Quantum Probability and Applications IV,
Lecture Notes in Math., **1442**(1990), 311–319.

First steps towards a Donsker and Varadhan theory in operator algebras

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1 Introduction and motivation

Let $\xi_1, \xi_2, \dots$ be a sequence of independent identically distributed random variables with mean $m = E(\xi_1)$. If $m < a < b$ then the law of large numbers states that

$$P\left(\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{n} \in [a, b]\right) \to 0 \quad \text{as} \quad n \to \infty.$$

However, much more is true, namely, that the convergence is exponentially fast. There is $\lambda(a, b) > 0$ such that

$$P\left(\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{n} \in [a, b]\right) \le Ce^{-n\lambda(a,b)}$$

with some constant C, and in fact, $\lambda(a, b)$ depends on $a - m$. The *large deviation principle* (LDP) is gathered from this example.

Let $\eta_1, \eta_2, \dots$ be a sequence of random variables and $I : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$. (η_n) is said to satisfy the LDP with rate function I if

(i) $\limsup_{n\to\infty} \frac{1}{n} \log P(\eta_n \in F) \le -\inf \{I(x) : x \in F\}$ for every closed set $F \subset \mathbb{R}$,

(ii) $\liminf_{n\to\infty} \frac{1}{n} \log P(\eta_n \in G) \ge -\inf \{I(x) : x \in G\}$ for every open set $G \subset \mathbb{R}$,

(iii) $I \ne +\infty$ and $\{t \in \mathbb{R} : I(t) \le L\}$ is compact for every $L \in \mathbb{R}$.

Cramér essentially proved in the 30's the following.

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Theorem 1 *If $\xi_1, \xi_2 \dots$ are identically distributed independent random variables and*

$$L(u) = E(\exp u\xi_1) < +\infty \qquad (u \in \mathbb{R})$$

then $\eta_n = (\xi_1 + \xi_2 + \dots + \xi_n)/n$ satisfies the LDP with rate function

$$I(x) = \sup\{ux - \log L(u) \,:\, u \in \mathbb{R}\}\,.$$

If you are pushed by the strong desire of noncommutativization everything possible then Theorem 1 does not give much chance. A reformulation due to Varadhan is more suitable.

Theorem 2 *Under the assumption of Theorem 1 the relation*

$$\lim_{n\to\infty} \frac{1}{n} \log E(\exp nf(\eta_n)) = \sup\{f(n) - I(u) \,:\, u \in \mathbb{R}\}$$

holds for every continuous bounded function $f : \mathbb{R} \to \mathbb{R}$.

In Theorem 2 probability, open and closed sets do not appear and a more functional analytic view point seems to be possible.

Personally I met LDP first when I visited Luigi Accardi at the 2nd University of Rome in 1984. We formulated Theorem 2 on the language of operator algebras as follows. Let $\mathcal{B}_n$ be a copy of a finite dimensional C^*-algebra $\mathcal{B}$ and take $\mathcal{A} = \otimes_{n\in\mathbb{Z}}\mathcal{B}_n$. Assume that φ is a product state on $\mathcal{A}$. Then φ and the embeddings $i_n : \mathcal{B}_n \to \mathcal{A}$ play the role of a Bernoulli sequence. For $b \in \mathcal{B}$ one can ask the existence of the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \varphi\left(\exp\left(nf\left(\frac{i_1(b) + i_2(b) + \dots + i_n(b)}{n}\right)\right)\right),$$

(where f is a continuous function $\mathbb{R} \to \mathbb{R}$) and we observed that

$$\liminf_{n\to\infty} \frac{1}{n} \log \varphi\left(\exp\left(nf\left(\frac{i_1(b) + i_2(b) + \dots + i_n(b)}{n}\right)\right)\right) \geq \omega(i_1(b)) - S_M(\varphi, \omega) \qquad (1)$$

whenever ω is a stationary state on $\mathcal{A}$ and $S_M(\varphi, \omega)$ is the mean relative entropy (see below). If you take simply $f = id$ then the lim inf in (1) is a limit with value $\log \varphi(\exp i_1(b))$. It can not be the sup of the right hand side (ω runs over stationary states), since the latter is a convex function of b. (Remember that $A \mapsto \log \varphi(\exp A)$ is not convex in general, being the exp function not operator convex). Hence the noncommutative generalization must go in another direction.

