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Introduction to quantum Fisher information

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Abstract

The subject of this paper is a mathematical transition from the Fisher information of classical statistics to the matrix formalism of quantum theory. If the monotonicity is the main requirement, then there are several quantum versions parametrized by a function. In physical applications the minimal is the most popular. There is a one-to-one correspondence between Fisher informations (called also monotone metrics) and abstract covariances. The skew information and the χ^2 -divergence are treated here as particular cases.

Keywords: Quantum state estimation, Fisher information, Cramér-Rao inequality, monotonicity, covariance, operator monotone function, skew information.

Introduction

Parameter estimation of probability distributions is one of the most basic tasks in information theory, and has been generalized to quantum regime [20, 22] since the description of quantum measurement is essentially probabilistic. First let us have a look at the classical Fisher information.

Let (X, \mathcal{B}, μ) be a probability space. If $\theta = (\theta^1, \dots, \theta^n)$ is a parameter vector in a neighborhood of $\theta_0 \in \mathbb{R}^n$, then we should have a smooth family μ_θ of probability

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measures with probability density f_θ :

$$\mu_\theta(H) = \int_H f_\theta(x) d\mu(x) \quad (H \in \mathcal{B}).$$

The Fisher information matrix at θ_0 is

$$\begin{aligned} J(\mu_\theta; \theta_0)_{ij} &:= \int_X f_{\theta_0}(x) \frac{\partial}{\partial \theta^i} \log f_\theta(x) \Big|_{\theta=\theta_0} \frac{\partial}{\partial \theta^j} \log f_\theta(x) \Big|_{\theta=\theta_0} d\mu(x) \\ &= \int_X \frac{1}{f_{\theta_0}(x)} \partial_i f_{\theta_0}(x) \Big|_{\theta=\theta_0} \partial_j f_{\theta_0}(x) \Big|_{\theta=\theta_0} d\mu(x) \\ &= - \int_X f_{\theta_0}(x) \partial_{ij} \log f_\theta(x) \Big|_{\theta=\theta_0} d\mu(x) \quad (1 \leq i, j \leq n). \end{aligned} \tag{1}$$

Note that $\log f_\theta(x)$ is usually called **log likelihood** and its derivative is the **score function**.

The Fisher information matrix is positive semidefinite. For example, if the parameter $\theta = (\theta^1, \theta^2)$ is two dimensional, then the Fisher information is a 2×2 matrix. From the Schwarz inequality

$$\begin{aligned} J(\mu_\theta; \theta_0)_{12}^2 &\leq \int_X \left[\frac{1}{\sqrt{f_{\theta_0}(x)}} \partial_1 f_{\theta_0}(x) \right]^2 d\mu(x) \int_X \left[\frac{1}{\sqrt{f_{\theta_0}(x)}} \partial_2 f_{\theta_0}(x) \right]^2 d\mu(x) \\ &= J(\mu_\theta; \theta_0)_{11} J(\mu_\theta; \theta_0)_{22} \end{aligned}$$

Therefore the matrix $J(\mu_\theta; \theta_0)$ is positive semidefinite.

Assume for the sake of simplicity, that θ is a single parameter. The random variable $\hat{\theta}$ is an **unbiased estimator** for the parameter θ if

$$\mathbb{E}_\theta(\hat{\theta}) := \int \hat{\theta}(x) f_\theta(x) d\mu(x) = \theta$$

for all θ . This means that the expectation value of the estimator is the parameter. The **Cramér-Rao inequality**

$$\text{Var}(\hat{\theta}) := \mathbb{E}_\theta((\hat{\theta} - \theta)^2) \geq \frac{1}{J(\mu_\theta; \theta)}$$

gives a lower bound for the variance of an unbiased estimator. (For more parameters we have an inequality between positive matrices.)

In the quantum formalism a probability measure is replaced by a positive matrix of trace 1. (Its eigenvalues form a probability measure, but to determine the so-called density matrix a basis of the eigenvectors is also deterministic.) If a parametrized family of density matrices D_θ is given, then there is a possibility for the quantum Fisher information. This quantity is not unique, the possibilities are determined by linear mappings.

The analysis of the linear mappings is the main issue of the paper. In physics $\theta \in \mathbb{R}$ mostly, but if it is an n -tuple, then Riemannian geometries appear. A coarse-graining gives a monotonicity of the Fisher informations and this is the second main subject of the present overview.

Fisher information has a big literature both in the classical and in the quantum case. The reference of the papers is not at all complete here. The aim is to have an introduction.

1 A general quantum setting

The Cramér-Rao inequality belongs to the basics of estimation theory in mathematical statistics. Its quantum analog appeared in the 1970's, see the book [20] of Helstrom and the book [22] of Holevo. Although both the classical Cramér-Rao inequality and its quantum analog are mathematically as trivial as the Schwarz inequality, the subject takes a lot of attention because it is located on the boundary of statistics, information and quantum theory. As a starting point we give a very general form of the quantum Cramér-Rao inequality in the simple setting of finite dimensional quantum mechanics. The paper [43] is followed here.

For $\theta \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$ a statistical operator $\rho(\theta)$ is given and the aim is to estimate the value of the parameter θ close to 0. Formally $\rho(\theta)$ is an $n \times n$ positive semidefinite matrix of trace 1 which describes a mixed state of a quantum mechanical system and we assume that $\rho(\theta)$ is smooth (in θ). Assume that an estimation is performed by the measurement of a self-adjoint matrix A playing the role of an observable. A is called **locally unbiased estimator** if

$$\frac{\partial}{\partial \theta} \text{Tr} \rho(\theta) A \Big|_{\theta=0} = 1. \quad (2)$$

This condition holds if A is an unbiased estimator for θ , that is

$$\text{Tr} \rho(\theta) A = \theta \quad (\theta \in (-\varepsilon, \varepsilon)). \quad (3)$$

To require this equality for all values of the parameter is a serious restriction on the observable A and we prefer to use the weaker condition (2).

