_{\mathrm{fb}}$.

The theory of quantum spin systems was established in the papers [26], [27]. Theorem 1.1 was proved in [19]. (For the limit in the sense of van Hove, the strong subadditivity is required, which was conjectured in [19] and proved in [21] somewhat later.) Although the mean entropy s is discontinuous with respect to the weak* topology, it is continuous for the norm topology ([12]).

The existence of the time evolution was obtained first [31] for a specific model and then in [27] for a general finite range interaction. The equivalence of the variational condition and the KMS condition is a result of [20] and [2]. Since the KMS condition is not used in the present paper, we do not discuss it. Note that the existence of the time evolution has been established in [22] under weaker condition than (1.17).

2. The Gibbs condition. The Gibbs condition provides a link between states of the quasilocal algebra $\mathcal{A}$ and the local canonical states. This condition is formulated by means of perturbation with a surface energy term. Given an interaction Φ and a finite region $\Lambda \subset \mathbb{Z}^\nu$ the surface energy is defined as

$$W_\Phi(\Lambda) = \sum \{\Phi(X) : X \not\subset \Lambda \text{ and } X \not\subset \Lambda^c\}\,. \tag{2.1}$$

It contains the contribution of all many-body interactions involving spins both from Λ and its complement Λ^c.

Lemma 2.1. For an interaction $\Phi \in \mathcal{F}_{\mathrm{fb}}^{-}$, the surface energy $W_\Phi(\Lambda)$ is defined and

$$\lim_{\Lambda \to \infty} \frac{\|W_\Phi(\Lambda)\|}{|\Lambda|} = 0.$$

Proof. For every $\varepsilon > 0$ choose $N > 0$ such that

$$\sum_{\substack{d(X)>N \\ X \ni 0}} \frac{\|\Phi(X)\|}{|X|} < \varepsilon.$$

We have

$$\begin{aligned}
\frac{1}{|\Lambda|} \sum_{\substack{X \cap \Lambda^c \neq \emptyset \\ X \cap \Lambda \neq \emptyset}} \|\Phi(X)\| &\le \frac{N_0}{|\Lambda|} \sum_{x \in \Lambda} \sum_{\substack{X \cap \Lambda^c \neq \emptyset \\ X \ni x}} \frac{\|\Phi(X)\|}{|X|} \\
&\le \frac{N_0}{|\Lambda|} \sum_{x \in \Lambda} \sum_{\substack{d(X)>N \\ X \ni x}} \frac{\|\Phi(X)\|}{|X|} + \frac{N_0}{|\Lambda|} \sum_{x \in \Lambda} \sum_{\substack{d(X)\le N \\ X \cap \Lambda^c \neq \emptyset \\ X \ni x}} \frac{\|\Phi(X)\|}{|X|} \\
&\le N_0 \varepsilon + \frac{N_0 |\Lambda_N|}{|\Lambda|} \|\Phi\| ,
\end{aligned}$$

where Λ_N denotes the set of all $x \in \Lambda$ with $\mathrm{dist}(x, \Lambda^c) \le N$, and $\Phi(X) = 0$ when $|X| > N_0$. Since $|\Lambda_N|/|\Lambda| \to 0$ as $\Lambda \to \infty$, we obtain the lemma. □

In order to state the Gibbs condition we recall the inner perturbation of a state. For a state ω and $h = h^* \in \mathcal{A}$ the perturbed state $[\omega^h]$ is defined as the unique minimizer of the weakly* lower semicontinuous strictly convex functional

$$\psi \mapsto S(\psi, \omega) + \psi(h) ,$$

where $S(\psi, \omega)$ is the relative entropy (cf. (1.15)). It follows that

$$S([\omega^h], \omega) \le \omega(h) - [\omega^h](h) \le 2\|h\| .$$

Since the chain rule $[[\omega^h]^k] = [\omega^{h+k}]$ holds, this implies

$$S([\omega^h], [\omega^k]) \le 2\|h - k\| . \tag{2.2}$$

More generally, one has

$$|S(\psi, \varphi) - S(\psi, [\varphi^h])| \le 2\|h\| . \tag{2.3}$$

We shall say that the state ω of $\mathcal{A}$ satisfies the Gibbs condition in the weak sense (with respect to β and Φ) if

$$[\omega^{-\beta W_\Phi(\Lambda)}] \Big| \mathcal{A}_\Lambda = \varphi_\Lambda^c \tag{2.4}$$

for any finite region $\Lambda \subset \mathbb{Z}^\nu$.

Theorem 2.2. If $\Phi \in \mathcal{F}_{\rm fb}^{-}$, then any translationally invariant state satisfying the Gibbs condition in the weak sense is thermodynamically stable.

