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A de Finetti-type Theorem with m-Dependent States

Dénes Petz*

Instituut voor Theoretische Fysica
Universiteit Leuven
B–3030 Leuven, Belgium[1]

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Summary

In this paper certain translation invariant states on the infinite tensor product C^*-algebra $\mathcal{A} = \bigotimes_{i=1}^{\infty} \mathcal{B}_i$ are considered. For $m \in \mathbb{Z}^+$ a state φ on $\mathcal{A}$ is m-dependent if

$$\varphi(a_1 \otimes a_2 \otimes \ldots \otimes a_l) = \varphi(a_1 \otimes \ldots \otimes a_k)\varphi(a_{k+m+1} \otimes \ldots \otimes a_l)$$

whenever $l > k+m$ and $a_{k+1} = a_{k+2} = \ldots = a_{k+m} = I$. The closed convex hull of the stationary m-dependent states is characterized by a symmetry condition. The case of $m = 0$ corresponds to independence and the result reduces to a C^*-algebraic version, due to E. Størmer, of the classical de Finetti's theorem on exchangeable sequences.

1 Introduction

De Finetti's celebrated theorem asserts that any exchangeable process is an average of independent identically distributed processes [9]. More precisely, let $\xi_1, \xi_2, \ldots$ be a sequence of $\{0,1\}$-valued random variables such that

$$P(\xi_1 = e_1,\ \xi_2 = e_2, \ldots,\ \xi_n = e_n) = P(\xi_1 = e_{\pi(1)},\ \xi_2 = e_{\pi(2)}, \ldots,\ \xi_n = e_{\pi(n)})$$

holds for all $n \in \mathbb{N}$, for all permutations π of $\{1, 2, \ldots, n\}$ and for every $e_1, e_2, \ldots, e_n \in \{0,1\}$). Then there exists a unique probability measure μ on $[0,1]$ such that

$$P(\xi_1 = e_1,\ \xi_2 = e_2, \ldots,\ \xi_n = e_n) = \int p^s (1-p)^{n-s}\, d\mu(p),$$

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[1]Present address: Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, H–1053 Budapest, Reáltanoda u. 13–15, Hungary. E-mail: petz@math.bme.hu

where $s = e_1 + e_2 + \ldots + e_n$ [8].

In the last two decades there has been a new strong interest in exchangeability and several extensions of de Finetti's original theorem have been obtained. Concerning this development we refer to the survey articles [2, 14] and below we describe results directly related to our generalization. Probabilists being averse to the C^*-algebraic language of the present paper may consult [1] to make themselves familiar with the algebraic formulation of de Finetti's theorem.

Hewitt and Savage extended de Finetti's theorem from $\{0,1\}$ to any compact Hausdorff space Ω [10]. For our purpose it is more suitable to consider measures on the infinite product space $X = \Omega \times \Omega \times \ldots$. Any permutation of the natural numbers gives rise to a transformation of X through the permutation of the coordinates. This transformation is continuous and induces a mapping on the space of continuous functions on X. The functional analytic formulation prefers positive functionals on the function space to measures. Denote by $\mathcal{B}(\mathcal{A})$ the C^*-algebra of all continuous functions on $\Omega(X)$. Then $\mathcal{A} = \mathcal{B} \otimes \mathcal{B} \otimes \ldots$ and the result of Hewitt and Sevage tells us in this language that every symmetric (i.e., permutation invariant) positive functional is a unique mixture of product ones. Allowing arbitrary C^*-algebra $\mathcal{B}$ Størmer proved in 1969 that the symmetric states form a simplex with a closed extreme boundary consisting of product states [17]. His result inspired similar theorems and found applications also in quantum statistical mechanics [5, 6, 7, 11, 15].