Let $[K, L]_\rho$ be an inner product (or quadratic cost function) on the linear space of self-adjoint matrices. This inner product depends on a density matrix and its meaning is not described now. When $\rho(\theta)$ is smooth in θ , as already was assumed above, then

$$\frac{\partial}{\partial \theta} \text{Tr} \rho(\theta) B \Big|_{\theta=0} = [B, L]_{\rho(0)} \quad (4)$$

with some $L = L^*$. From (2) and (4), we have $[A, L]_{\rho(0)} = 1$ and the Schwarz inequality yields

$$[A, A]_{\rho(0)} \geq \frac{1}{[L, L]_{\rho(0)}}. \quad (5)$$

This is the celebrated **inequality of Cramér-Rao type** for the locally unbiased estimator.

The right-hand-side of (5) is independent of the estimator and provides a lower bound for the quadratic cost. The denominator $[L, L]_{\rho(0)}$ appears to be in the role of Fisher information here. We call it **quantum Fisher information** with respect to the cost function $[\cdot, \cdot]_{\rho(0)}$. This quantity depends on the tangent of the curve $\rho(\theta)$. If the densities $\rho(\theta)$ and the estimator A commute, then

$$L = \rho_0^{-1} \frac{d\rho(\theta)}{d\theta} \quad \text{and} \quad [L, L]_{\rho(0)} = \text{Tr} \rho_0^{-1} \left(\frac{d\rho(\theta)}{d\theta} \right)^2 = \text{Tr} \rho_0 \left(\rho_0^{-1} \frac{d\rho(\theta)}{d\theta} \right)^2.$$

Now we can see some similarity with (1).

The quantum Fisher information was defined as $[L, L]_{\rho(0)}$, where

$$\left. \frac{\partial}{\partial \theta} \rho(\theta) \right|_{\theta=0} = L.$$

This L is unique, but the quantum Fisher information depends on the inner product $[\cdot, \cdot]_{\rho(0)}$. This is not unique, there are several possibilities to choose a reasonable inner product $[\cdot, \cdot]_{\rho(0)}$. Note that $[A, A]_{\rho(0)}$ should have the interpretation of “variance” (if $\text{Tr} \rho_0 A = 0$.)

Another approach is due to Braunstein and Caves [4] in physics, but Nagaoka considered a similar approach [34].

1.1 From classical Fisher information via measurement

The observable A has a spectral decomposition

$$A = \sum_{i=1}^k \lambda_i E_i.$$

(Actually the property $E_i^2 = E_i$ is not so important, only $E_i \geq 0$ and $\sum_i E_i = I$. Hence $\{E_i\}$ can be a so-called POVM as well.) On the set $X = \{1, 2, \dots, k\}$ we have probability distributions

$$\mu_\theta(\{i\}) = \text{Tr} \rho(\theta) E_i.$$

Indeed,

$$\sum_{i=1}^k \mu_\theta(\{i\}) = \text{Tr} \rho(\theta) \sum_{i=1}^k E_i = \text{Tr} \rho(\theta) = 1.$$

Since

$$\mu_\theta(\{i\}) = \frac{\text{Tr} \rho(\theta) E_i}{\text{Tr} D E_i} \text{Tr} D E_i$$

we can take

$$\mu(\{i\}) = \text{Tr } DE_i$$

where D is a statistical operator. Then

$$f_\theta(\{i\}) = \frac{\text{Tr } \rho(\theta) E_i}{\text{Tr } DE_i} \quad (6)$$

and we have the classical Fisher information defined in (1):

$$\sum_i \frac{\text{Tr } \rho(\theta) E_i}{\text{Tr } DE_i} \left[\frac{\text{Tr } \rho(\theta)' E_i}{\text{Tr } DE_i} : \frac{\text{Tr } \rho(\theta) E_i}{\text{Tr } DE_i} \right]^2 \text{Tr } DE_i = \sum_i \frac{[\text{Tr } \rho(\theta)' E_i]^2}{\text{Tr } \rho(\theta) E_i}$$

(This does not depend on D .) In the paper [4] the notation

$$F(\rho(\theta); \{E(\xi)\}) = \int \frac{[\text{Tr } \rho(\theta)' E(\xi)]^2}{\text{Tr } \rho(\theta) E(\xi)} d\xi$$

is used, this is an integral form, and for Braunstein and Caves the quantum Fisher information is the supremum of these classical Fisher informations [4].

Theorem 1.1 *Assume that D is a positive definite density matrix, $B = B^*$ and $\text{Tr } B = 0$. If $\rho(\theta) = D + \theta B + o(\theta^2)$, then the supremum of*

$$F(\rho(0); \{E_i\}) = \sum_i \frac{[\text{Tr } BE_i]^2}{\text{Tr } DE_i} \quad (7)$$

over the measurements $A = \sum_{i=1}^k \lambda_i E_i$ is

$$\text{Tr } B \mathbb{J}_D^{-1}(B), \quad \text{where} \quad \mathbb{J}_D C = (DC + CD)/2. \quad (8)$$

Proof: The linear mapping \mathbb{J}_D is invertible, so we can replace B in (7) by $\mathbb{J}_D(C)$. We have to show

$$\begin{aligned} & \sum_i \frac{[\text{Tr } \mathbb{J}_D(C) E_i]^2}{\text{Tr } DE_i} \\ &= \frac{1}{4} \sum_i \frac{(\text{Tr } CDE_i)^2 + (\text{Tr } DCE_i)^2 + 2(\text{Tr } CDE_i)(\text{Tr } DCE_i)}{\text{Tr } DE_i} \leq \text{Tr } DC^2. \end{aligned}$$

This follows from

$$\begin{aligned} (\text{Tr } CDE_i)^2 &= \left(\text{Tr } (E_i^{1/2} C D^{1/2})(D^{1/2} E_i^{1/2}) \right)^2 \\ &\leq \text{Tr } E_i C D C \text{Tr } D^{1/2} E_i D^{1/2} = \text{Tr } E_i C D C \text{Tr } DE_i. \end{aligned}$$

and

$$(\text{Tr } DCE_i)^2 = \left(\text{Tr } (E_i^{1/2} D^{1/2})(D^{1/2} C E_i^{1/2}) \right)^2$$

$$\leq \operatorname{Tr} D^{1/2} E_i D^{1/2} \operatorname{Tr} D^{1/2} C E_i C D^{1/2} = \operatorname{Tr} E_i C D C \operatorname{Tr} D E_i.$$

So $F(\rho(0); \{E_i\}) \leq \operatorname{Tr} D C^2$ holds for any measurement $\{E_i\}$.