Over the pure noncommutativization of the LDP (or more precisely, Theorem 2), a motivation arises from quantum statistical mechanics. I briefly recall the notion of a quantum lattice system. To each $n \in \mathbb{Z}$ a copy $\mathcal{B}_n$, of the 2×2 matrices is associated. If $I \subset \mathbb{Z}$ then A_I denotes $\otimes_{n\in I}\mathcal{B}_n$ and I write $\mathcal{A}$ for $\mathcal{A}_{\mathbb{Z}}$. Having fixed some selfadjoint matrices x and h, one defines the local Hamiltonians by the formula

$$H_{[m,n]} = \sum_{i=m}^{n} h_i + \frac{1}{n-m} \sum_{i,j=m}^{n} x_i x_j.$$

(This interaction is called mean field [4].) The mean free energy at the inverse temperature β is given as

$$F_n(\beta) = \frac{1}{n} \log \mathrm{Tr}\, \exp \left(\sum_{i=1}^{n} \log D_i - n\beta \left(\frac{x_1 + x_2 + \cdots + x_n)}{n} \right)^2 \right), \qquad (2)$$

where $D_i = \exp(-\beta h_i)/\mathrm{Tr}\, \exp(-\beta h_i)$. The similarity of (1) and (2) is striking if f is the square function. (For the sake of simplicity, you may take $\beta = -1$.) On the other hand, statistical mechanics suggests the correction of (1) in order to get an equality (which is, in fact, the Gibbs variational principle).

This lecture is based on the joint work [7] with Raggio and Verbeure and benefited also from the generalization [9] of Raggio and Werner. In the latter paper the interested reader may find more physical interpretations. Concerning large deviations and (classical) statistical mechanics I refer to the book [3].

2 Preliminaries

Let $\mathcal{A}$ be a C^*-algebra with a state φ. Performing the GNS-construction we arrive at a cyclic vector $\Phi \in \mathcal{H}_\varphi$ and a representation $\pi_\varphi : \mathcal{A} \to B(\mathcal{H}_\varphi)$. If Φ is separating for the von Neumann algebra $\pi_\varphi(\mathcal{A})''$ then we say that φ is separating. (Typical examples of separating states are the KMS-states.) Having a cyclic and separating vector Φ (with respect to $\pi_\varphi(\mathcal{A})''$), the corresponding modular operator Δ is at our disposal. The formula

$$\Phi^h = \exp\left(\frac{1}{2}(\log \Delta + \pi_\varphi(h)\right)\Phi$$

defines the perturbed vector for $h = h^* \in \mathcal{A}$ and

$$\varphi^h(a) = \langle \pi_\varphi(a)\Phi^h, \Phi^h \rangle \qquad (a \in \mathcal{A})$$

is the (unnormalized) perturbed functional. In the next mainly the quantity $\varphi^h(1) = \|\Phi^h\|^2$ will occur. It is known that $h \mapsto \log \varphi^h(I)$ is a convex continuous function on $\mathcal{A}^{sa}$ [1].

Whenever F is a convex function on a normed space X then its conjugate (or Legendre–Fenchel transform, see [10]) is defined on X^* as

$$F^*(X^*) = \sup\{X^*(x) - F(x) \,:\, x \in X\}.$$

The conjugate of $F : h \mapsto \log \varphi^h(I)$ is the relative entropy.

PROPSITION 1 *If $\omega \in \mathcal{A}_h^*$ then*

$$F^*(\omega) = \begin{cases} S(\varphi, \omega) : & \textit{if } \omega \textit{ is a state} \\ +\infty & \textit{otherwise.} \end{cases}$$

For the sake of a simple presentation I prove only that $F^*(\omega) = +\infty$ if ω is not a state. You may take the other part of the proposition as the definition of the relative entropy. (See [8] for the details; concerning the definition and general properties of the relative entropy I refer to the survey papers [2] and [5].)