Proof. We shall prove that for a translationally invariant state φ satisfying (2.4)

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} S(\varphi_\Lambda, \varphi_\Lambda^{\rm c}) = 0\,.$$

According to (2.3) we have

$$S(\varphi, [\varphi^{-\beta W_\Phi(\Lambda)}]) \le 2\beta \|W_\Phi(\Lambda)\|$$

and from the monotonicity of relative entropy

$$S(\varphi_\Lambda, \varphi_\Lambda^{\rm c}) = S(\varphi|\mathcal{A}_\Lambda, [\varphi^{-\beta W_\Phi(\Lambda)}]|\mathcal{A}_\Lambda) \le S(\varphi, [\varphi^{-\beta W_\Phi(\Lambda)}]) \le 2\beta\|W_\Phi(\Lambda)\|\,.$$

The right-hand side divided by $|\Lambda|$ tends to 0 by Lemma 2.1 and this makes the proof complete. □

The state ω of $\mathcal{A}$ satisfies the Gibbs condition (with respect to β and Φ) if the cyclic vector Ω of the corresponding GNS-representation is separating for the generated von Neumann algebra $\pi_\omega(\mathcal{A})''$ and if, for any finite region $\Lambda \subset \mathbb{Z}^\nu$, the perturbed state $[\omega^{-\beta W_\Phi(\Lambda)}]$ has the following factorization property:

$$[\omega^{-\beta W_\Phi(\Lambda)}](ab) = \varphi_\Lambda^{\rm c}(a) \times \psi(b) \tag{2.5}$$

for every $a \in \mathcal{A}_\Lambda,\ b \in \mathcal{A}_{\Lambda^c}$, where $\varphi_\Lambda^{\rm c}$ is the canonical state on $\mathcal{A}_\Lambda$, and ψ is certain state of $\mathcal{A}_{\Lambda^c}$. (The state ψ depends on the region Λ.)

Lemma 2.3. Let φ be a Gibbs state for a relatively short range interaction $\Phi \in \mathcal{F}_{\rm fb}^{-}$ at the inverse temperature β. Let D_Λ be the density of $\varphi|\mathcal{A}_\Lambda$ and $D_\Lambda^{\rm c}$ be that of the local canonical state. Then the operator inequality

$$\log D_\Lambda^{\rm c} - \log D_\Lambda \le 2\beta\|W_\Phi(\Lambda)\| \tag{2.6}$$

holds for every finite region $\Lambda \subset \mathbb{Z}^\nu$.

Proof. It suffices to show that for every state ω on $\mathcal{A}_\Lambda$

$$S(\omega, \varphi_\Lambda) \le S(\omega, \varphi_\Lambda^{\rm c}) + 2\beta\|W_\Phi(\Lambda)\|$$

which is proven by the following chain of inequalities.

$$\begin{aligned} S(\omega, \varphi_\Lambda) &\le S(\omega\otimes\psi, \varphi) \le S(\omega\otimes\psi, [\varphi^{-\beta W_\Phi(\Lambda)}]) + 2\beta\|W_\Phi(\Lambda)\| \\ &= S(\omega\otimes\psi, \varphi_\Lambda^{\rm c}\otimes\psi) + 2\beta\|W_\Phi(\Lambda)\| = S(\omega, \varphi_\Lambda^{\rm c}) + 2\beta\|W_\Phi(\Lambda)\|\,, \end{aligned}$$

where the state ψ is the same as in definition (2.5). □

Theorem 2.4. Let φ be an ergodic Gibbs state for a relatively short range interaction $\Phi \in \mathcal{F}_{\rm fb}^{-}$. Then for the density D_Λ of $\varphi|\mathcal{A}_\Lambda$

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \pi_\varphi(-\log D_\Lambda) = s(\varphi)I$$

strongly in the GNS-representation of φ.

Proof. First we prove that for the density D_Λ^c of the local canonical state φ_Λ^c

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|}\pi_\varphi(-\log D_\Lambda^c) = s(\varphi)I \tag{2.7}$$

strongly. What we need is the mean ergodic theorem (cf. [8]) which tells

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|}\pi_\varphi\Big(\sum_{x\in\Lambda}\gamma_x(A_\Phi)\Big) = \varphi(A_\Phi)I \quad \text{strongly}\,, \tag{2.8}$$

where A_Φ is the same as in Theorem 1.2. Namely,

$$A_\Phi = \sum_{X\ni 0}\frac{\Phi(X)}{|X|}.$$

Since we have

$$0 \le \frac{1}{|\Lambda|}\Big\|\sum_{x\in\Lambda}\gamma_x(A_\Phi) - H(\Lambda)\Big\| \le \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\sum_{\substack{X\cap\Lambda^c\neq\emptyset\\ X\ni x}}\frac{\|\Phi(X)\|}{|X|} \to 0$$

as in the proof of Lemma 2.1, (2.8) implies the strong convergence

$$\frac{1}{|\Lambda|}\pi_\varphi(H(\Lambda)) \to \varphi(A_\Phi)I\,. \tag{2.9}$$

Now in the light of the relation

$$\begin{aligned} -\frac{1}{|\Lambda|}\pi_\varphi(\log D_\Lambda^c) &= \frac{\beta\pi_\varphi(H(\Lambda))}{|\Lambda|} + \frac{1}{|\Lambda|}\log\operatorname{Tr} e^{-\beta H(\Lambda)} \\ &\to \beta\varphi(A_\Phi)I + p(\beta,\Phi)I = s(\varphi)I\,, \end{aligned}$$

the strong limit (2.7) has been verified.