Next we want to analyze the condition for equality. Let $\mathbb{J}_D^{-1} B = C = \sum_k \lambda_k P_k$ be the spectral decomposition. In the Schwarz inequalities the condition of equality is

$$D^{1/2} E_i^{1/2} = c_i D^{1/2} C E_i^{1/2}$$

which is

$$E_i^{1/2} = c_i C E_i^{1/2}.$$

So $E_i^{1/2} \leq P_{j(i)}$ for a spectral projection $P_{j(i)}$. This implies that all projections P_i are the sums of certain E_i 's. (The simplest measurement for equality corresponds to the observable C .) \square

Note that \mathbb{J}_D^{-1} is in Example 1. It is an exercise to show that for

$$D = \begin{bmatrix} r & 0 \\ 0 & 1-r \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ \bar{b} & -a \end{bmatrix}$$

the optimal observable is

$$C = \begin{bmatrix} \frac{a}{r} & 2b \\ 2\bar{b} & -\frac{a}{1-r} \end{bmatrix}.$$

The quantum Fisher information (8) is a particular case of the general approach of the previous session, \mathbb{J}_D is in Example 1 below, this is the minimal quantum Fisher information which is also called **SLD Fisher information**. The inequality between (7) and (8) is a particular case of the monotonicity, see [40, 42] and Theorem 1.2 below.

If $D = \operatorname{Diag}(\lambda_1, \dots, \lambda_n)$, then

$$F_{\min}(D; B) := \operatorname{Tr} B \mathbb{J}_D^{-1}(B) = \sum_{ij} \frac{2}{\lambda_i + \lambda_j} |B_{ij}|^2.$$

In particular,

$$F_{\min}(D; i[D, X]) = \sum_{ij} \frac{2(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} |X_{ij}|^2$$

and for commuting D and B we have

$$F_{\min}(D; B) = \operatorname{Tr} D^{-1} B^2.$$

The minimal quantum Fisher information corresponds to the inner product

$$[A, B]_\rho = \frac{1}{2} \operatorname{Tr} \rho(AB + BA) = \operatorname{Tr} A \mathbb{J}_\rho(B).$$

Assume now that $\theta = (\theta^1, \theta^2)$. The formula (6) is still true. If

$$\partial_i \rho(\theta) = B_i,$$

then the classical Fisher information matrix $F(\rho(0); \{E_k\})_{ij}$ has the entries

$$F(\rho(0); \{E_k\})_{ij} = \sum_k \frac{\text{Tr } B_i E_k \text{Tr } B_j E_k}{\text{Tr } \rho(0) E_k} \quad (9)$$

and the quantum Fisher information matrix is

$$\begin{bmatrix} \text{Tr } B_1 \mathbb{J}_D^{-1}(B_1) & \text{Tr } B_1 \mathbb{J}_D^{-1}(B_2) \\ \text{Tr } B_2 \mathbb{J}_D^{-1}(B_1) & \text{Tr } B_2 \mathbb{J}_D^{-1}(B_2) \end{bmatrix}. \quad (10)$$

Is there any inequality between the two matrices?

Let $\beta(A) = \sum_k E_k A E_k$. This is a completely positive trace preserving mapping. In the terminology of Theorem 2.3 the matrix (10) is J_1 and

$$J_2 = F(\rho(0); \{E_k\}).$$

The theorem states the inequality $J_2 \leq J_1$.

1.2 The linear mapping \mathbb{J}_D

Let $D \in \mathbf{M}_n$ be a positive invertible matrix. The linear mapping $\mathbb{J}_D^f : \mathbf{M}_n \rightarrow \mathbf{M}_n$ is defined by the formula

$$\mathbb{J}_D^f = f(\mathbb{L}_D \mathbb{R}_D^{-1}) \mathbb{R}_D,$$

where $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$,

$$\mathbb{L}_D(X) = DX \quad \text{and} \quad \mathbb{R}_D(X) = XD.$$

(The operator $\mathbb{L}_D \mathbb{R}_D^{-1}$ appeared in the modular theory of von Neumann algebras.)

Lemma 1.1 *Assume that $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous and $D = \text{Diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Then*

$$(\mathbb{J}_D^f B)_{ij} = \lambda_j f\left(\frac{\lambda_i}{\lambda_j}\right) B_{ij}$$

Moreover, if $f_1 \leq f_2$, then $0 \leq \mathbb{J}_D^{f_1} \leq \mathbb{J}_D^{f_2}$.

Proof: Let $f(x) = x^k$. Then

$$\mathbb{J}_D^f B = D^k B D^{1-k}$$

and

$$(\mathbb{J}_D^f B)_{ij} = \lambda_i^k \lambda_j^{1-k} B_{ij} = \lambda_j f\left(\frac{\lambda_i}{\lambda_j}\right) B_{ij}.$$

This is true for polynomials and for any continuous f by approximation. \square

It follows from the lemma that

$$\langle A, \mathbb{J}_D^f B \rangle = \langle B^*, \mathbb{J}_D^f A^* \rangle \quad (11)$$

if and only if

$$\lambda_j f\left(\frac{\lambda_i}{\lambda_j}\right) = \lambda_i f\left(\frac{\lambda_j}{\lambda_i}\right),$$

which means $xf(x^{-1}) = f(x)$. Condition (11) is equivalent to the property that $\langle X, \mathbb{J}_D^f Y \rangle \in \mathbb{R}$ when X and Y are self-adjoint.

The functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ used here are the **standard operator monotone functions** defined as

- (i) if for positive matrices $A \leq B$, then $f(A) \leq f(B)$,
- (ii) $xf(x^{-1}) = f(x)$ and $f(1) = 1$.

These functions are between the arithmetic and harmonic means [26, 44]:

$$\frac{2x}{x+1} \leq f(x) \leq \frac{1+x}{2}.$$

Given f ,

$$m_f(x, y) = yf\left(\frac{x}{y}\right)$$

is the corresponding mean and we have

$$(\mathbb{J}_D^f B)_{ij} = m_f(\lambda_i, \lambda_j) B_{ij}. \quad (12)$$

Hence

$$\mathbb{J}_D^f B = X \circ B$$

is a Hadamard product with $X_{ij} = m_f(\lambda_i, \lambda_j)$. Therefore the linear mapping \mathbb{J}_D^f is positivity preserving if and only if the above X is positive.

The inverse of \mathbb{J}_D^f is the mapping

$$\frac{1}{f}(\mathbb{L}_D \mathbb{R}_D^{-1}) \mathbb{R}_D^{-1}$$

which acts as $B \mapsto Y \circ B$ with $Y_{ij} = 1/m_f(\lambda_i, \lambda_j)$. So $(\mathbb{J}_D^f)^{-1}$ is positivity preserving if and only if Y is positive.

A necessary condition for the positivity of \mathbb{J}_D^f is $f(x) \leq \sqrt{x}$, while the necessary condition for the positivity of $(\mathbb{J}_D^f)^{-1}$ is $f(x) \geq \sqrt{x}$. So only $f(x) = \sqrt{x}$ is the function which can make both mappings positivity preserving.

Example 1 If $f(x) = (x + 1)/2$ (arithmetic mean), then

$$\mathbb{J}_D B = \frac{1}{2}(DB + BD) \quad \text{and} \quad \mathbb{J}_D^{-1} B = \int_0^\infty \exp(-tD/2) B \exp(-tD/2) dt.$$

This is from the solution of the equation $DB + BD = 2B$. □

Example 2 If $f(x) = 2x/(x + 1)$ (harmonic mean), then

$$\mathbb{J}_D B = \int_0^\infty \exp(-tD^{-1}/2) B \exp(-tD^{-1}/2) dt$$

and

$$\mathbb{J}_D^{-1} B = \frac{1}{2}(D^{-1}B + BD^{-1}).$$

This function f is the minimal and it generates the maximal Fisher information which is also called **right information matrix**. □

Example 3 For the logarithmic mean

$$f(x) = \frac{x - 1}{\log x} \tag{13}$$

we have

$$\mathbb{J}_D(B) = \int_0^1 D^t B D^{1-t} dt \quad \text{and} \quad \mathbb{J}_D^{-1}(B) = \int_0^\infty (D + t)^{-1} B (D + t)^{-1} dt$$

This function induces an important Fisher information. □

Example 4 For the geometric mean $f(x) = \sqrt{x}$ and

$$\mathbb{J}_D(B) = D^{1/2} B D^{1/2} \quad \text{and} \quad \mathbb{J}_D^{-1}(B) = D^{-1/2} B D^{-1/2}.$$

□

\mathbb{J}_D^f is the largest if f is the largest which is described in Example 1 and the smallest is in Example 2.

Theorem 1.2 Let $\beta : \mathbf{M}_n \rightarrow \mathbf{M}_m$ be a completely positive trace preserving mapping and $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a matrix monotone function. Then

$$\beta^*(\mathbb{J}_{\beta(D)}^f)^{-1} \beta \leq (\mathbb{J}_D^f)^{-1} \tag{14}$$

and

$$\beta \mathbb{J}_D^f \beta^* \leq \mathbb{J}_{\beta(D)}^f. \tag{15}$$

Actually (14) and (15) are equivalent and they are equivalent to the matrix monotonicity of f [43].

In the rest f is always assumed to be a standard matrix monotone function. Then $\text{Tr } \mathbb{J}_D B = \text{Tr } DB$.

Example 5 Here we want to study \mathbb{J}_D^f , when D can have 0 eigenvalues. Formula (12) makes sense. For example, if $D = \text{Diag}(0, \lambda, \lambda, \mu)$ ($\lambda, \mu > 0, \lambda \neq \mu$), then

$$\mathbb{J}_D^f B = \begin{bmatrix} 0 & m(0, \lambda)B_{12} & m(0, \mu)B_{13} & m(0, \mu)B_{14} \\ m(0, \lambda)B_{21} & \lambda B_{22} & m(\lambda, \mu)B_{23} & m(\lambda, \mu)B_{24} \\ m(0, \mu)B_{31} & m(\lambda, \mu)B_{32} & \mu B_{34} & \mu B_{34} \\ m(0, \mu)B_{41} & m(\lambda, \mu)B_{42} & \mu B_{43} & \mu B_{43} \end{bmatrix}.$$

If $f(0) > 0$, then this matrix has only one 0 entry. If $f(0) = 0$, then

$$\mathbb{J}_D^f B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda B_{22} & m(\lambda, \mu)B_{23} & m(\lambda, \mu)B_{24} \\ 0 & m(\lambda, \mu)B_{32} & \mu B_{34} & \mu B_{34} \\ 0 & m(\lambda, \mu)B_{42} & \mu B_{43} & \mu B_{43} \end{bmatrix}.$$

and the kernel of \mathbb{J}_D is larger. We have

$$\langle B, \mathbb{J}_D^f B \rangle = \sum_{ij} m_f(\lambda_i, \lambda_j) |B_{ij}|^2$$

and some terms can be 0 if D is not invertible.

The inverse of \mathbb{J}_D^f exists in the generalized sense

$$[(\mathbb{J}_D^f)^{-1} B]_{ij} = \begin{cases} \frac{1}{m_f(\lambda_i, \lambda_j)} B_{ij} & \text{if } m_f(\lambda_i, \lambda_j) \neq 0, \\ 0 & \text{if } m_f(\lambda_i, \lambda_j) = 0. \end{cases}$$

(This is the Moore-Penrose generalized inverse.) □

It would be interesting to compare the functions which non-zero at 0 with the others.

