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ON ENTROPY FUNCTIONALS OF STATES OF OPERATOR ALGEBRAS

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For two finite probability distributions $(p_1, p_2, \ldots, p_n)$ and $(q_1, q_2, \ldots, q_n)$ the quantity

$$\sum_{k=1}^{n} p_k(\log p_k - \log q_k) \tag{1}$$

was introduced in 1951 by Kulback and Leibler. They called it information for discrimination [12, 13]. Some years later Rényi suggested the name information gain [23]. As a natural analogue of (1) Umegaki defined the relative entropy of two density matrices in 1962 [27] by the formula

$$\operatorname{Tr} \varrho(\log \varrho - \log \varphi) \tag{2}$$

and this notion was extended by Araki [2] to states of C*-algebras as follows.

Let the von Neumann algebra $\mathcal{M}$ act on a Hilbert space $\mathcal{H}$ and let the normal state ω be given by a vector $\Omega \in \mathcal{H}$. Let φ be another normal state. Then there exists a positive selfadjoint operator $\Delta(\varphi, \omega)$ such that

(i) $\left\| \Delta(\varphi, \omega)^{1/2} a\Omega \right\|^2 = \varphi(apa^*)$ for every $a \in \mathcal{M}$ and for the support projection p of ω,

(ii) the support of $\Delta(\varphi, \omega)$ is in the closure of $\mathcal{M}\Omega$,

(iii) $\mathcal{M}\Omega$ is a core for the restriction of $\Delta(\varphi, \omega)^{1/2}$ to the closure of $\mathcal{M}\Omega$.

For normal states Araki defined the relative entropy as

$$S(\omega, \varphi) = -\langle \log \Delta(\varphi, \omega)\Omega, \Omega \rangle \tag{3}$$

which turns out to be independent of the representation. For positive functionals of an arbitrary C*-algebra $\mathcal{A}$ the relative entropy may be determined through the GNS-construction. Let $(\mathcal{H}, \Phi, \pi)$ stand for the GNS-triplet for the unital C*-algebra $\mathcal{A}$ and the positive functional ϕ of $\mathcal{A}$. Let ψ be another positive functional on $\mathcal{A}$. We write $\bar{\psi}$ for the normal state of $\pi(\mathcal{A})''$ such that

$$\bar{\psi}(\pi(a)) = \psi(a) \qquad (a \in \mathcal{A})$$

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if a normal functional with this property exists. Let

$$S(\psi,\phi)=\begin{cases} S(\bar{\psi},\bar{\phi}) & \text{if } \bar{\psi} \text{ exists,}\\ +\infty & \text{otherwise.}\end{cases}$$

Properties of the relative entropy functional were established in many papers and the highlight of this development was Lieb's convexity theorem [15]. The notion received much attention in quantum mechanics [16]. Concerning the details we refer to the survey papers [3] and [19].

The aim of the present paper is to characterize the relative entropy functional through its well-known properties and to prove some results related to a net of mappings approximating the identity. As a frame we consider nuclear C*-algebras [10, p. 858] and injective von Neumann algebras [24, p. 143]. Such algebras are well-aproximated by finite dimensional ones and we shall benefit from the characterization of the relative entropy functional on matrix algebras [22].

Our crucial postulate for the relative entropy includes the notion of conditional expectation. Let us recall that in the setting of operator algebras conditional expectation (or projection of norm one) is defined as a positive unital idempotent linear mapping onto a subalgebra [25, p. 131].

Now we list properties of the relative entropy functional needed in the characterization. Let us recall that a separating state gives rise to a separating cyclic vector in the GNS Hilbert space for the generated von Neumann algebra.

(i) Conditional expectation property: Assume that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$ and there exists a projection of norm one E of $\mathcal{B}$ onto $\mathcal{A}$ which leaves invariant the separating state φ. Then for every state ω of $\mathcal{B}$ the equality $S(\omega,\varphi)=S(\omega|\mathcal{A},\varphi|\mathcal{A},)+S(\omega,\omega\circ E,)$ holds.

(ii) Monotonicity property: For every completely unital positive mapping α of a C*-algebra $\mathcal{A}$ into $\mathcal{B}$ we have $S(\omega,\varphi)\geqq S(\omega\circ\alpha,\varphi\circ\alpha)$.