Now we set

$$a_\Lambda = -\frac{1}{|\Lambda|}\log D_\Lambda \quad \text{and} \quad b_\Lambda = -\frac{1}{|\Lambda|}\log D_\Lambda^c + \frac{2\beta\|W_\Phi(\Lambda)\|}{|\Lambda|}\,.$$

Lemma 2.1 and relation (2.7) imply that $\pi_\varphi(b_\Lambda) \to s(\varphi)I$ strongly. Moreover $0 \le a_\Lambda \le b_\Lambda$ by Lemma 2.3. Since (b_Λ) is uniformly bounded, to prove the theorem it suffices to show that $\omega\big(\pi_\varphi(b_\Lambda - a_\Lambda)\big) \to 0$ for a faithful normal state ω of $\pi_\varphi(\mathcal{A})''$. Due to the Gibbs condition on φ, the normal state $\tilde\varphi$ induced by φ on $\pi_\varphi(\mathcal{A})''$ is faithful and

$$\lim_{\Lambda\to\infty}\varphi(b_\Lambda - a_\Lambda) = s(\varphi) - s(\varphi) = 0$$

which completes the proof. □

The Gibbs condition appeared in [5] as the appropriate version of the Dobrushin-Lanford-Ruelle equilibrium condition for quantum systems. The perturbation theory of states was developed earlier in [1], see [29] for its extension. The variational approach to state perturbation (used above) is from [10] (see also Chapter 12 of

[23]). When $\|\Phi\|_\lambda$ is finite, then the Gibbs condition and the variational principle are equivalent ([2]). So, for $\Phi \in \mathcal{F}_{\rm fb}^-$, the weak form of the Gibbs condition implies the Gibbs condition itself.

Theorem 2.4 resembles the McMillan theorem of information theory (see [6] for an orientation and [18] for more sophisticated results). Our Theorem 2.4 is conceptually close to [13], where random fields were discussed. It states the convergence of entropy operators, contrary to Theorem 1.1, where the expectation values converge. (The entropy operator appeared in [11] and a restrictedly noncommutative McMillan type convergence theorem was obtained in [24].)

3. Macroscopic uniformity. Macroscopic uniformity is a basic feature of statistical mechanical systems and it is illustrated here, following [15], by the example of equilibrium canonical ensemble. Let the Hamiltonian of the finite system confined to the region Λ be the selfadjoint operator $H(\Lambda)$. At the inverse temperature β the equilibrium statistical operator is

$$D_\Lambda^c \equiv \frac{e^{-\beta H(\Lambda)}}{\mathrm{Tr}_\Lambda\, e^{-\beta H(\Lambda)}} \tag{3.1}$$

which has spectral decomposition $\sum_i \lambda_i p_i$ with $\lambda_1 \geq \lambda_2 \geq \ldots > 0$. (For the sake of simplicity we have omitted β and Λ from the notation. The projections p_i are assumed to be of rank one.) Choose and fix $0 < \kappa < 1$. Let $n(\kappa)$ be the smallest integer such that

$$\sum_{i=1}^{n(\kappa)} \lambda_i \geq 1 - \kappa$$

and set

$$P(\kappa) = \sum_{i=1}^{n(\kappa)} p_i\,. \tag{3.2}$$

The projection $P(\kappa)$ (or rather its range) is called the high probability subspace and its orthocomplement $I - P(\kappa)$ is the low probability subspace. It is believed that in the thermodynamic limit

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \log \mathrm{Tr}\, P(\kappa) \equiv \log W \tag{3.3}$$

exists and it is independent of the particular probability level κ. (The book [15] says that the high probability subspace is sharply defined in this case.) Limit theorems of type (3.3) originate from the work of Shannon and led to the asymptotic equipartition property of information theory (see [6]). (3.3) tells that asymptotically the probability is concentrated on a small proportion of (eigen)states and it expresses macroscopic uniformity for statistical mechanics. The formula was asserted on p. 76 of [15] as a "tentative theorem" which was expected to hold if the interaction falls off rapidly enough with the distance. Grandy says also that the right-hand side of (3.3) is an entropy. In this section we clarify (3.3) in the model of quantum spin systems. For example, we show that the tentative theorem is a rigorous result for extremal equilibrium states.

When ψ is a state on a matrix algebra $\mathcal{M}$, we define for $0 < \kappa < 1$

$$\beta_\kappa(\psi) = \min\{\log \operatorname{Tr} q : q \in \mathcal{M} \text{ is a projection, } \psi(q) \geq 1 - \kappa\}. \tag{3.4}$$

If $\lambda_1 \geq \lambda_2 \geq \dots$ is the eigenvalue list of the density matrix of ψ (in decreasing order and counting multiplicities), then $\beta_\kappa(\psi)$ is given by $\log \operatorname{Tr} P(\kappa)$, where $P(\kappa)$ is determined as in (3.2).

