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Markov property of Gaussian states of canonical commutation relation algebras

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Abstract

The Markov property of Gaussian states of CCR-algebras is studied. The detailed description is given by the representing block matrix. The proof is short and allows infinite dimension. The relation to classical Gaussian Markov triplets is also described. The minimizer of relative entropy with respect to a Gaussian Markov state has the Markov property. The appendix contains formulas for the relative entropy.

Key words and phrases: Weyl unitaries, von Neumann entropy, CCR algebra, Markov triplet, Gaussian state, relative entropy, block matrix.

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Introduction

The notion of Gaussian (or quasi-free) state was developed in the framework of the C*-algebraic approach to the canonical commutation relation (CCR) [17, 11, 6, 19]. The CCR-algebra is generated by the Weyl unitaries (satisfying a commutation relation, therefore Weyl algebra is an alternative terminology). The Gaussian states on CCR can be regarded as analogues of Gaussian distributions in classical probability: The n-point functions can be computed from the 2-point functions and in a kind of central limit theorem the limiting state is quasi-free and it maximizes the von Neumann entropy when the 2-point function is fixed [20]. The Gaussian states are quite tractable, for example the von Neumann entropy has an explicit expression [7, 6].

The Markov property was invented by Accardi in the non-commutative (or quantum probabilistic) setting [1, 2]. This Markov property is based on a completely positive, identity preserving map, so-called quasi-conditional expectation and it was formulated in the tensor product of matrix algebras. (A slightly different formulation is in [9].) A state of a tensor product system is Markovian if and only if the von Neumann entropy increase is constant. This property and a possible definition of the Markov condition was suggested in [21]. A remarkable property of the von Neumann entropy is the strong subadditivity [16, 10, 18, 23] which plays an important role in the investigations of quantum system's correlations. The above mentioned constant increase of the von Neumann entropy is the same as the equality for the strong subadditivity of von Neumann entropy.

A CCR (or Weyl) algebra is parameterized by a Hilbert space, we use the notation $\mathrm{CCR}(\mathcal{H})$ when $\mathcal{H}$ is the Hilbert space. Assume that φ_{123} is a state on the composite system $\mathrm{CCR}(\mathcal{H}_1)\otimes\mathrm{CCR}(\mathcal{H}_2)\otimes\mathrm{CCR}(\mathcal{H}_3)$. Denote by $\varphi_{12}, \varphi_{23}$ the restriction to the first two and to the second and third factors, similarly φ_2 is the restriction to the second factor. The Markov property is defined as

$$S(\varphi_{123}) - S(\varphi_{12}) = S(\varphi_{23}) - S(\varphi_2),$$

where S denotes the von Neumann entropy [18]. When φ_{123} is Gaussian, it is given by a positive operator (corresponding to the 2-point function) and the main goal of the present paper is to describe the Markov property in terms of this operator. The technique will be use block matrix methods and linear analysis. The paper [24] studies a similar question for the CAR algebra, [3] is about multivariate Gaussian distributions and [13] is about quasi-free states under the finite dimensional condition. Although the multivariate Gaussian case (in classical probability) is rather different from the present non-commutative setting, we use the same block matrix formalism (and the paper [3] was actually a preparation of this problem). A Gaussian state is described by a block matrix and the Markov property is formulated by the entries. A Markovian Gaussian state induces multivariate Gaussian restrictions, but they are very special in that framework. Given a Markovian Gaussian state, the relative entropy can be minimized under a fixed initial condition. It is proven that the minimizer is a Markov state as well.

The paper is organized as follows. The preliminary section contains some crucial properties of the Weyl unitaries in the CCR algebra and the Gaussian states. The Markov condition is not discussed in details in the setting of non-commutative C*-algebras. In the next section we investigate the Gaussian Markov triplets. The essential necessary and sufficient condition described in the block matrix approach: the block matrix should be block diagonal. There are nontrivial Markovian quasi-free states which are not a product in the time localization. However, the first and the third subalgebras are always independent. Note that the proof does not require the finite dimension of the Hilbert space (contrary to [13]). In a subsection the Markovian quasi-free state is compared with the spin chain and with the classical probabilistic vector-valued Gaussian. The minimization of the relative entropy with respect a Gaussian state on CCR is also discussed under two conditions. The minimizer is Markovian similarly to the probabilistic case [4]. The appendix is devoted to the concept of relative entropy in general CCR case and it is computed for the Gaussian states.

1 Preliminaries

1.1 CCR-algebras

Let $\mathcal{H}$ be a Hilbert space. Assume that for every $f \in \mathcal{H}$ a unitary operator $W(f)$ is given such that the relations

$$W(f_1)W(f_2) = W(f_1 + f_2)\exp(\mathrm{i}\,\sigma(f_1, f_2)), \tag{1}$$

$$W(-f) = W(f)^* \tag{2}$$

hold for $f_1, f_2, f \in \mathcal{H}$ with $\sigma(f_1, f_2) := \mathrm{Im}\langle f_1, f_2\rangle$. The C*-algebra generated by these unitaries is unique and denoted by $\mathrm{CCR}(\mathcal{H})$ [19, 25].