(iii) Direct sum property: Assume that $\mathcal{B}=\mathcal{B}_1\oplus\mathcal{B}_2$ and $\varphi_{12}(a\oplus b)=\lambda\varphi_1(a)+(1-\lambda)\varphi_2(b)$ and $\omega_{12}(a\oplus b)=\lambda\omega_1(a)+(1-\lambda)\omega_2(b)$ for every $a\in\mathcal{B}_1, b\in\mathcal{B}_2$ and some $0<\lambda<1$. Then $S(\omega_{12},\varphi_{12})=\lambda S(\omega_1,\varphi_1)+(1-\lambda)S(\omega_2,\varphi_2)$.

(iv) Nilpotence property: $S(\varphi,\varphi)=0$.

(v) Lower semicontinuity: The function $(\omega,\varphi)\mapsto S(\omega,\varphi)$ is weak* lower semicontinuous on the state space of a nuclear C*-algebra $\mathcal{B}$ (when φ is assumed to be separating).

The properties (i)–(v) are well-known for the relative entropy functional. Among them the conditional expectation property is the most crucial (it was obtained in [21] in full generality, cf. [20]). The monotonicity has been proven by Uhlmann [26] and weak* lower semicontinuity is a consequence of Kosaki's formula [11] stated here for further use:

$$(4)\qquad S(\omega,\varphi)=$$
$$=\sup\sup\left\{\omega(I)\log n-\int_{1/n}^{\infty}\omega\big(y(t)^*y(t)\big)+t^{-1}\varphi\big(x(t)x(t)^*\big)\frac{dt}{t}\right\}$$

where the first sup is taken over all natural numbers n, the second one is over all step functions $x:(1/n,\infty)\to\mathcal{A}$ with finite range and $y(t)=I-x(t)$.

THEOREM 1 [22]. *If a real valued functional $S'(\varphi,\omega)$ defined for separating states φ and arbitrary states ω of finite dimensional C^*-algebras posesses the properties* (i)–(v) *then there exists a constant $C\in\mathbf{R}$ such that*

$$S'(\varphi,\omega)=C\,\mathrm{Tr}\,D_\omega(\log D_\omega-\log D_\varphi)\,.$$

The proof consists of several steps. It is shown that for larger and larger class of states

$$S'(\varphi,\omega)=C\,S(\varphi,\omega)$$

must hold.

A C*-algebra $\mathcal{A}$ is said to be nuclear if, for every C*-algebra $\mathcal{B}$, there is only one C*-norm on $\mathcal{A}\odot\mathcal{B}$. Finite dimensional and abelian C*-algebras are nuclear. A C*-algebra $\mathcal{A}$ is called AF-algebra if it contains an increasing sequence of finite dimensional subalgebras such that their union is norm dense in $\mathcal{A}$. It can be proved that the inductive limit of nuclear C*-algebras is nuclear itself. In particular, every AF-algebra is nuclear. Let $(\alpha_i:\mathcal{A}_i\to\mathcal{A})_i$ be a net of unital completely positive mapping defined on finite dimensional algebras. We shall call $(\alpha_i)_i$ a norm approximating net if for each i there exists a unital completely positive mapping $\beta_i:\mathcal{A}\to\mathcal{A}_i$ such that

$$\|\alpha_i\circ\beta_i(a)-a\|\to 0\quad(a\in\mathcal{A}).$$

(The net $\alpha_i\circ\beta_i$ approximates the identity of $\mathcal{A}$ in the topology of pointwise norm convergence.) The class of nuclear C*-algebras is characterized by the existence of a norm approximating net. A C*-algebra admitting the existence of a norm approximating net is often called semidiscrete. The equivalence of nuclearity and semidiscretness was proved in [4]. The works [8] and [14] review this subject. Most physically important C*-algebras are nuclear. For example, the algebra of the canonical commutation relation is a nonseparable nuclear C*-algebra.

THEOREM 2. *Let $\mathcal{A}$ be a nuclear C^*-algebra with states φ and ω. If $(\alpha_i:\mathcal{A}_i\to\mathcal{A})_i$ is a norm approximating net then*

$$S(\omega,\varphi)=\lim_i S(\omega\circ\alpha_i,\varphi\circ\alpha_i).$$

Consequently, $S(\omega, \varphi)$ *is the lowest upper bound of the quantities* $S(\omega \circ \alpha, \varphi \circ \alpha)$ *where* α *ranges all completely positive unital mappings from a finite dimentional algebra into* $\mathcal{A}$.