Theorem 3.1. Let φ be an ergodic Gibbs state for a relatively short range interaction $\Phi \in \mathcal{F}_{\mathrm{fb}}^-$. Then

$$\lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \beta_\kappa(\varphi_\Lambda) = s(\varphi) \tag{3.5}$$

for every $0 < \kappa < 1$.

Proof. For each $\delta > 0$ and finite region $\Lambda \subset \mathbb{Z}^\nu$, let p_Λ denote the spectral projection of $-|\Lambda|^{-1} \log D_\Lambda$ corresponding to the interval $(s(\varphi) - \delta, s(\varphi) + \delta)$. Then

$$\exp(|\Lambda|(s(\varphi) - \delta)) D_\Lambda p_\Lambda \leq p_\Lambda \leq \exp(|\Lambda|(s(\varphi) + \delta)) D_\Lambda p_\Lambda \tag{3.6}$$

and Theorem 2.4 says that $\pi_\varphi(p_\Lambda) \to I$ strongly as $\Lambda \to \infty$.

Choose (Λ_n), $\Lambda_n \to \infty$, such that

$$\lim_{n \to \infty} \frac{1}{|\Lambda_n|} \beta_\kappa(\varphi_{\Lambda_n}) = \liminf_{\Lambda \to \infty} \frac{1}{|\Lambda|} \beta_\kappa(\varphi_\Lambda). \tag{3.7}$$

Then we can choose projections $q_n \in \mathcal{A}_{\Lambda_n}$, $n \in \mathbb{N}$, such that $\varphi(q_n) \geq 1 - \kappa$ and

$$\log \operatorname{Tr} q_n \leq \beta_\kappa(\varphi_{\Lambda_n}) + 1. \tag{3.8}$$

Taking a subsequence of (q_n), we may suppose that $\pi_\varphi(q_n) \to y \in \pi_\varphi(\mathcal{A})''$ weakly. Hence

$$\lim_{n \to \infty} \varphi(p_n q_n) = \tilde{\varphi}(y) = \lim_{n \to \infty} \varphi(q_n) \geq 1 - \kappa. \tag{3.9}$$

and from (3.6) we get

$$\operatorname{Tr} q_n \geq \operatorname{Tr} p_n q_n \geq \exp\big(|\Lambda_n|(s(\varphi) - \delta)\big) \varphi(p_n q_n). \tag{3.10}$$

By (3.9) and (3.10)

$$\liminf_{n \to \infty} \frac{1}{|\Lambda_n|} \log \operatorname{Tr} q_n \geq s(\varphi) - \delta$$

which together with (3.7) and (3.8) implies that

$$\liminf_{\Lambda \to \infty} \frac{\beta_\kappa(\varphi_\Lambda)}{|\Lambda|} \geq s(\varphi) - \delta\,.$$

The other half of the proof, i.e.

$$\limsup_{\Lambda \to \infty} \frac{\beta_\kappa(\varphi_\Lambda)}{|\Lambda|} \leq s(\varphi) + \delta\,,$$

is a consequence of the second inequality in (3.6):

$$\frac{\beta_\kappa(\varphi_\Lambda)}{|\Lambda|} \le \frac{\log \operatorname{Tr} p_\Lambda}{|\Lambda|} \le s(\varphi) + \delta$$

holds if Λ is so large that $\varphi(p_\Lambda) \ge 1 - \kappa$. $\square$

Relation (3.5) cannot hold for any Gibbs state. If $\varphi^1 \le \mu\varphi^2$ then $\varphi^1(q_\Lambda) \ge 1-\kappa$ implies $\varphi^2(q_\Lambda) \ge 1 - \kappa'$ for some κ' and

$$\beta_\kappa(\varphi^1_\Lambda) \ge \beta_{\kappa'}(\varphi^2_\Lambda)\,.$$

Hence the left-hand side of (3.5) is constant on a segment $\varphi_\lambda = \lambda\varphi^1 + (1-\lambda)\varphi^2$ $(0 < \lambda < 1)$ (provided that it exists independently of κ) but the right-hand side is affine according to Theorem 1.1.

Lemma 3.2. Let φ be a Gibbs state for $\Phi \in \mathcal{F}^-_{\mathrm{fb}}$ and let $a_n \in \mathcal{A}_{\Lambda_n}$ be a positive contraction for a sequence (Λ_n) of regions such that $\Lambda_n \to \infty$.

(i) If $\inf_n \varphi(a_n) \ge \delta > 0$, then $\lim_n |\Lambda_n|^{-1} \log \varphi^c_{\Lambda_n}(a_n) = 0$.