Our aim in the present article is to carry out a generalization. In order to have the extremal states to be m-dependent we use a weaker notion of symmetry. If $m \in \mathbb{Z}^+$ then we say that a stationary state φ on $\mathcal{A}$ is m-dependent if

$$\varphi(a_1 \otimes a_2 \otimes \ldots \otimes a_l) = \varphi(a_1 \otimes \ldots \otimes a_k)\varphi(a_{k+m+1} \otimes \ldots \otimes a_l)$$

whenever $l > k + m$ and $a_{k+1} = a_{k+2} = \ldots = a_{k+m} = I$. This notion comes from probability theory where m-dependent sequences of random variables are frequently studied as slight extension of independence [12].

The method we use is based on results from harmonic analysis on abelian semigroups and originated from the beautiful paper of Ressel [16].

2 Preliminaries

Let $\mathcal{B}$ be a unital C^*-algebra and for each $n \in \mathbb{N}$ let $\mathcal{B}_n$ a copy of $\mathcal{B}$. Write $\mathcal{A}$ for the infinite projective tensorproduct $\bigotimes_{i=1}^{\infty} \mathcal{B}_i$ [13, 18]. If I is a subset of $\mathbb{N}$ then we denote by $\mathcal{A}_I$ the C^*-subalgebra generated by $\cup\{\mathcal{B}_n : n \in I\}$. If π is a permutation of $\mathbb{N}$ then there is an automorphism α_π of $\mathcal{A}$ such that

$$\alpha_\pi(i_k(b)) = i_{\pi(k)}(b) \qquad (b \in \mathcal{B},\ k \in \mathbb{N}),$$

where $i_k : \mathcal{B} \to \mathcal{B}_k \subset \mathcal{A}$ is the embedding of $\mathcal{B}$ as a k^{th} factor. After Hewitt, Savage and Størmer we call a state φ of $\mathcal{A}$ symmetric if $\varphi \circ \alpha_\pi = \varphi$ for all finite permutations of $\mathbb{N}$. Evidently, a symmetric state is shift invariant. We write α for the right shift endomorphism of $\mathcal{A}$.

Since the permutation group is generated by transpositions, the permutation invariance may be formulated in a slightly weaker way.

$$\varphi(a_1 a\alpha(b)a_2) = \varphi(a_1 b\alpha(a)a_2) \tag{1}$$

whenever $a_1 \in \mathcal{A}_{[1,l-1]}$, $a, b \in \mathcal{B}_l$ and $a_2 \in \mathcal{A}_{[l+2,\infty]}$. Condition (1) requires the invariance of φ under the permutation exchanging l with $l+1$ and leaving all other coordinates fixed.

Let $m \in \mathbb{Z}^+$ be fixed. We say that the state φ of $\mathcal{A}$ is m-dependent if

$$\varphi(ab) = \varphi(a)\varphi(b) \tag{2}$$

whenever $a \in \mathcal{A}_{[1,n]}$ and $b \in \mathcal{A}_{[n+m+1,\infty]}$. In probability theory m-dependent sequences of random variables are easily obtained by taking functions of an independent sequence [12]. (For example, if (η_n) is independent then $\xi_n = \eta_n + \eta_{n+1} + \ldots + \eta_{n+m}$ forms an m-dependent sequence.) In the theory of operator algebras it is a bit more complicated to show an m-dependent state. In this paper we shall consider only α-invariant m-dependent states.

We say that φ is m-symmetric if the following two conditions are satisfied.

$$\varphi(a_1 a\alpha^{k+m}(b)a_2) = \varphi(a_1 b\alpha^{k+m}(a)a_2) \tag{3}$$

if $a_1 \in \mathcal{A}_{[1,l-1]}, a, b \in \mathcal{A}_{[l+m,l+m+k]}$, $a_2 \in \mathcal{A}_{[l+2m+2k+1,\infty]}$ and

$$\varphi(a_1, \alpha^k(a)) \text{ is independent of } k \tag{4}$$

if $a \in \mathcal{A}_{[1,l-1]}$, $b \in \mathcal{A}_{[l+m,\infty]}$ and $k, l \in \mathbb{Z}^+$. Below we refer to conditions (3) and (4) simply as symmetry conditions. Note that an m-symmetric state is necessarily α-invariant. Indeed, (4) tells that it is so on $a \in \mathcal{A}_{[m+1,\infty]}$ and combining this with (3) we get the stationarity. We shall see that the appropriate symmetry conditions needed to characterize the closed convex hull of m-dependent states are (3) and (4).