PROOF. Since $S(\omega \circ \alpha, \varphi \circ \alpha) \leqq S(\omega, \varphi)$ holds for any completely positive mapping α, we show that given a generating net $(\alpha_i : \mathcal{A}_i \to \mathcal{A})_i$ and numbers $0 \leqq u < S(\omega, \varphi)$, $0 < \varepsilon$, for large enough i

$$S(\omega \circ \alpha_i, \varphi \circ \alpha_i) \geqq u - \varepsilon \tag{5}$$

holds.

The main ingredient of the proof will be Kosaki's formula (4). There exists an $n \in \mathbf{N}$ and a step function $x : [1/n, \infty) \to \mathcal{A}$ such that it has finite range, $x(t) = I$ for large t and

$$\log n - \int_{1/n}^{\infty} t^{-1}\omega\big(y(t)^* y(t)\big) + t^{-2}\varphi\big(x(t)x(t)^*\big)\, dt \geqq u.$$

For large i we have

$$\log n - \int_{1/n}^{\infty} t^{-1}\omega(\alpha_i \circ \beta_i\big(y(t)^* y(t)\big) + t^{-2}\varphi(\alpha_i \circ \beta_i\big(x(t)x(t)^*\big)\, dt \geqq u - \varepsilon$$

where $\|\alpha_i \circ \beta_i(a) - a\| \to 0$ for every $a \in \mathcal{A}$. So writing $x_i(t)$ and $y_i(t)$ for $\beta_i\big(x_i(t)\big)$ and $\beta_i\big(y_i(t)\big)$, respectively, we obtain from the Schwarz inequality

$$\log n - \int_{1/n}^{\infty} t^{-1}(\omega \circ \alpha_i)\big(y_i(t)^* y_i(t)\big) + t^{-2}(\varphi \circ \alpha_i)\big(x_i(t)x_i(t)^*\big)\, dt \geqq u - \varepsilon$$

and Kosaki's formula yields (5) for large enough i. ☐

THEOREM 3. *If a real valued functional* $S'(\varphi, \omega)$ *defined for separating states* φ *and arbitrary states* ω *of nuclear* C^**-algebras posesses the properties* (i)–(v) *then there exists a constant* $C \in \mathbf{R}$ *such that*

$$S'(\varphi, \omega) = C\, S(\varphi, \omega). \tag{6}$$

PROOF. Theorem 1 tells us that S' must be a constant multiple on finite dimensional algebras. The rest is in Theorem 2. For an arbitrary nuclear C*-algebra let $(\alpha_i : \mathcal{A}_i \to \mathcal{A})_i$ be a norm approximating net and let $\beta_i : \mathcal{A} \to \mathcal{A}_i$ be the corresponding completely positive mappings from the definition of such a net. From the monotonicity

$$S'(\omega, \varphi) \geqq \limsup_i S'(\omega \circ \alpha_i, \varphi \circ \alpha_i) \geqq \limsup_i S'(\omega \circ \alpha_i \circ \beta_i, \varphi \circ \alpha_i \circ \beta_i).$$

According to the weak* lower semicontinuity we have

$$S'(\omega,\varphi) \leqq \liminf_i S'(\omega\circ\alpha_i\circ\beta_i,\varphi\circ\alpha_i\circ\beta_i)\,.$$

Therefore

$$S'(\omega,\varphi) = \lim_i S'(\omega\circ\alpha_i,\varphi\circ\alpha_i)$$

and (6) must holds. □

Let $\mathcal{M}$ be a von Neumann algebra and let $(\alpha_i : \mathcal{A}_i \to \mathcal{M})$ be a net of unital completely positive mappings with finite dimensional algebras $(\mathcal{A}_i)_i$. We shall call $(\alpha_i)_i$ a weak* approximating net if for each i there exists a normal unital completely positive mapping $\beta_i : \mathcal{M} \to \mathcal{A}_i$ such that

$$\lim_i \psi(\alpha_i\circ\beta_i a) = \psi(a)$$

for every $a \in \mathcal{M}$ and for every $\psi \in \mathcal{M}_*$.

Assume that a von Neumann algebra $\mathcal{M}$ contains an ascending net $(\mathcal{N}_i)_i$ of finite dimensional subalgebras so that $\cup_i\mathcal{N}_i$ is strongly dense in $\mathcal{M}$. Let κ_i be the embedding of $\mathcal{N}_i$ into $\mathcal{M}$. Then $(\kappa_i)_i$ is a strong approximating net. Indeed, for the generalized conditional expectation $E_i : \mathcal{M} \to \mathcal{M}_i$ we have $\kappa_i\circ E_i \to \mathrm{id}$ strongly, due to the martingale convergence theorem [9] and [18].