If the surface energies $W_\Phi(\Lambda_n)$ are norm bounded, then

(ii) $\inf_n \varphi^c_{\Lambda_n}(a_n) > 0$ implies $\inf_n \varphi(a_n) > 0$,
(iii) $\lim_n \varphi(a_n) = 1$ implies $\lim_n \varphi^c_{\Lambda_n}(a_n) = 1$.

Proof. Let

$$F(s,t) = s\log\frac{s}{t} + (1-s)\log\frac{1-s}{1-t} \qquad (0 \le s,t \le 1), \tag{3.11}$$

which is the relative entropy of the probability distribution $(s, 1-s)$ with respect to $(t, 1-t)$. Assume that opposite to the claim $\lim_n |\Lambda_n|^{-1}\log\varphi^c_{\Lambda_n}(a_n) = 0$, we have $\varphi^c_{\Lambda_n}(a_n) \le e^{-|\Lambda_n|\eta}$ for some $\eta > 0$. Then by the monotonicity of relative entropy

$$S(\varphi_{\Lambda_n}, \varphi^c_{\Lambda_n}) \ge F(\varphi_{\Lambda_n}(a_n), \varphi^c_{\Lambda_n}(a_n))$$

which implies

$$\begin{aligned}\limsup_n \frac{1}{|\Lambda_n|} S(\varphi_{\Lambda_n}, \varphi^c_{\Lambda_n}) &\ge \liminf_n \frac{1}{|\Lambda_n|} F(\varphi_{\Lambda_n}(a_n), \varphi^c_{\Lambda_n}(a_n)) \\ &\ge \eta \liminf_n \varphi(a_n) > 0\,.\end{aligned}$$

This contradicts to the Gibbs condition (see (1.16) and Theorem 2.2), hence the first claim is proven.

To prove (ii) and (iii), we apply a similar argument.

$$F(\varphi^c_{\Lambda_n}(a_n), \varphi_{\Lambda_n}(a_n)) \le S(\varphi^c_{\Lambda_n}, \varphi_{\Lambda_n}) \le S([\varphi^{-\beta W_\Phi(\Lambda_n)}], \varphi) \le 2\beta\|W_\Phi(\Lambda_n)\|.$$

Hence the boundedness of $F(\varphi^c_{\Lambda_n}(a_n), \varphi_{\Lambda_n}(a_n))$ yield (ii) and (iii). $\square$

Theorem 3.3. Let φ be a Gibbs state with respect to $\Phi \in \mathcal{F}^-_{\mathrm{fb}}$. Assume that the surface energies $W_\Phi(\Lambda)$ are uniformly bounded, i.e. $\|W_\Phi(\Lambda)\| \le \alpha$ whenever Λ is a parallelepiped for some $\alpha < \infty$. Then for every $0 < \kappa < 1$,

$$\lim_{\Lambda\to\infty}\frac{\beta_\kappa(\varphi_\Lambda)}{|\Lambda|}=s(\varphi)=\lim_{\Lambda\to\infty}\frac{\beta_\kappa(\varphi^c_\Lambda)}{|\Lambda|}\,.$$

Proof. We note first that under the conditions the Gibbs state is unique and ergodic. Hence the first stated equality is covered by Theorem 3.1. To prove the second, our argument will be similar to the proof of Theorem 3.1.

For a $\delta>0$, let p_Λ be the spectral projection of $-|\Lambda|^{-1}\log D^c_\Lambda$ corresponding to the interval $(s(\varphi)-\delta, s(\varphi)+\delta)$. We know from the proof of Theorem 2.4 that $\pi_\varphi(p_\Lambda)\to I$ strongly as $\Lambda\to\infty$ (cf. (2.7)). Now let us follow the lines of the proof of Theorem 3.1. Choose (Λ_n), $\Lambda_n\to\infty$, such that

$$\lim_{n\to\infty}\frac{1}{|\Lambda_n|}\beta_\kappa(\varphi^c_{\Lambda_n})=\liminf_{\Lambda\to\infty}\frac{1}{|\Lambda|}\beta_\kappa(\varphi^c_\Lambda),$$

and projections $q_n\in\mathcal{A}_{\Lambda_n}$ such that $\varphi^c_{\Lambda_n}(q_n)\ge 1-\kappa$ and $\log\mathrm{Tr}\,q_n\le\beta_\kappa(\varphi^c_{\Lambda_n})+1$. Then $\inf_n\varphi(q_n)>0$ by Lemma 3.2 (ii). Assuming that $\pi_\varphi(q_n)\to y\in\pi_\varphi(\mathcal{A})''$ weakly, we get

$$\lim_{n\to\infty}\varphi(p_nq_np_n)=\tilde\varphi(y)=\lim_{n\to\infty}\varphi(q_n)>0$$

and $\lim_n|\Lambda_n|^{-1}\log\varphi^c_{\Lambda_n}(p_nq_n)=0$ by Lemma 3.2 (i) because $\varphi^c_{\Lambda_n}(p_nq_np_n)=\varphi^c_{\Lambda_n}(p_nq_n)$. Since

$$\mathrm{Tr}\,q_n\ge\mathrm{Tr}\,p_nq_n\ge\exp(|\Lambda_n|(s(\varphi)-\delta))\varphi^c_{\Lambda_n}(p_nq_n),$$

we have

$$\liminf_{\Lambda\to\infty}\frac{1}{|\Lambda|}\beta_\kappa(\varphi^c_\Lambda)\ge s(\varphi)-\delta.$$