Now we review a few things from harmonic analysis on semigroups. Let S be an abelian semigroup, written additively, with neutral element written 0. A function $f : S \to \mathbb{R}$ is completely positive definite if

$$\sum_{j,k=1}^{n} c_j c_k f(s + s_j + s_k) \geq 0 \tag{5}$$

for all $n \geq 1$, $s, s_1, s_2, \ldots, s_n \in S$ and $c_1, c_2, \ldots, c_n \in \mathbb{R}$. A semicharacter $\rho : S \to \mathbb{R}$ means a function such that

$$\rho(s+t) = \rho(s)\rho(t) \quad \text{and} \quad \rho(0) = 1.$$

A nonnegative semicharacter is completely positive definite. An introduction to positive definite functions on abelian semigroups is found in [3]. For our purpose a theorem of Ressel has central importance [16]. Let F be a completely positive definite bounded function on S. Then there exists a unique Radon measure μ over the space of nonnegative bounded semicharacters S_+^* such that

$$F(s) = \int \rho(s)\, d\mu(\rho).$$

3 Result

Let $\mathcal{A}_{[1,n]}$, $\mathcal{A}$ and α be as above. We denote by $\mathcal{S}_m$ the set of all m-symmetric states. On the state space of $\mathcal{A}$ we consider the weak* topology.

THEOREM 1 *The closed extremal boundary of the compact convex set $\mathcal{S}_m$ consists of the (α-invariant) m-dependent states. Every $\psi \in \mathcal{S}_m$ admits an integral decomposition*

$$\psi(a) = \int \rho(a)\, d\mu(\rho) \qquad (a \in \mathcal{A})$$

with a unique probability Radon measure on the (α-invariant) m-dependent states.

LEMMA 1 *If $\psi \in \mathcal{S}_m$ and $a_1 \in \mathcal{A}^+_{[1,n]}$, $a_2 \in \mathcal{A}_{[1,k]}$ then*

$$\psi(a_1 \alpha^{n+m}(a_2^*) \alpha^{n+k+2m}(a_2)) \geq 0.$$

Since

$$a_1 \left(\sum_{l=1}^{t} \alpha^{l(k+m)+n}(a_2^*) \right) \left(\sum_{l=1}^{t} \alpha^{l(k+m)+n}(a_2) \right) \geq 0$$

we have

$$T = \psi \left(a_1 \sum_{l=1}^{t} \alpha^{l(k+m)+n}(a_2^*) \sum_{l=1}^{t} \alpha^{l(k+m)+n}(a_2) \right) \geq 0.$$

Due to the symmetry conditions (3) and (4)

$$T = t\psi(a_1 \alpha^{n+m}(a_2^* a_2)) + (t^2 - t)\psi(a_1 \alpha^{n+m}(a_2^*) \alpha^{n+k+2m}(a_2))$$

and division by t^2 and letting $t \to \infty$ gives the Lemma.

To prove the Theorem we set S for the free abelian semigroup generated by the positive contractions in $\mathcal{A}_\infty = \cup\{\mathcal{A}_{[1,n]} : n \in \mathbb{N}\}$. A typical element of S is a formal sum

$$s = a_1 \dot{+} a_2 \dot{+} \dots \dot{+} a_k \qquad (a_i \in \mathcal{A}^{+,1}_\infty,\ k \in \mathbb{N}),$$

where the sequence $(a_1, a_2, \dots, a_k)$ is determined by s up to a permutation. For a finite sequence $(a_1, a_2, \dots, a_k)$ in $\mathcal{A}^{+,1}_\infty$ we set

$$G_0(a_1, a_2, \dots, a_k) = \alpha^{m+l}(a_1) \dots \alpha^{k(m+l)}(a_k)$$

if l is the smallest integer such that $a_1, a_2, \dots, a_k \in \mathcal{A}_{[1,l]}$. If $s = a_1 \dot{+} a_2 \dot{+} \dots \dot{+} a_k$ and $\psi \in \mathcal{S}_m$ then we may define