The C*-algebra $\mathrm{CCR}(\mathcal{H})$ is not separable, but nuclear [8], therefore its tensor product with any other C*-algebra is uniquely defined [14, Chap. 11]. The relation (1) shows that $W(f_1)$ and $W(f_2)$ commute if f_1 and f_2 are orthogonal. It follows that $\mathrm{CCR}(\mathcal{H}_1)\otimes \mathrm{CCR}(\mathcal{H}_2)$ is isomorphic to $\mathrm{CCR}(\mathcal{H}_1 \oplus \mathcal{H}_2)$.

The C*-algebra $\mathrm{CCR}(\mathcal{H})$ has a very natural state

$$\omega(W(f)) := \exp\big(-\|f\|^2/2\big) \tag{3}$$

which is called Fock state. The GNS-representation of $\mathrm{CCR}(\mathcal{H})$ is called Fock representation and it leads to the the Fock space $\mathcal{F}(\mathcal{H})$ with cyclic vector Φ, also called vacuum vector. Since ω is actually a product state, the GNS Hilbert space is a tensor product. We shall identify the abstract unitary $W(f)$ with the representing unitary acting on the Fock space $\mathcal{F}(\mathcal{H})$. The map $t \mapsto W(tf)$ is a strongly continuous 1-parameter group of unitaries and according to the Stone theorem we have

$$W(tf) = \exp(\mathrm{i}tB(f)) \quad \text{and} \quad \frac{\partial}{\partial t}\Big|_{t=0} W(tf) = \mathrm{i}B(f)$$

for a self-adjoint operator $B(f)$, called field operator. The distribution of a field operator is Gaussian with respect to the Fock state. The usual (Bose) creation operator is defined by

$$a^+(f) = \frac{1}{\sqrt{2}}(B(f) - \mathrm{i}B(\mathrm{i}f)),$$

and the annihilation operator $a(f)$ is its adjoint.

1.2 Gaussian states

The Fock state (3) can be generalized by choosing a positive operator $A \in B(\mathcal{H})$:

$$\omega_A(W(f)) := \exp\big(- \|f\|^2/2 - \langle f, Af\rangle\big). \tag{4}$$

This is called Gaussian or (gage invariant) quasi-free state. By derivation we get

$$\begin{aligned} \omega_A(B(f)B(g)) &= -\mathrm{i}\sigma(f,g) + \frac{1}{2}\Big(\langle f, (I+2A)g\rangle + \langle g, (I+2A)f\rangle\Big) \\ &= \mathrm{Re}\langle f, (I+2A)g\rangle - \mathrm{i}\mathrm{Im}\langle f, g\rangle, \end{aligned}$$

and all higher order correlation functions are expressed by this two-point functions [6]. Moreover,

$$\omega_A(a^+(f)a(g)) = \langle g, Af\rangle\,. \tag{5}$$

For $0 \le A \in B(\mathcal{H})$, the statistical operator of the quasi-free state ω_A of $\mathrm{CCR}(\mathcal{H})$ in the Fock representation has the form

$$D_A = \frac{\Gamma(A(I+A)^{-1}))}{\mathrm{Tr}\,\Gamma(A(I+A)^{-1}))}, \tag{6}$$

where Γ is the second quantization of operators [6]. Therefore the von Neumann entropy is

$$S(\omega_A) = \mathrm{Tr}\,\kappa(A), \quad \text{where} \quad \kappa(t) = -t\log t + (t+1)\log(t+1). \tag{7}$$

Assume that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and write the positive mapping $A \in B(\mathcal{H})$ in the form of block matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

If $f \in \mathcal{H}_1$, then

$$\omega_A(W(f \oplus 0)) = \exp\big(- \|f\|^2/2 - \langle f, A_{11}f\rangle\big).$$

Therefore the restriction of the quasi-free state ω_A to $\mathrm{CCR}(\mathcal{H}_1)$ is the quasi-free state $\omega_{A_{11}}$.

1.3 Classical Markov triplets

Let $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$ be a finite set with probability distribution $p(x_1, x_2, x_3)$ ($x_i \in \mathcal{X}_i, 1 \le i \le 3$). The Markov property is defined by conditional probabilities in the stochastic setting:

$$p(x_3|x_1, x_2) = p(x_3|x_2)$$

which means

$$\frac{p(x_1, x_2, x_3)}{p(x_1, x_2)} = \frac{p(x_2, x_3)}{p(x_2)}$$

or

$$\log p(x_1, x_2, x_3) + \log p(x_2) = \log p(x_2, x_3) + \log p(x_1, x_2).$$

The expectation value gives the Shannon entropy equality

$$S(p_{123}) + S(p_2) = S(p_{12}) + S(p_{23})$$

which is actually an equivalent form. This relation can be used for the definition of Markov property in the CCR-setting.