Injective von Neumann algebras admit the existence of a weak* approximating net (for the identity, see [8]) and completely similarly to the previous proofs one obtains the following.

THEOREM 4. *Let $\mathcal{M}$ be an injective von Neumann algebra with normal states φ and ω. Then $S(\omega,\varphi)$ is the supremum of all the quantities $S(\omega\circ\alpha,\varphi\circ\alpha)$ where α runs over all completely positive unital mappings from a finite dimensional algebra into $\mathcal{M}$.*

THEOREM 5. *If a real valued functional $S'(\varphi,\omega)$ defined for separating states φ and arbitrary states ω of injective von Neumann algebras posesses the properties* (i)–(v) *then the functional S' is a constant multiple of Araki's relative entropy.*

While this characterization of the relative entropy on injective algebras is based on finite dimensional approximation, we note that another characterization was given in [6] which benefited from the fact that an injective von Neumann algebra is the range of a conditional expectation of some $B(\mathcal{H})$.

Let $\mathcal{M} \subset B(\mathcal{H})$ be an injective von Neumann algebra and let φ,ω be normal states on $\mathcal{M}$. Then $S(\omega,\varphi) = \mathrm{ent}_{\mathcal{M}}(\omega,\varphi)$ where $\mathrm{ent}_{\mathcal{M}}$ is defined in the following way:

(i) $\mathrm{ent}_{B(\mathcal{H})}(\sigma,\rho) = S(\sigma,\rho)$ when σ and ρ are normal.

(ii) $\mathrm{ent}_{B(\mathcal{H})}(\sigma,\rho) = \sup\{\,F(\sigma,\rho) : F$ is w^* lower semicontinuous, convex, and coincides with $S(\sigma,\rho)$ when σ and ρ are normal$\}$.

(iii) $\mathrm{ent}_{\mathcal{M}}(\sigma,\rho) = \inf\{\mathrm{ent}_{B(\mathcal{H})}(\sigma',\rho') : \sigma'|\mathcal{M} = \sigma \text{ and } \rho'|\mathcal{M} = \rho\}$.

The following definition for the entropy of states of arbitrary C^*-algebras was proposed in [17]:

$$S(\varphi) = \sup\left\{\textstyle\sum_i \lambda_i S(\varphi_i,\varphi) : \sum_i \lambda_i \varphi_i = \varphi\right\}. \tag{7}$$

Here the supremum is over all decompositions of φ into finite (or equivalently countable) convex combinations of other states. This definition was generalized in [5]. Let $\alpha : \mathcal{B} \to \mathcal{A}$ be a completely positive unital map and φ a state of $\mathcal{A}$. The quantity

$$H_\varphi(\alpha) = \sup\left\{\textstyle\sum_i \lambda_i S(\varphi_i \circ \alpha, \varphi \circ \alpha) : \sum_i \lambda_i \varphi_i = \varphi\right\}. \tag{8}$$

can be called the entropy of the mapping α.

THEOREM 6. *Let $\mathcal{A}$ be a nuclear C^*-algebra with an approximating net $(\alpha_i : \mathcal{A}_i \to \mathcal{A})_i$. Then for every state φ of $\mathcal{A}$*

$$S(\varphi) = \lim_i H_\varphi(\alpha_i)$$

holds.

PROOF. By the definition of the entropy we can find a finite convex decomposition $\sum_{k=1}^n \lambda_k \varphi_k$ of φ for an $\varepsilon > 0$ so that

$$S(\varphi) \leqq \sum_{k=1}^n \lambda_k S(\varphi_k,\varphi) + \varepsilon.$$

For i big enough we have

$$S(\varphi_k,\varphi) \leqq S(\varphi_k \circ \alpha_i, \varphi \circ \alpha_i) + \varepsilon \quad (k = 1,2,\dots,n)$$

due to Theorem 2. Hence

$$S(\varphi) \leqq \sum_{k=1}^n \lambda_k S(\varphi_k \circ \alpha_i, \varphi \circ \alpha_i) + 2\varepsilon \leqq H_\varphi(\alpha_i) + 2\varepsilon$$

for large i. Since $H_\varphi(\alpha_i) \leqq S(\varphi)$, the proof is complete. □

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