2 Fisher information and covariance

Assume that f is a standard matrix monotone function. The operators \mathbb{J}_D^f are used to define Fisher information and the covariance. (The latter can be called also quadratic cost.) The operator \mathbb{J}_D^f depends on the function f , but f will be not written sometimes.

Let $A = A^*, B = B^* \in \mathbf{M}_n$ be observables and $D \in \mathbf{M}_n$ be a density matrix. The covariance of A and B is

$$\text{Cov}_D^f(A, B) := \langle A, \mathbb{J}_D^f(B) \rangle - (\text{Tr } DA)(\text{Tr } DB). \quad (16)$$

Since

$$\text{Cov}_D^f(A, A) = \langle (A - I\text{Tr} DA), \mathbb{J}_D(A - I\text{Tr} DA) \rangle$$

and $\mathbb{J}_D \geq 0$, we have for the variance $\text{Var}_D^f(A) := \text{Cov}_D^f(A, A) \geq 0$.

The monotonicity (15) gives

$$\text{Var}_D^f(\beta^* A) \leq \text{Var}_{\beta D}^f(A).$$

for a completely positive trace preserving mapping β .

The usual **symmetrized covariance** corresponds to the function $f(t) = (t + 1)/2$:

$$\text{Cov}_D(A, B) := \frac{1}{2} \text{Tr} (D(A^*B + BA^*)) - (\text{Tr} DA^*)(\text{Tr} DB).$$

Let A_1, A_2, \dots, A_k be self-adjoint matrices and let D be a statistical operator. The covariance is a $k \times k$ matrix $C(D)$ defined as

$$C(D)_{ij} = \text{Cov}_D^f(A_i, A_j). \quad (17)$$

$C(D)$ is a positive semidefinite matrix and positive definite if the observables A_1, A_2, \dots, A_k are linearly independent. It should be remarked that this matrix is only a formal analogue of the classical covariance matrix and it is not related to a single quantum measurement [33].

The variance is defined by \mathbb{J}_D and the Fisher information is formulated by the inverse of this mapping:

$$\gamma_D(A, B) = \text{Tr} A \mathbb{J}_D^{-1}(B^*). \quad (18)$$

Here A and B are self-adjoint. If A and B are considered as tangent vectors at the footpoint D , then $\text{Tr} A = \text{Tr} B = 0$. In this approach $\gamma_D(A, B)$ is an inner product in a Riemannian geometry [2, 21]. It seems that this approach is not popular in quantum theory. It happens also that the condition $\text{Tr} D = 1$ is neglected and only $D > 0$. Then formula (18) can be extended [29].

If $DA = AD$ for a self-adjoint matrix A , then

$$\gamma_D(A, A) = \text{Tr} D^{-1} A^2$$

does not depend on the function f . (The dependence is characteristic on the orthogonal complement, this will come later.)

Theorem 2.1 *Assume that $(A, B) \mapsto \gamma_D(A, B)$ is an inner product for $A, B \in \mathbf{M}_n$, for positive definite density matrix $D \in \mathbf{M}_n$ and for every n . Suppose the following properties:*

- (i) *For commuting D and $A = A^*$ we have $\gamma_D(A, A) = \text{Tr} D^{-1} A^2$.*

(ii) If $\beta : \mathbf{M}_n \rightarrow \mathbf{M}_m$ is a completely positive trace preserving mapping, then

$$\gamma_{\beta(D)}(\beta(A), \beta(A)) \leq \gamma_D(A, A). \quad (19)$$

(iii) If $A = A^*$ and $B = B^*$, then $\gamma_D(A, B)$ is a real number.

(iv) $D \mapsto \gamma_D(A, B)$ is continuous.

Then

$$\gamma_D(A, B) = \langle A, (\mathbb{J}_D^f)^{-1} B \rangle \quad (20)$$

for a standard matrix monoton function f .

Example 6 In quantum statistical mechanics, perturbation of a density matrix appears. Suppose that $D = e^H$ and $A = A^*$ is the perturbation

$$D_t = \frac{e^{H+tA}}{\text{Tr } e^{H+tA}} \quad (t \in \mathbb{R}).$$

The quantum analog of formula (1) would be

$$-\text{Tr } D_0 \frac{\partial^2}{\partial t^2} \log D_t \Big|_{t=0}.$$

A simple computation gives

$$\int_0^1 \text{Tr } e^{sH} A e^{(1-s)H} A ds - (\text{Tr } DA)^2$$

This is a kind of variance. □

Let $\mathcal{M} := \{D_\theta : \theta \in G\}$ be a smooth m -dimensional **manifold** of $n \times n$ density matrices. Formally $G \subset \mathbb{R}^m$ is an open set including 0. If $\theta \in G$, then $\theta = (\theta_1, \theta_2, \dots, \theta_m)$. The **Riemannian structure** on \mathcal{M} is given by the inner product (18) of the tangent vectors A and B at the foot point $D \in \mathcal{M}$, where $\mathbb{J}_D : \mathbf{M}_n \rightarrow \mathbf{M}_n$ is a positive mapping when \mathbf{M}_n is regarded as a Hilbert space with the Hilbert-Schmidt inner product. (This means $\text{Tr } A \mathbb{J}_D(A)^* \geq 0$.)

Assume that a collection $A = (A_1, \dots, A_m)$ of self-adjoint matrices is used to estimate the true value of θ . The expectation value of A_i with respect to the density matrix D is $\text{Tr } DA_i$. A is an **unbiased estimator** if

$$\text{Tr } D_\theta A_i = \theta_i \quad (1 \leq i \leq m). \quad (21)$$

(In many cases unbiased estimator $A = (A_1, \dots, A_m)$ does not exist, therefore a weaker condition is more useful.)

The **Fisher information** matrix of the estimator A is a positive definite matrix

$$J(D)_{ij} = \text{Tr } L_i \mathbb{J}_D(L_j), \quad \text{where } L_i = \mathbb{J}_D^{-1}(\partial_i D_\theta).$$

Both $C(D)$ and $J(D)$ depend on the actual state D .