$$F(s) = \psi(G_0(a_1, a_2, \dots, a_k))$$

because the symmetry conditions on ψ provide that the right hand side is independent of the ordering of $(a_1, a_2, \dots, a_k)$. For the sake of a simpler notation we choose and fix for every $s \in S$ an ordered decomposition $s = a_1 \dot{+} a_2 \dot{+} \dots \dot{+} a_k$ and we set

$$G(s) = G_0(a_1, a_2, \dots, a_k).$$

We have $G(s) \in \mathcal{A}_{[1,k(m+l)+l]}$ with l described above and we call the number $k(m+l)+l = L(s)$ the rank of s. The symmetry conditions yield that

$$F(s_1 \dot{+} s_2 \dot{+} s_3) = \psi(G(s_1)\alpha^t(G(s_2))\alpha^u(G(s_3)))$$

whenever $t \geq L(s_1)$, $u \geq t + L(s_2)$ and $s_1, s_2, s_3 \in S$.

Now we are going to check that F is completely positive definite. We have to show that

$$\sum_{j,k=1}^{n} c_j c_k F(s \dot{+} s_j \dot{+} s_k) \geq 0$$

for all $n \geq 1$, $s, s_1, s_2, \dots s_n \in S$ and $c_1, c_2, \dots, c_n \in \mathbb{R}$. We choose $l \in \mathbb{N}$ such that

$$l \geq L(s), L(s_1), \dots, L(s_n)$$

and have

$$F(s \dot{+} s_i \dot{+} s_j) = \psi(G(s)\alpha^{il}(G(s_i))\alpha^{(j+n)l}(G(s_j))).$$

Taking $b_0 = G(s)$ and $b_i = \alpha^{il}(g(s_i))$ we obtain

$$\sum_{j,k=1}^{n} c_j c_k F(s \dot{+} s_i \dot{+} s_k) = \psi\left(b_0\left(\sum_{i=1}^{n} c_i b_i\right)\alpha^{nl}\left(\sum_{j=1}^{n} c_j b_j\right)\right)$$

which is nonnegative thanks to the Lemma. Indeed, $b_0, b_1, \dots, b_n$ are positive operators and $b_1, \dots, b_n$ are contained in $\mathcal{A}_{[m+l,nl]}$.

The function $F : S \to [0,1]$ is proven to be completely positive definite. According to Proposition 1 of [16] there is a unique Radon probability measure μ on the compact subset $K = \{\rho \in S^* : 0 \leq \rho \leq 1\}$ of the semicharacters such that

$$F(s) = \int \rho(s)\, d\mu(\rho) \qquad (s \in S).$$

To each m-dependent state φ there corresponds a semicharacter $j(\varphi)$ in an obvious way:

$$j(\varphi)(a_1 \dot{+} a_2 \dot{+} \dots \dot{+} a_k) = \prod_{i=1}^{k} \varphi(a_i).$$

So j is an embedding of the m-dependent states into K. We have to prove that the measure μ is actually concentrated on the range of j.

Following Ressel we fix $a_1, a_2 \in \mathcal{A}_\infty^{+,1}$ such that $a_1 + a_2 = I$. Then

$$\begin{aligned}
\int (\rho(a_1) + \rho(a_2))^n\, d\mu(\rho) &= \sum_{k=0}^{n} \binom{n}{k} \int \rho(a_1)^k \rho(a_2)^{n-k}\, d\mu(\rho) \\
&= \sum_{k=0}^{n} \binom{n}{k} \int \rho(ka_1 \dot{+} (n-k)a_2)\, d\mu(\rho) \\
&= \sum_{k=0}^{n} \binom{n}{k} F(ka_1 \dot{+} (n-k)a_2) \\
&= \psi((a_1 + a_2)\alpha^{l+m}(a_1 + a_2) \dots \alpha^{(n-1)(l+m)}(a_1 + a_2)) = 1
\end{aligned}$$