For $h = t\,I$ we have

$$\omega(h) - F(h) = (\omega(I) - 1)t$$

and in the case of $\omega(I) \neq 1$, $F^*(\omega)$ must be $+\infty$. If $h_0 \in \mathcal{A}_0 \subset \mathcal{A}$ then the monotonicity theorem ([6]) tells that

$$\omega(h_0) - \log \varphi^{h_0}(I) \geq \omega(h_0) - \log(\varphi \mid \mathcal{A}_0)^{h_0}(I).$$

When ω is not positive, there exists $0 \leq h_0$ with $\omega(h_0) < 0$. Let $\mathcal{A}_0$ be the commutative subalgebra generalized by $\{h_0\}$. Then

$$\omega(t\,h_0) - \log(\varphi \mid \mathcal{A}_0)^{t\,h_0}(I) = t\,\omega(h_0) - \log \varphi(e^{t\,h_0}) \to +\infty.$$

Let $\mathcal{A}_1$ be a C^*-algebra with a separating state ρ. I write $\mathcal{A}_n$ for the n-fold minimal C^*-tensor product of $\mathcal{A}_1$ with itself and $\mathcal{A}$ for the inductive limit C^*-algebra. φ stands for the product state $\rho \otimes \rho \otimes \dots$ of $\mathcal{A}$. Note that φ is also separating. For a shift invariant state ψ of $\mathcal{A}$ the limit of

$$\frac{1}{n} S(\varphi \mid \mathcal{A}_n,\ \psi \mid \mathcal{A}_n)$$

exists and is called the mean relative entropy of φ and ψ; in notation $S_M(\varphi, \psi)$. In fact,

$$S_M(\varphi, \psi) = \sup \left\{ \frac{1}{n} S(\varphi \mid \mathcal{A}_n,\ \psi \mid \mathcal{A}_n) \,:\, n \in \mathbb{N} \right\},$$

therefore $\psi \mapsto S_M(\varphi, \psi)$ is weak* lower semicontinuous and convex ([8]).

It is worthwhile to state in this section a theorem of Størmer which will play an important role below. The result says that if ψ is a state of A invariant under all finite permutations of the factors of the infinite tensorproduct then ψ is an average (i.e., integral) of product states [11].

3 The perturbational limit principle

Let $\mathcal{A}$ be a C^*-algebra with a separating state φ. Assume that for every $n \in \mathbb{N}$ a completely positive unital mapping α_n of a C^*-algebra $\mathcal{B}$ into $\mathcal{A}$ is given and the invariance $\varphi \circ \alpha_n = \varphi \circ \alpha_m$ $(n, m \in \mathbb{N})$ holds. Motivated by Section 1, I say that (α_n) satisfies the *perturbational limit principle* (PLP) if there exists a lower semicontinuous function I from the state space $\mathcal{S}(\mathcal{B})$ of $\mathcal{B}$ into $\mathbb{R}^+ \cup \{+\infty\}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \varphi^{n\,f(\alpha_n(a))}(I) = \sup \{ f(\nu(a)) - I(\nu) \,:\, \nu \in \mathcal{S}(\mathcal{B}) \} \tag{3}$$

holds for every continuous function $f : \mathbb{R} \to \mathbb{R}$ and for every $a = a^* \in \mathcal{B}$.

Fix $a = a^* \in \mathcal{B}$ and assume that $|f(t) - g(t)| < \varepsilon$ for $|t| \leq \|a\|$. Then $\|f(\alpha_n(a)) - g(\alpha_n(a))\| \leq \varepsilon$ and

$$\varphi^{n\, f(\alpha_n(a))}(I) \leq \varphi^{n\, g(\alpha_n(a))}(I) \cdot e^{n\varepsilon}.$$

It follows that the left hand side of (3) is continuous in f. Since the right hand side is obviously continuous, due to the Weierstrass approximation theorem (3) holds for every continuous function f whenever it holds for all polynomials.

PROPSITION 2 *If* (3) *holds for every polynomial f and for every element a of a norm dense set $\mathcal{D}$ in $\mathcal{B}^{sa}$ then the PLP holds.*

It should be showed that (3) is true for a polynomial f and an arbitrary $b \in \mathcal{B}^{sa}$. The proof is an application of the Golden–Thompson–Araki inequality [1]. If $\varepsilon > 0$ is given then for a small $\delta > 0$ then $\|b_1 - b_2\| < \delta$ implies $\|f(\alpha_n(b_1)) - f(\alpha_n(b_2))\| < \varepsilon$ and we have

$$\left| \limsup_{n\to\infty} \frac{1}{n} \log \varphi^{n\, f(\alpha_n(b_1))}(I) - \liminf_{n\to\infty} \frac{1}{n} \log \varphi^{n\, f(\alpha_n(b_2))}(I) \right| \leq \varepsilon.$$

Hence the uniform continuity of the functional calculus was used and I state it for the sake of completeness as a separate lemma.