The next theorem is the the **Cramér-Rao inequality** for matrices. The point is that the right-hand-side does not depend on the estimators.

Theorem 2.2 *Let $A = (A_1, \dots, A_m)$ be an unbiased estimator of θ . Then for the above defined matrices the inequality*

$$C(D_\theta) \geq J(D_\theta)^{-1}$$

holds.

Proof: In the proof the block-matrix method is used and we restrict ourselves for $m = 2$ for the sake of simplicity and assume that $\theta = 0$. Instead of D_0 we write D .

The matrices A_1, A_2, L_1, L_2 are considered as vectors and from the inner product $\langle A, B \rangle = \text{Tr } A \mathbb{J}_D(B)^*$ we have the positive matrix

$$X := \begin{bmatrix} \text{Tr } A_1 \mathbb{J}_D(A_1) & \text{Tr } A_1 \mathbb{J}_D(A_2) & \text{Tr } A_1 \mathbb{J}_D(L_1) & \text{Tr } A_1 \mathbb{J}_D(L_2) \\ \text{Tr } A_2 \mathbb{J}_D(A_1) & \text{Tr } A_2 \mathbb{J}_D(A_2) & \text{Tr } A_2 \mathbb{J}_D(L_1) & \text{Tr } A_2 \mathbb{J}_D(L_2) \\ \text{Tr } L_1 \mathbb{J}_D(A_1) & \text{Tr } L_1 \mathbb{J}_D(A_2) & \text{Tr } L_1 \mathbb{J}_D(L_1) & \text{Tr } L_1 \mathbb{J}_D(L_2) \\ \text{Tr } L_2 \mathbb{J}_D(A_1) & \text{Tr } L_2 \mathbb{J}_D(A_2) & \text{Tr } L_2 \mathbb{J}_D(L_1) & \text{Tr } L_2 \mathbb{J}_D(L_2) \end{bmatrix}.$$

From the condition (21), we have

$$\text{Tr } A_i \mathbb{J}_D(L_i) = \frac{\partial}{\partial \theta_i} \text{Tr } D_\theta A_i = 1$$

for $i = 1, 2$ and

$$\text{Tr } A_i \mathbb{J}_D(L_j) = \frac{\partial}{\partial \theta_j} \text{Tr } D_\theta A_i = 0$$

if $i \neq j$. Hence the matrix X has the form

$$\begin{bmatrix} C(D) & I_2 \\ I_2 & J(D) \end{bmatrix}, \quad (22)$$

where

$$C(D) = \begin{bmatrix} \text{Tr } A_1 \mathbb{J}_D(A_1) & \text{Tr } A_1 \mathbb{J}_D(A_2) \\ \text{Tr } A_2 \mathbb{J}_D(A_1) & \text{Tr } A_2 \mathbb{J}_D(A_2) \end{bmatrix}$$

and

$$J(D) = \begin{bmatrix} \text{Tr } L_1 \mathbb{J}_D(L_1) & \text{Tr } L_1 \mathbb{J}_D(L_2) \\ \text{Tr } L_2 \mathbb{J}_D(L_1) & \text{Tr } L_2 \mathbb{J}_D(L_2) \end{bmatrix}.$$

The positivity of (22) implies the statement of the theorem. \square

We have have the orthogonal decomposition

$$\{B = B^* : [D, B] = 0\} \oplus \{i[D, A] : A = A^*\} \quad (23)$$

of the self-adjoint matrices and we denote the two subspaces by \mathcal{M}_D and \mathcal{M}_D^c , respectively.

Example 7 The Fisher information and the covariance are easily handled if D is diagonal, $D = \text{Diag}(\lambda_1, \dots, \lambda_n)$ or formulated by the matrix units $E(ij)$

$$D = \sum_i \lambda_i E(ii).$$

The general formulas in case of diagonal D are

$$\gamma_D(A, A) = \sum_{ij} \frac{1}{\lambda_j f(\lambda_i/\lambda_j)} |A_{ij}|^2, \quad \text{Cov}_D(A, A) = \sum_{ij} \lambda_j f(\lambda_i/\lambda_j) |A_{ij}|^2.$$

Moreover,

$$\gamma_D^f(\text{i}[D, X], \text{i}[D, X]) = \sum_{ij} \frac{(\lambda_i - \lambda_j)^2}{\lambda_j f(\lambda_i/\lambda_j)} |X_{ij}|^2. \quad (24)$$

Hence for diagonal D all Fisher informations have simple explicit formula.

The description of the commutators is more convenient if the eigenvalues are different. Let

$$S_1(ij) := E(ij) + E(ji), \quad S_2(ij) := -\text{i}E(ij) + \text{i}E(ji)$$

for $i < j$. (They are the generalization of the Pauli matrices σ_1 and σ_2 .) We have

$$\text{i}[D, S_1(ij)] = (\lambda_i - \lambda_j)S_2(ij), \quad \text{i}[D, S_2(ij)] = (\lambda_j - \lambda_i)S_1(ij).$$

In Example 1 we have $f(x) = (1+x)/2$. This gives the minimal Fisher information described in Theorem 1.1:

$$\gamma_D(A, B) = \int_0^\infty \text{Tr} A \exp(-tD/2) B \exp(-tD/2) dt.$$

The corresponding covariance is the symmetrized $\text{Cov}_D(A, B)$. This is maximal among the variances.