Since $\varphi^c_\Lambda(p_\Lambda)\to 1$ by Lemma 3.2 (iii), the other half is as in the proof of Theorem 3.1. □

In the one-dimensional case we have sligthly more than the content of Theorem 3.3. Namely, for $\Phi\in\mathcal{F}$

$$\lim_{n\to\infty}\frac{\beta_\kappa(\varphi_{[1,n]})}{n}=\lim_{n\to\infty}\frac{\beta_\kappa(\varphi^c_{[1,n]})}{n}=\lim_{n\to\infty}\frac{S(\varphi_{[1,n]})}{n}=\lim_{n\to\infty}\frac{S(\varphi^c_{[1,n]})}{n}\,. \tag{3.12}$$

The last equality follows from the existence of $\lambda>0$ such that

$$\lambda^{-1}\varphi_{[1,n]}\le\varphi^c_{[1,n]}\le\lambda\varphi_{[1,n]} \tag{3.13}$$

for every $n\in\mathbb{N}$ in case of a finite range interaction (see [5] or [4]):

$$\begin{aligned}|n^{-1}S(\varphi^c_{[1,n]})-s(\varphi)|&\le\varphi^c_{[1,n]}(|-n^{-1}\log D^c_{[1,n]}-s(\varphi)I|)\\&\le\lambda\varphi(|-n^{-1}\log D^c_{[1,n]}-s(\varphi)I|)\end{aligned}$$

and this tends to 0 according to (2.7).

We make a final comment on the last equality of (3.12). The upper semicontinuity of the entropy density yields that

$$\limsup_{\Lambda\to\infty} \frac{S(\varphi^c_\Lambda)}{|\Lambda|} \le s(\varphi) \tag{3.14}$$

for a Gibbs state φ in any dimension and at any temperature.

4. Relative entropy densities. For a translationally invariant state ω the limit

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} S(\omega_\Lambda, \varphi^c_\Lambda) \equiv h(\omega|\beta, \Phi) \tag{4.1}$$

exists due to the existence of the mean entropy and the pressure. (In fact, the existence of the limit can be obtained directly by a superadditivity argument and an alternative proof is provided in this way for the existence of the pressure.) The macroscopic quantity $h(\omega|\beta,\Phi)$ characterizes thermodynamically stable states. In (4.1) we replace the relative entropy by its variant $S_{\rm co}$ and we show that the limit remains unchanged. This is the content of the section.

Let us recall the Legendre transformation formula

$$S(\psi_1, \psi_2) = \max\{\operatorname{Tr} D_1 h - \log \operatorname{Tr} \exp(php + \log D_2) : h = h^*\}\,, \tag{4.2}$$

where D_i is the density of ψ_i, $i = 1, 2$ and p is the support projection of D_2. A variant of the relative entropy is defined as

$$S_{\rm co}(\psi_1, \psi_2) = \max\{\operatorname{Tr} D_1 h - \log \operatorname{Tr} (D_2 \exp h) : h = h^*\} \tag{4.3}$$

and

$$S(\psi_1, \psi_2) \ge S_{\rm co}(\psi_1, \psi_2) \tag{4.4}$$

due to the Golden-Thompson trace inequality. (It was proved in [25] that the inequality in (4.4) is strict except for the case of commuting D_1 and D_2.) $S_{\rm co}$ may be related to measurements described by projection-valued measures.

$$\begin{aligned} S_{\rm co}(\psi_1, \psi_2) = \max\Big\{ &\sum_i \psi_1(p_i)\big(\log \psi_1(p_i) - \log \psi_2(p_i)\big) : \\ &\qquad p_i \text{ are projections and } \sum_i p_i = I \Big\}\,. \end{aligned} \tag{4.4}$$

So (4.4) also follows from the monotonicity of relative entropy.

Lemma 4.1. Let $e_1, \dots, e_m$ be projections in a matrix algebra $\mathcal{M}$ with $\sum_{k=1}^m e_k = 1$ and set $E(a) = \sum_{k=1}^m e_k a e_k$ $(a \in \mathcal{M})$. Then for every density matrix D in $\mathcal{M}$,

$$S(D, E(D)) \le \log m.$$

Proof. By the joint convexity of the relative entropy, it suffices to establish the upper bound for a density D of rank one. Then

$$S(D, E(D)) = -\operatorname{Tr} E(D) \log E(D)\,,$$

which is the entropy of $E(D)$. Since each $e_k D e_k$ is of rank one or zero, the rank of $E(D)$ is at most m and the claim follows. □

Theorem 4.2. Assume that the translationally invariant interaction Φ is of relatively short range and of finite body. Then for a tranlationally invariant state ω of $\mathcal{A}$ we have

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} S_{\rm co}(\omega_\Lambda, \varphi^c_\Lambda) = h(\omega|\beta, \Phi)\,,$$

where ω_Λ is the restriction of ω to $\mathcal{A}_\Lambda$ and φ^c_Λ is the local canonical state (with respect to β and Φ.)