LEMMA 1 *Let f be a polynomial. Then for every $\varepsilon > 0$ and $K > 0$ there exists a $\delta > 0$ such that for every selfadjoint operator A, B with $\|A\|, \|B\| < K$ and $\|A - B\| < \delta$ the estimate*

$$\|f(A) - f(B)\| \leq \varepsilon$$

holds.

The argument that proved Proposition 2 yields also the following.

PROPSITION 3 *Let $\alpha_n, \beta_n : \mathcal{B} \to \mathcal{A}$ be completely positive unital mappings. Assume that $\varphi \circ \alpha_n = \varphi \circ \beta_m$ $(n, m \in \mathbb{N})$ and $\|\alpha_n(a) - \beta_n(a)\| \to 0$ for every $a \in \mathcal{B}^{sa}$. Then (α_n) satisfies the PLP if and only if (β_n) does so.*

Proposition 3 supplies us immediately with some trivial examples. Let $\mathcal{A} = \mathcal{B}$ and $\alpha_n = \frac{1}{n}(a + (n-1)\varphi(a)I)$. Then the PLP holds with the rate function

$$I(\nu) = \begin{cases} 0 & \nu = \varphi \\ +\infty & \text{otherwise.} \end{cases}$$

I note that taking $a = I$ it follows from (3) that $I(\nu) \geq 0$ and $I \neq +\infty$. It is also clear that if the PLP holds then the rate function is uniquely determined. Set

$$F(a) = \lim_{n\to\infty} \frac{1}{n} \log \varphi^{n\, \alpha_n(a)}(I).$$

Then F is a lower semicontinuous convex function and I is the restriction of the conjugate of F to $\mathcal{S}(\mathcal{B})$.

In the rest of the lecture I consider only $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_1 \otimes \dots$ and $\varphi = \rho \otimes \rho \otimes \dots$.

4 Perturbational limit theorems

Let $S_n \subset \operatorname{Aut}\mathcal{A}$ be the group of permutations of the first n factors in $\mathcal{A}_0 \otimes \mathcal{A}_s \otimes \ldots$. A state ψ of $\mathcal{A}$ is called symmetric if it is invariant under $\cup_{n=1}^{\infty} S_n$; the set of all symmetric states is denoted by $I(\mathcal{A})$. The mapping $E_n : \mathcal{A} \to \mathcal{A}$ defined by

$$E_n(a) = \frac{1}{n!} \sum_{\alpha \in S_n} \alpha(a) \qquad (a \in \mathcal{A})$$

is a projection of norm one onto the fixed point algebra $\{a \in \mathcal{A}_{n-1} : \alpha(a) = a$ for every $\alpha \in S_n\}$. When a happens to be in $\mathcal{A}_k$ and $n > k$ then we have also

$$E_n(a) = \frac{(n-k)!}{n!} \sum_{\alpha \in S_n/S_k} \alpha(a).$$

Note that for $a \in \mathcal{A}_0$ we have

$$E_1(a) = \frac{1}{n}(a + \gamma(a) + \cdots + \gamma^{n-1}(a)),$$

where γ is the right shift endomorphism of $\mathcal{A}$.

LEMMA 2 *Let $a \in \mathcal{A}_k$, $l \in \mathbb{N}$, $k < n \in \mathbb{N}$ and $\psi \in I(\mathcal{A})$. Then*

$$\left| \psi\left(E_n(a)^l\right) - \psi\left(a\gamma^{k+1}(a) \cdots \gamma^{l(k+1)}(a)\right) \right| \leq \frac{c(l,k)}{n} \|a\|^l.$$

The proof is elementary combinatorics. One should consider the multinomial expansion of

$$\left(\sum_{\alpha \in S_n | S_{k+1}} \alpha(a) \right)^2,$$

in which the typical term is $\alpha_{\pi_1}(a)\alpha_{\pi_2}(a)\ldots\alpha_{\pi_l}(a)$, where π_i's are such permutations of $\{0, 1, \ldots, n-1\}$ such that

$$\pi_i(\{0, 1, \ldots, k\}) \cap \pi_j(\{0, 1, 2, \ldots, k\}) = \emptyset$$

for every $i \neq j$. Due to the symmetry condition the state ψ takes at such terms the value $\psi(a\gamma^{k+1}(a)\ldots\gamma^{l(k+1)}(a))$.