as a consequence of symmetry. So we find that $\int(\rho(a_1) + \rho(a_2))^n \, d\mu(\rho) = 1$ for every $n \in \mathbb{N}$, which implies easily that

$$H^2(a_1, a_2) = \{\rho \in K \, : \, \rho(a_1) + \rho(a_2) = 1\}$$

if of measure 1. Similarly, for $b_1, b_2, b_3 \in \mathcal{A}_\infty^{+,1}$ with $b_1 + b_2 + b_3 = I$ we have $\rho(b_1) + \rho(b_2) + \rho(b_3) = 1$ for almost all $\rho \in K$. Since

$$H^3(b_1, b_2, b_3) = \{\rho \in K \, : \, \rho(b_1) + \rho(b_2) + \rho(b_3) = 1\}$$

is closed we obtain that

$$K_1 = \cap \{H^2(a_1, a_2) \, : \, a_1 + a_2 = I\} \cap \{H^3(b_1, b_2, b_3) \, : \, b_1 + b_2 + b_3 = I\}$$

is a closed set of full measure.

Now let $\rho \in K_1$. If $a + b + c = I$ then $\rho(a) + \rho(b) + \rho(c) = 1 = \rho(a + b) + \rho(c)$. This gives $\rho(a + b) = \rho(a) + \rho(b)$ and, in particular, ρ is monotone, $\rho(0) = 0$ and $\rho(I) = 1$. By induction $\rho(\lambda a) = \lambda \rho(a)$ for all rational $\lambda \in [0, 1]$. Due to the monotonicity this must hold also for irrational λ. It turns out that ρ comes from the restriction of a state on $\mathcal{A}$ to the positive contractions $\mathcal{A}_\infty^{+,1}$ of $\mathcal{A}_\infty$.

Let us fix an $m' \geq m$ and $a \in \mathcal{A}_\infty^{+,1}$. We define three endomorphisms of S. Being S free any mapping given on the generators extends to a homomorphism in a unique way. Set

$$h(b) = \alpha(b) \qquad (b \in \mathcal{A}_\infty^{+,1})$$

$$h_a(b) = b \dotplus a \qquad (b \in \mathcal{A}_\infty^{+,1})$$

$$k_a^{m'}(b)(a) = b\alpha^{l+m'}(a) \qquad (b \in \mathcal{A}_{[l,\infty)}^{+,1} \text{ but } b \notin \mathcal{A}_{[l-1,\infty)}^{+,1}).$$

It is straightforward to see that

$$F \circ h = F \quad \text{and} \quad F \circ h_a = F \circ k_a^{m'}.$$

From the uniqueness of the decomposing measure we obtain that μ must be concentrated on the closed set

$$K_2 = \{\rho \in K_1 \, : \, \rho \circ h_a = \rho \circ k_a^{m'} \quad \text{for every} \quad m' \geq m, \ a \in \mathcal{A}_\infty^{+,1} \quad \text{and} \quad \rho \circ h = \rho\}.$$

For $\rho \in K_2$ the α-invariance is evident and

$$\rho(a_1 \alpha^{l+m'}(a)) = \rho \circ k_a^{m'}(a_1) = \rho \circ h_a(a_1) = \rho(a \dotplus a_1) = \rho(a)\rho(a_1)$$

provides the m-dependency (2). So we proved that the range of the embedding j is contained in K_2 and the proof of the theorem is complete. □

4 Discussion

First we formulate our theorem in terms of random variables. Let $\xi_1, \xi_2, \dots$ be a stationary sequence of random variables. In order to make clear the m-symmetry we introduce a convenient notation. G with some subscript will always be an open set. If $I = \{n, n+1, \dots, n+t\}$ and $J = \{k, k+1, \dots, k+t\}$ are intervals in $\mathbb{N}$ of the same length then we write $\xi(I) \in G(J)$ for the set