From Example 2 we have the maximal Fisher information

$$\gamma_D(A, B) = \frac{1}{2} \text{Tr} D^{-1} (AB + BA)$$

The corresponding covariance is a bit similar to the minimal Fisher information:

$$\text{Cov}_D(A, B) = \int_0^\infty \text{Tr} A \exp(-tD^{-1}/2) B \exp(-tD^{-1}/2) dt - \text{Tr} DA \text{Tr} DB.$$

Example 3 leads to the Bogoliubov-Kubo-Mori inner product as Fisher information [41, 42]:

$$\gamma_D(A, B) = \int_0^\infty \text{Tr} A(D+t)^{-1} B(D+t)^{-1} dt$$

It is also called BKM Fisher information, the characterization is in the paper [14] and it is also proven that this gives a large deviation bound of consistent superefficient estimators [17]. \square

Let $\mathcal{M} := \{\rho(\theta) : \theta \in G\}$ be a smooth k -dimensional manifold of invertible density matrices. The *quantum score operators* (or logarithmic derivatives) are defined as

$$L_i^f(\theta) := (\mathbb{J}_{\rho(\theta)}^f)^{-1}(\partial_{\theta_i}\rho(\theta)) \quad (1 \leq i \leq m) \quad (25)$$

and

$$J(\theta)_{ij} := \text{Tr} L_i^f(\theta) \mathbb{J}_{\rho(\theta)}^f(L_j(\theta)) = \text{Tr} (\mathbb{J}_{\rho(\theta)}^f)^{-1}(\partial_{\theta_i}\rho(\theta))(\partial_{\theta_j}\rho(\theta)) \quad (1 \leq i, j \leq k) \quad (26)$$

is the **quantum Fisher information matrix** (depending on the function f). The function $f(x) = (x + 1)/2$ yields the symmetric logarithmic derivative (SLD) Fisher information.

Theorem 2.3 *Let $\beta : \mathbf{M}_n \rightarrow \mathbf{M}_m$ be a completely positive trace preserving mapping and let $\mathcal{M} := \{\rho(\theta) \in \mathbf{M}_n : \theta \in G\}$ be a smooth k -dimensional manifold of invertible density matrices. For the Fisher information matrix $J_1(\theta)$ of \mathcal{M} and for Fisher information matrix $J_2(\theta)$ of $\beta(\mathcal{M}) := \{\beta(\rho(\theta)) : \theta \in G\}$ we have the monotonicity relation*

$$J_2(\theta) \leq J_1(\theta).$$

Proof: We set $B_i(\theta) := \partial_{\theta_i}\rho(\theta)$. Then $\mathbb{J}_{\beta(\rho(\theta))}^{-1}\beta(B_i(\theta))$ is the score operator of $\beta(\mathcal{M})$ and we have

$$\begin{aligned} \sum_{ij} J_2(\theta)_{ij} a_i \bar{a}_j &= \text{Tr} \mathbb{J}_{\beta(\rho(\theta))}^{-1} \beta \left(\sum_i a_i B_i(\theta) \right) \beta \left(\sum_j \bar{a}_j B_j(\theta) \right) \\ &= \left\langle \sum_i a_i B_i, (\beta^* \mathbb{J}_{\beta(\rho(\theta))}^{-1} \beta) \sum_j \bar{a}_j B_j(\theta) \right\rangle \\ &\leq \left\langle \sum_i a_i B_i, \mathbb{J}_{\rho(\theta)}^{-1} \sum_j \bar{a}_j B_j(\theta) \right\rangle \\ &= \text{Tr} \mathbb{J}_{\rho(\theta)}^{-1} \left(\sum_i a_i B_i(\theta) \right) \left(\sum_j \bar{a}_j B_j(\theta) \right) \\ &= \sum_{ij} J_1(\theta)_{ij} a_i \bar{a}_j, \end{aligned}$$

where (14) was used. □

The monotonicity of the Fisher information matrix in some particular cases appeared already in the literature: [38] treated the case of the Kubo-Mori inner product and [4] considered the symmetric logarithmic derivative and measurement in the role of coarse graining.

Example 8 The function

$$f_\beta(t) = \beta(1 - \beta) \frac{(x - 1)^2}{(x^\beta - 1)(x^{1-\beta} - 1)} \quad (27)$$

is operator monotone if $0 < \beta < 2$. Formally $f(1)$ is not defined, but as a limit it is 1. The property $xf(x^{-1}) = f(x)$ also holds. Therefore this function determines a Fisher information [39]. If $\beta = 1/2$, then the variance has a simple formula:

$$\text{Var}_D A = \frac{1}{2} \text{Tr} D^{1/2} (D^{1/2} A + A D^{1/2}) A - (\text{Tr} D A)^2.$$

□

Example 9 The functions $x^{-\alpha}$ and $x^{\alpha-1}$ are matrix monotone decreasing and so is their sum. Therefore

$$f_\alpha(x) = \frac{2}{x^{-\alpha} + x^{\alpha-1}}$$

is a standard operator monotone function.

$$\gamma_\sigma^f(\rho, \rho) = 1 + \text{Tr} (\rho - \sigma) \sigma^{-\alpha} (\rho - \sigma) \sigma^{\alpha-1}$$

may remind us to the abstract Fisher information, however now ρ and σ are positive definite density matrices. In the paper [46]

$$\chi_\alpha^2(\rho, \sigma) = \text{Tr} (\rho - \sigma) \sigma^{-\alpha} (\rho - \sigma) \sigma^{\alpha-1}$$

is called **quantum χ^2 -divergence**. (If ρ and σ commute, then the formula is independent of α .) Up to the constant 1, this is an interesting and important particular case of the monotone metric. The general theory (19) implies the monotonicity of the χ^2 -divergence. □

3 Extended monotone metrics

As an extension of the papers [5, 40] Kumagai made the following generalization [29]. Now H_n^+ denotes the strictly positive matrices in \mathbf{M}_n . Formally $K_\rho(A, B) \in \mathbb{C}$ is defined for all $\rho \in H_n^+$, $A, B \in \mathbf{M}_n$ and $n \in \mathbb{N}$ and it is assumed that

- (i) $(A, B) \mapsto K_\rho(A, B)$ is an inner product on \mathbf{M}_n for every $\rho \in H_n^+$ and $n \in \mathbb{N}$.
- (ii) $\rho \mapsto K_\rho(A, B)$ is continuous.
- (iii) For a trace-preserving completely positive mapping β

$$K_{\beta(\rho)}(\beta(A), \beta(A)) \leq K_\rho(A, A)$$

holds.