It follows from Lemma 2 that $F_f^a(\psi) = \lim_{n\to\infty} \psi(f(E_n(a)))$ exists for all polynomial f and $a \in \mathcal{A}_k^{sa}$. By approximation the existence of $F_f^a(\psi)$ extends to every continuous f and $a \in \mathcal{A}^{sa}$.

LEMMA 3 *We have for every continuous $f : \mathbb{R} \to \mathbb{R}$, $a \in \mathcal{A}^{sa}$ and $\psi \in I(\mathcal{A})$ the following relation*

$$\liminf_{n\to\infty} \frac{1}{n} \log \varphi^{n\, f(E_n(a))}(I) \geq F_f^a(\psi) - S_M(\varphi, \psi)$$

holds.

According to Proposition 1

$$\log \varphi^{n\,f(E_n(a))}(I) \geq \psi(n\,f(E_n(a))) - S(\varphi \mid \mathcal{A}_{n-1}, \psi \mid \mathcal{A}_{n-1})$$

holds. Divided by n and letting $n \to \infty$ the assertion is concluded.

Let us denote by $\overline{\omega}_n$ the state $\varphi^{n\,f(E_n(a))}/\varphi^{n\,f(E_n(a))}(I)$ restricted to $\mathcal{A}_{n-1}$. We write $\tilde{\omega}_n$ for the periodical state $\overline{\omega}_n \otimes \overline{\omega}_n \otimes \dots$ of $\mathcal{A}$. $\tilde{\omega}$ is not stationary but S_m-invariant if $n > m$. Since $\tilde{\omega}_n$ is invariant under γ^n, the state

$$\omega_n = \frac{1}{n}(\tilde{\omega}_n + \tilde{\omega}_n \circ \gamma + \dots + \tilde{\omega}_n \circ \gamma^{n-1})$$

will be γ invariant. It is easy to see that the sequences (ω_n) and $(\tilde{\omega}_n)$ have the same weak* limit points, and all of them are symmetric.

Lemma 4 *Let $N_0 \subset \mathbb{N}$ be an infinite subset, ψ a weak* limit point of $\{\omega_n : n \in \mathbb{N}_0\}$, $\varepsilon > 0$ and f a polynomial. Then*

$$\left| E^a_f(\psi) - \overline{\omega}_n(f(E_n(a))) \right| < \varepsilon$$

is valid for infinitely many n in N_0.

Let $f(u) = \sum_{t=0}^{l} c_t u^t$. Remember that due to Lemma 2

$$E^a_f(\psi) = \sum_{t=0}^{l} c_t \psi\left(a\gamma^{k+1}(a)\dots\gamma^{t(k+1)}(a)\right)$$

and ψ is also a limit point of $\{\overline{\omega}_n : n \in N_0\}$. Therefore

$$\left| \sum_{t=0}^{l} c_t \left(\psi\left(a\gamma^{k+1}(a)\dots\gamma^{t(k+1)}(a)\right) - \tilde{\omega}_n\left(a\gamma^{k+1}(a)\dots\gamma^{t(k+1)}(a)\right)\right) \right| < \delta$$

for infinitely many $n \in N_0$. Since for n big enough

$$(\tilde{\omega}_n - \overline{\omega}_n)\left(a\gamma^{k+1}(a)\dots\gamma^{t(k+1)}(a)\right) = 0$$

and thanks to Lemma 2

$$\left| \tilde{\omega}_n\left(a\gamma^{k+1}(a)\dots\gamma^{t(k+1)}(a)\right) - \tilde{\omega}_n\left(E_n(a)^t\right) \right| < \delta;$$

the proof is completed. □

Theorem 3 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $a \in \mathcal{A}^{sa}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \varphi^{n\,f(E_n(a))}(I) = \sup\left\{F^a_f(\psi) - S_M(\varphi, \psi) : \psi \in I(\mathcal{A})\right\}.$$