$$\bigcap_{i=0}^{t} \{\xi_{n+i} \in G_{k+i}\}.$$

The notion of m-symmetry includes two kinds of invariance. On the one hand,

$$\begin{aligned} &P(\xi(I_1) \in G(I_1),\ \xi(I_2) \in G(J_2),\ \xi(I_3) \in G(J_3),\ \xi(I_4) \in G(I_4)) \\ &\quad = P(\xi(I_1) \in G(I_1),\ \xi(I_2) \in G(J_3),\ \xi(I_3) \in G(J_2),\ \xi(I_4) \in G(I_4)) \end{aligned}$$

whenever $I_1 < I_2 < I_3 < I_4$ are intervals in $\mathbb{N}$ such that there is a gap of length not smaller than m between the consecutive ones and J_2, J_3 are also intervals with $|I_i| = |J_i|$. On the other hand, it is required that

$$P(\xi(I_1) \in G(J_1),\ \xi(I_2 + t) \in G(J_2))$$

does not depend on $t \in \mathbb{Z}^+$ whenever $I_1 < I_2$, $|I_i| = |J_i|$ and there is a gap of at least m numbers between I_1 and I_2. The Theorem asserts that a stationary m-symmetric sequence is a mixture of stationary m-dependent sequences. We have not seen this result in the literature of probability theory.

If we compare symmetry with m-symmetry then the essential point is the following. While symmetry is characterized by transposition of single coordinates in the definition of m-symmetry transposition of blocks of coordinates occurs (under the condition that there is an interval of length m between the two blocks).

Now we show an example of an m-dependent stationary state on an infinite tensor-product of C^*-algebras. Let $l^2(\mathbb{Z})$ be the complex Hilbert space of the double infinite sequences. Then there is a unique C^*-algebra $\tilde{\mathcal{A}}$ determined by the following conditions

(i) for every $f \in l^2(\mathbb{Z})$ a unitary $W(f)$ in $\tilde{\mathcal{A}}$ and the linear span of $\{W(f) : f \in l^2(\mathbb{Z})\}$ is dense in $\tilde{\mathcal{A}}$.

(ii) $W(-f) = W(f^*)$ $(f \in l^2(\mathbb{Z}))$

(iii) $W(f)W(g) = W(f+g)\exp(i \operatorname{Im} \langle f, g\rangle)$ $(f, g \in l^2(\mathbb{Z}))$.

We fix a function $h : \mathbb{Z} \to \mathbb{C}$ such that $\operatorname{supp} h \subset [0, m]$ and the Fourier transform of h is nonnegative. Define the convolution operator A on $l^2(\mathbb{Z})$ by

$$(Af)(n) = \sum_{i=-\infty}^{\infty} h(n-i)g(i).$$

Then

$$\alpha(f,g) = \text{Re}\,(\langle f,g\rangle + \langle Af,g\rangle) \qquad (f,g \in l^2(\mathbb{Z}))$$

gives a positive semidefinite real bilinear form on $l^2(\mathbb{Z})$ and there is a state φ on $\tilde{\mathcal{A}}$ such that

$$\varphi(W(f)) = \exp\left(-\frac{1}{2}\alpha(f,f)\right) \qquad (f \in l^2(\mathbb{Z})).$$

(Concerning the details of the construction of $\tilde{\mathcal{A}}$ and φ we refer to 5.2 of [4].) For $n \in \mathbb{Z}^+$ let $\mathcal{B}_n$ be the subalgebra generated by $\{W(f) : \text{supp}\, f \subset \{n\}\}$. Then

$$\bigotimes_{n=1}^{\infty} \mathcal{B}_n = \mathcal{A} = C^*\{W\{f\} : \text{supp}\, f \subset \mathbb{N}\}$$

and φ is m-dependent on $\mathcal{A}$. Indeed, by straightforward checking

$$\varphi(W(f)W(g)) = \varphi(W(f))\varphi(W(g))$$

if $\text{supp}\, f \subset [1,l]$ and $\text{supp}\, g \subset [l+m',\infty)$ with $m' > m$.

Using the notation of the preliminaries, we assume now that $\mathcal{B} = B(\mathcal{H})$ for a complex Hilbert space $\mathcal{H}$. Hudson and Moody showed that if the symmetric state φ of $\mathcal{A}$ restricted to every $\mathcal{A}_{[1,n]}$ is normal (that is, φ is locally normal), then the decomposing measure is concentrated on the normal states [10]. Their proof goes through without change in the m-symmetric generalization.