Proof. The inverse temperature β does not play any role in the theorem, so let $\beta = 1$. Since $S_{\rm co}$ is smaller than the usual relative entropy, we have to show that

$$\liminf_{\Lambda\to\infty} \frac{1}{|\Lambda|} S_{\rm co}(\omega_\Lambda, \varphi^c_\Lambda) \ge \lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} S(\omega_\Lambda, \varphi^c_\Lambda). \tag{4.5}$$

In order to estimate $S(\omega_\Lambda, \varphi^c_\Lambda)$ from above in terms of $S_{\rm co}$, we shall replace the density D^c_Λ of φ^c_Λ by a positive operator D_0 in such a way that the relevant entropy quantities can be controlled.

Let $C \equiv \Lambda(N)$ be a cube with edge length $N \in \mathbb{N}$. Given a region $\Lambda \subset \mathbb{Z}^\nu$ we resolve Λ into disjoint translated images $C_1, C_2, \dots, C_L$ of C. Grouping the terms in the local Hamiltonian $H(\Lambda)$, we write

$$H(\Lambda) = \sum_{l=1}^{L} H(C_l) + \Delta(\Lambda, C).$$

It can be seen that

$$\frac{\|\Delta(\Lambda, C)\|}{|\Lambda|} \le \frac{1}{|C|} \sum_{\substack{X\cap C^c \ne \emptyset \\ X \cap C \ne \emptyset}} \|\Phi(X)\| + \frac{|\Lambda \setminus \cup_{l=1}^L C_l|}{|\Lambda|} |||\Phi|||$$

and hence

$$\limsup_{\Lambda\to\infty} \frac{\|\Delta(\Lambda, C)\|}{|\Lambda|} \le \frac{1}{|C|} \sum_{\substack{X\cap C^c \ne \emptyset \\ X \cap C \ne \emptyset}} \|\Phi(X)\|\,, \tag{4.6}$$

where the right-hand side tends to 0 as $N \to \infty$ (cf. the proof of Lemma 2.1).

For a while we fix N and Λ and we put $\delta = \|\Delta(\Lambda, C)\|$. Since

$$-\sum_{l=1}^{L} H(C_l) - \delta \le -H(\Lambda) \le -\sum_{l=1}^{L} H(C_l) + \delta,$$

we can choose a unitary $U \in \mathcal{A}_\Lambda$ such that $UH(\Lambda)U^*$ commutes with $\sum_{l=1}^L H(C_l)$ and

$$-\sum_{l=1}^{L} H(C_l) - \delta \le -UH(\Lambda)U^* \le -\sum_{l=1}^{L} H(C_l) + \delta.$$

Then for the density D^c_Λ of φ^c_Λ, we have

$$UD^c_\Lambda U^* = \frac{\exp(-UH_\Lambda U^*)}{\mathrm{Tr}\, e^{-H(\Lambda)}} \le \frac{\exp(-\sum_{l=1}^L H(C_l) + \delta)}{\mathrm{Tr}\ \exp(-\sum_{l=1}^L H(C_l) - \delta)} = e^{2\delta} \otimes_{l=1}^L D^c_{C_l}, \quad (4.7)$$

and similarly

$$UD^c_\Lambda U^* \ge e^{-2\delta} \otimes_{l=1}^L D^c_{C_l}. \quad (4.8)$$

Let $\mu_1, \dots, \mu_M$ be the eigenvalue list of D^c_C. Then the set of eigenvalues of $\otimes_{l=1}^L D^c_{C_l}$ is

$$\Big\{ \prod_{m=1}^M \mu_m^{l_m} : l_1, \dots, l_M \ge 0,\ \sum_{m=1}^M l_m = L \Big\},$$

which is arranged as $\{\eta_1, \dots, \eta_K\}$ and $K \le (L+1)^M$.