In the paper [29] such $K_\rho(A, B)$ is called **extended monotone metric** and the description is

$$K_\rho(A, B) = b(\text{Tr} \rho) \text{Tr} A^* \text{Tr} B + c \langle A, (\mathbb{J}_\rho^f)^{-1}(B) \rangle,$$

where $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is matrix monotone, $f(1) = 1$, $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $c > 0$. Note that

$$(A, B) \mapsto b(\text{Tr } \rho) \text{Tr } A^* \text{Tr } B \quad \text{and} \quad (A, B) \mapsto c \langle A, (\mathbb{J}_\rho^f)^{-1} B \rangle$$

satisfy conditions (ii) and (iii) with constant $c > 0$. The essential point is to check

$$b(\text{Tr } \rho) \text{Tr } A^* \text{Tr } A + c \langle A, (\mathbb{J}_\rho^f)^{-1} A \rangle \geq 0.$$

In the case of 1×1 matrices this is

$$b(x)|z|^2 + \frac{c}{x}|z|^2 \geq 0$$

which gives the condition $xb(x) + c > 0$. If this is true, then

$$\begin{aligned} & \left(\sum_i \lambda_i \right) b \left(\sum_i \lambda_i \right) \left| \sum_i A_{ii} \right|^2 + c \left(\sum_i \lambda_i \right) \sum_{ij} \frac{1}{m_f(\lambda_i, \lambda_j)} |A_{ij}|^2 \\ & \geq -c \left| \sum_i A_{ii} \right|^2 + c \left(\sum_i \lambda_i \right) \sum_{ij} \frac{1}{m_f(\lambda_i, \lambda_j)} |A_{ij}|^2 \\ & \geq -c \left| \sum_i A_{ii} \right|^2 + c \left(\sum_i \lambda_i \right) \sum_i \frac{1}{\lambda_i} |A_{ii}|^2. \end{aligned}$$

The positivity is the inequality

$$\left(\sum_i \lambda_i \right) \sum_i \frac{1}{\lambda_i} |A_{ii}|^2 \geq \left| \sum_i A_{ii} \right|^2$$

which is a consequence of the Schwarz inequality.

4 Skew information

The Wigner-Yanase-Dyson skew information is the quantity

$$I_p(D, A) := -\frac{1}{2} \text{Tr} [D^p, A][D^{1-p}, A] \quad (0 < p < 1).$$

Actually, the case $p = 1/2$ is due to Wigner and Yanase [47] and the extension was proposed by Dyson. The convexity of $I_p(D, A)$ in A is a famous result of Lieb [30]

It was observed in [39] that the Wigner-Yanase-Dyson skew information is connected to the Fisher information which corresponds to the function (28). For this function we have

$$\gamma_D(\mathfrak{i}[D, A], \mathfrak{i}[D, A]) = \frac{1}{2\beta(1-\beta)} \text{Tr} ([\rho^\beta, A][\rho^{1-\beta}, A]). \quad (28)$$

Apart from a constant factor this expression is the skew information proposed by Wigner and Yanase [47]. In the limiting cases $p \rightarrow 0$ or 1 we have the function (13) corresponding to the Kubo-Mori-Bogoliubov case.

Let f be a standard function and $A = A^* \in \mathbf{M}_n$. The quantity

$$I_D^f(A) := \frac{f(0)}{2} \gamma_D^f(\mathfrak{i}[D, A], \mathfrak{i}[D, A])$$

was called **skew information** in [16] in this general setting. So the skew information is nothing else but the Fisher information restricted to \mathcal{M}_D^c , but it is parametrized by the commutator. Skew information appeared twenty years before the concept of quantum Fisher information. Skew information appears in a rather big literature, for example, connection with uncertainty relations [3, 10, 9, 13, 25, 31, 32].

If $D = \text{Diag}(\lambda_1, \dots, \lambda_n)$ is diagonal, then

$$\gamma_D^f(\mathfrak{i}[D, A], \mathfrak{i}[D, A]) = \sum_{ij} \frac{(\lambda_i - \lambda_j)^2}{\lambda_j f(\lambda_i/\lambda_j)} |A_{ij}|^2.$$

This implies that the identity

$$I_D^f(A) = \text{Cov}_D(A, A) - \text{Cov}_D^{\tilde{f}}(A, A) \quad (29)$$

holds if $\text{Tr } DA = 0$ and

$$\tilde{f}(x) := \frac{1}{2} \left((x+1) - (x-1)^2 \frac{f(0)}{f(x)} \right). \quad (30)$$

It was proved in [8] that for a standard function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$, \tilde{f} is standard as well. Another proof is in [45] which contains the following theorem.

Theorem 4.1 *Assume that $X = X^* \in \mathcal{M}$ and $\text{Tr } DX = 0$. If f is a standard function such that $f(0) \neq 0$, then*

$$\left. \frac{\partial^2}{\partial t \partial s} S_F(D + ti[D, X], D + si[D, X]) \right|_{t=s=0} = f(0) \gamma_D^f(\mathfrak{i}[D, X], \mathfrak{i}[D, X])$$

for the standard function $F = \tilde{f}$.

All skew informations are obtained from an f -divergence (or quasi-entropy) by differentiation.

Example 10 The function

$$f(x) = \left(\frac{1 + \sqrt{x}}{2} \right)^2 \quad (31)$$

gives the Wigner-Yanase skew information

$$I^{WY}(D, A) = I_{1/2}(D, A) = -\frac{1}{2} \text{Tr} [D^{1/2}, A]^2.$$

The skew information coming from the minimal Fisher information and it is often denoted as $I^{SLD}(D, A)$. The simple mean inequalities

$$\left(\frac{1 + \sqrt{x}}{2}\right)^2 \leq \frac{1 + x}{2} \leq 2 \left(\frac{1 + \sqrt{x}}{2}\right)^2$$

imply

$$I^{WY}(D, A) \leq I^{SLD}(D, A) \leq 2I^{WY}(D, A).$$

□

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