In [7] Fannes, Lewis and Verbeure consider states on infinite tensorproduct $\mathcal{C} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \ldots$ which are symmetric in the $\mathcal{B}$ factors. From the Størmer theorem they deduced a characterization of there states of $\mathcal{C} \otimes \mathcal{B} \otimes \ldots$. The m-symmetric version of their result may be obtained similarly.

In our proof the stationarity of all states was exploited very much. Fannes treated nonstationary states in a de Finetti-type theorem [5, 6]. We can imagine that by a stronger control of the permutations his theorem may be recaptured by this analytical approach and also an m-symmetric version might be obtained.

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References

[1] Accardi, L., Pistone, G.: De Finetti's theorem, sufficiency and Dobrushin's theory. In: Koch, G., Spizzichino, F. (eds.) *Exchangeability in probability and statistics.* pp. 125–156. Amsterdam New York: North Holland 1982.

[2] Aldous, D.: *Exchangeability and related topics.* (Lect. Notes Math., Vol. 1196), Berlin Heidelberg New York: Springer 1985.

[3] BERG, C., CHRISTENSEN, J.P.R., RESSEL, P.: *Harmonic analysis on semigroups.* Berlin Heidelberg New York: Springer 1984.

[4] BRATTELI, O., ROBINSON, D.W.: *Operator algebras and quantum statistical mechanics II.* Berlin Heidelberg New York: Springer 1981.

[5] FANNES, M.: Non-translation invariant product states of classical lattice systems. *Publ. Res. Inst. Math. Sci.* **23**, pp. 895–908, (1987).

[6] FANNES, M.: An application of de Finetti's theorem. In: Accardi, L., Walderfels, W., von (eds.) *Quantum probability and applications III.* Proceedings, Oberwolfach 1987 (Lect. Notes Math., Vol. 1303, pp. 89–102) Berlin Heidelberg New York: Springer 1988.

[7] FANNES, M., LEWIS, J.T., VERBEURE, A.: Symmetric states of composite systems. *Lett. Math. Phys.* **15**, pp. 255–260, (1988).

[8] FELLER, W.: *An introduction to probability theory and its applications, Vol. II.* New York: Wiley 1966.

[9] FINETTI, B. de: Funzione caratteristica di un fenomeno aleatosio. *Atti Accad. Naz. Linoei, VI. Ser., Mem. Cl. Sci. Fis. Mat. Nat.* **4**, pp. 251–299, (1931).

[10] HEWITT, E., SAVAGE, L.F.: Symmetric measures on Cartesian products. *Trans. Am. Math. Soc.* **80**, pp. 470–501, (1955).

[11] HUDSON, R.L., MOODY, G.R.: Locally normal symmetric states and an analog of de Finetti's theorem. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 33, pp. 343–351, (1976).

[12] IBRAGIMOV, I.A., LINNIK, YU.V.: *Independent and stationary sequences of random variables.* Groningen: Valters-Noordhoff 1971.

[13] KADISON, R.V., RINGROSE, J.R.: *Fundamentals of the theory of operator algebras I, II.* Orlando: Academic Press 1986.

[14] KINGMAN, J., Uses of exchangeability. *Ann. Probab.* **6**, pp. 183–197, (1978).

[15] PETZ, D., RAGGIO, G.A., VERBEURE, A.: Asymptotics of Varadhan-type and the Gibbs variational principle. *Commun. Math. Phys.* **121**, pp. 271–282, (1989).

[16] RESSELL, P.: De-Finetti-type theorems: an analytical approach. *Ann. Probab.* **13**, pp. 898–922, (1985).

[17] STØRMER, E.: Symmetric states on infinite tensor products of C^*-algebras. *J. Funct. Anal.* **3**, pp. 48–68, (1969).

[18] TAKESAKI, M.: *Theory of operator algebras I.* Berlin Heidelberg New York: Springer 1979.