Taking the spectral decomposition of D^c_Λ let us write $D^c_\Lambda = \sum_{j\in J} \lambda_j q_j$ where q_j are projections. We may choose a partition $J_1, J_2, \dots J_K$ of J such that

$$e^{-2\delta}\eta_k \le \lambda_j \le e^{2\delta}\eta_k \quad \text{if} \quad j \in J_k. \quad (4.9)$$

Now define

$$e_k = \sum_{j\in J_k} q_j \quad (1 \le k \le K) \quad \text{and} \quad D_0 = \sum_{k=1}^K \eta_k e_k.$$

Then D^c_Λ commutes with D_0 by definition and

$$e^{-2\delta} D_0 \le D^c_\Lambda \le e^{2\delta} D_0$$

holds due to (4.9). D_0 is an approximation of D^c_Λ and we estimate the relative entropy $S(\omega_\Lambda, \varphi^c_\Lambda)$ using this D_0 as follows. We write D for the density of ω_Λ.

$$\begin{aligned} S(\omega_\Lambda, \varphi^c_\Lambda) &\le S(D, e^{-2\delta} D_0) = S(D, D_0) + 2\delta \\ &= S(E(D), D_0) + S(D, E(D)) + 2\delta \\ &\le S_{\mathrm{co}}(D, D_0) + S(D, E(D)) + 2\delta \\ &\le S_{\mathrm{co}}(D, e^{-2\delta} D^c_\Lambda) + S(D, E(D)) + 2\delta \\ &= S_{\mathrm{co}}(\omega_\Lambda, \varphi^c_\Lambda) + S(D, E(D)) + 4\delta \end{aligned}$$

where $E(D) = \sum_{k=1}^K e_k D e_k$ as in Lemma 4.1. (Here the first and the third inequalities are based on the monotonicity of the relative entropy in the second variable, and the conditional expectation property and the relation between S and S_{co} were utilized. Moreover the second inequality can be seen from the commutativity of $E(D)$ and D_0.) Hence

$$\frac{S(\omega_\Lambda, \varphi^c_\Lambda)}{|\Lambda|} \le \frac{S_{\mathrm{co}}(\omega_\Lambda, \varphi^c_\Lambda)}{|\Lambda|} + \frac{\log K}{|\Lambda|} + \frac{4\delta}{|\Lambda|}$$

due to Lemma 4.1. The last term of the right-hand side are arbitrarily small by (4.6) when N is large enough. The last but one term converges to 0 as $\Lambda \to \infty$ because $\log K \le M \log(1 + |\Lambda|)$. □

Theorem 4.2 yields the following characterization of Gibbs states. When for any net of selfadjoint $h_\Lambda \in \mathcal{A}_\Lambda$ the condition

$$\limsup_{\Lambda} \Big(\omega(h_\Lambda) - \frac{1}{|\Lambda|}\varphi^c_\Lambda(\exp(|\Lambda| h_\Lambda))\Big) \le 0 \tag{4.10}$$

holds for a translationally invariant state ω, then ω is a Gibbs state.

Let $\Phi \in \mathcal{F}^-$, $\beta > 0$ and let φ be a Gibbs state. The mean relative entropy $S_M(\omega, \varphi)$ is defined for a translationally invariant state ω as

$$S_M(\omega, \varphi) = \limsup_{\Lambda \to \infty} \frac{S(\omega_\Lambda, \varphi_\Lambda)}{|\Lambda|} . \tag{4.11}$$

The existence of the limit in (4.11) is not known to us. We can see that for a commuting interaction (i.e., $[H(X), H(Y)] = 0$ for every X, Y) (4.11) is a limit and equals to $h(\omega|\Phi, \beta)$ which is well-known for Gibbs fields (see [14]). We can arrive at the same conclusion for $\Phi \in \mathcal{F}$ in the one-dimensional case. Then

$$|S(\omega_{[1,n]}, \varphi_{[1,n]}) - S(\omega_{[1,n]}, \varphi^c_{[1,n]})| \le \log \lambda$$

according to (3.13). This argument extends to the multi-dimensional case at small inverse temperature. It was proved in [5] (see also [4]) that for $\Phi \in \mathcal{F}^-_{\mathrm{fb}}$ and for small β there are constants λ_Λ such that

$$\lambda_\Lambda^{-1} \varphi_\Lambda \le \varphi^c_\Lambda \le \lambda_\Lambda \varphi_\Lambda \tag{4.12}$$

and

$$\lim_{\Lambda \to \infty} \frac{\log \lambda_\Lambda}{|\Lambda|} = 0 . \tag{4.13}$$

Therefore,

$$\frac{|S(\omega_\Lambda, \varphi_\Lambda) - S(\omega_\Lambda, \varphi^c_\Lambda)|}{|\Lambda|} \le \frac{\log \lambda_\Lambda}{|\Lambda|}$$

and we obtain

$$\lim_{\Lambda \to \infty} \frac{S(\omega_\Lambda, \varphi_\Lambda)}{|\Lambda|} = h(\omega|\beta, \Phi) . \tag{4.14}$$

Hence

$$S_M(\omega, \varphi) = 0 \tag{4.15}$$

implies that ω is a Gibbs state. The converse holds generally. When ω and φ are Gibbs states then Lemma 2.3 gives (4.15).

The quantity S_{co} showed up in [9] and [25]. Theorem 4.2 is continuation of our previous results in some sense. In [16] we showed that the relative entropy and S_{co} have the same asymptotics when the reference state is a product. The content of Theorem 4.2 is similar but the reference is the local canonical state. Entropy densities for states of another type (including quantum Markov states) will be discussed in [17].

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