A part of the theorem is contained in Lemma 3. Set

$$C \equiv \limsup_{n\to\infty} \frac{1}{n} \log \varphi^{n\, f(E_n(a))}(I).$$

There is an infinite subset N_0 of the integers such that

$$\overline{\omega}_n(f(E_n(a))) - \frac{1}{n} S(\varphi \mid \mathcal{A}_{n-1}, \overline{\omega}_n) > C - \varepsilon$$

for every $n \in N_0$. Let ψ be a weak* limit point of $\{\omega_n : n \in \mathbb{N}_0\}$. According to Lemma 4 there is an infinite subset N_1 of N_0 such that

$$E_f^a(\psi) \geq \overline{\omega}_n(f(E_n(a))) - \varepsilon$$

for every $n \in N_1$. By the lower semicontinuity of $\omega \mapsto S_M(\varphi, \omega)$

$$S_M(\varphi, \psi) \leq S_M(\varphi, \omega_m) + \varepsilon$$

for some $m \in N_1$. Since $S_M(\varphi, \omega_m) = \frac{1}{m} S(\varphi \mid \mathcal{A}_{n-1}, \overline{\omega}_m)$ we estimate as

$$\begin{aligned} E_f^a(\psi) - S_M(\varphi, \psi) &\geq \overline{\omega}(f(E_m(a))) - S_M(\varphi, \omega_m) - 2\varepsilon \\ &= \overline{\omega}_m(f(E_m(a))) - \frac{1}{m} S(\varphi(\mathcal{A}_{m-1}, \overline{\omega}_m)) \\ &\geq C - 3\varepsilon. \end{aligned}$$

The proof is complete. □

Theorem 3 in the case $a \in \mathcal{A}_0^{sa}$ was obtained in [7]. Essentially the same argument worked here in the more general situation.

COROLLARY 1 *Let* $a \in \mathcal{A}_k^{sa}$. *Then*

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \log \varphi^{n\, f(E_n(a))}(I) \\ &\quad = \sup \left\{ f(\nu(a)) - \frac{1}{k+1} S(\varphi \mid \mathcal{A}_k, \nu) \ := \rho \otimes \ldots \otimes \rho \in \mathcal{A}_k^* \text{ with some } \rho \in \mathcal{S}(\mathcal{A}_0) \right\}. \end{aligned}$$

If μ is a Radon measure on $\mathcal{S}(\mathcal{A}_0)$ then

$$\omega = \int (\rho \otimes \rho \otimes \ldots)\, d\mu(\rho) \in I(\mathcal{A})$$

and due to Størmer theorem we get all elements of $I(\mathcal{A})$ this way. We may assume again that f is a polynomial. If so then

$$E_f^a(\omega) = \lim_{n\to\infty} \omega(f(E_n(a))) = \int f(\rho \otimes \ldots \otimes \rho(a))\, d\mu(\rho).$$

(The tensorproduct is of $(k+1)$-fold.) On the other hand,

$$S_M(\varphi, \omega) = \int S(\varphi \mid \mathcal{A}_0, \rho)\, d\mu(\rho) = \frac{1}{k+1} \int S(\varphi \mid \mathcal{A}_k, \rho \otimes \ldots \otimes \rho)\, d\mu(\rho).$$

Therefore,

$$E_f^a(\omega) - S_M(\varphi,\omega) \le \sup\left\{ f(\rho\otimes\ldots\otimes\rho(a)) - \frac{1}{k+1}S(\varphi)\mathcal{A}_k,\rho\otimes\ldots\otimes\rho) \;:\; \rho\in\mathcal{S}(\mathcal{A}_0)\right\}$$

and the converse inequality is obvious.

The corollary tells that $\alpha_n : \mathcal{A}_0 \to \mathcal{A}$, $\alpha_n = E_n \mid \mathcal{A}_0$ satisfies the PLP and the rate function is the relative entropy. I note that

$$\sup\{f(\rho(a)) - S(\varphi \mid \mathcal{A}_0,\rho) \;:\; \rho\in\mathcal{S}(\mathcal{A}_0)\} = \sup\{f(u) - \mathcal{F}(u) \;:\; u\in\mathbb{R}\},$$

where $\mathcal{F} : \mathbb{R}\to\mathbb{R}^+\cup\{+\infty\}$ is the Legendre transform of the function $t\mapsto \log\varphi^{ta}(I)$. This is important from physical point of view (see [7, 9] and references therein).

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