2 Markov property on CCR

2.1 Matrix characterization

Assume that the Hilbert space $\mathcal{H}$ has the orthogonal decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. Then

$$\mathrm{CCR}(\mathcal{H}) = \mathrm{CCR}(\mathcal{H}_1) \otimes \mathrm{CCR}(\mathcal{H}_2) \otimes \mathrm{CCR}(\mathcal{H}_3)$$

and the equality in the strong subadditivity of the von Neumann entropy can be the definition of the Markov property [21].

We study the Markov property of a Gaussian state $\omega_A \equiv \omega_{123}$, where A is a positive operator acting on $\mathcal{H}$. This operator has the block-matrix form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{bmatrix}.$$

and we set

$$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}, \qquad C = \begin{bmatrix} A_{22} & A_{23} \\ A_{23}^* & A_{33} \end{bmatrix}.$$

In connection with the strong subadditivity of the von Neumann entropy, the definition of the Markov property is

$$\mathrm{Tr}\,\kappa(A) + \mathrm{Tr}\,\kappa(A_{22}) = \mathrm{Tr}\,\kappa(B) + \mathrm{Tr}\,\kappa(C), \qquad (8)$$

where $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ is defined in (7). Our aim is to characterize the Markov property in terms of the block-matrix A.

Denote by P_i the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_i$, $1 \le i \le 3$. Of course, $P_1 + P_2 + P_3 = I$ and we use also the notation $P_{12} := P_1 + P_2$ and $P_{23} := P_2 + P_3$.

Theorem 2.1 *Assume that $A \in B(\mathcal{H})$ is a positive invertible operator and the Gaussian state $\omega_A \equiv \omega_{123}$ on $\mathrm{CCR}(\mathcal{H})$ has finite von Neumann entropy. Then the following conditions are equivalent.*

- *(a)* $S(\omega_{123}) + S(\omega_2) = S(\omega_{12}) + S(\omega_{23})$
- *(b)* $\mathrm{Tr}\,\kappa(A) + \mathrm{Tr}\,\kappa(P_2AP_2) = \mathrm{Tr}\,\kappa(P_{12}AP_{12}) + \mathrm{Tr}\,\kappa(P_{23}AP_{23})$
- *(c) There is a projection $P \in B(\mathcal{H})$ such that $P_1 \le P \le P_1 + P_2$ and $PA = AP$.*

Proof.

(a) and (b) are different only in notation. Condition (c) tells that the matrix A has a special form:

$$A = \begin{bmatrix} A_{11} & \begin{bmatrix} a & 0 \end{bmatrix} & 0 \\ \begin{bmatrix} a^* \\ 0 \end{bmatrix} & \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} & \begin{bmatrix} 0 \\ b \end{bmatrix} \\ 0 & \begin{bmatrix} 0 & b^* \end{bmatrix} & A_{33} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & a \\ a^* & c \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} d & b \\ b^* & A_{33} \end{bmatrix} \end{bmatrix}, \qquad (9)$$

where the parameters a, b, c, d (and 0) are operators. This is a block diagonal matrix, $A = \mathrm{Diag}\,(A_1, A_2)$,

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

and the projection P is

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

in this setting.

The Hilbert space $\mathcal{H}_2$ is decomposed as $\mathcal{H}_2^L \oplus \mathcal{H}_2^R$, where $\mathcal{H}_2^L$ is the range of the projection PP_2. Therefore,

$$\mathrm{CCR}(\mathcal{H}) = \mathrm{CCR}(\mathcal{H}_1 \oplus \mathcal{H}_2^L) \otimes \mathrm{CCR}(\mathcal{H}_2^R \oplus \mathcal{H}_3) \qquad (10)$$

and ω_{123} becomes a product state $\omega_L \otimes \omega_R$. This shows that the implication $(c) \Rightarrow (a)$ is obvious.

The essential part is the proof $(a) \Rightarrow (c)$ which is based on a result of [3]:

$$\mathrm{Tr}\,\log(A) + \mathrm{Tr}\,\log(A_{22}) \le \mathrm{Tr}\,\log(B) + \mathrm{Tr}\,\log(C) \qquad (11)$$

holds and the necessary and sufficient condition for equality is $A_{13} = A_{12}A_{22}^{-1}A_{23}$.

The integral representation

$$\kappa(x) = \int_1^\infty t^{-2} \log(tx+1)\, dt \tag{12}$$

shows that the function κ is operator monotone (since the logarithm is so, [5]) and (12) implies the inequality

$$\operatorname{Tr} \kappa(A) + \operatorname{Tr} \kappa(A_{22}) \le \operatorname{Tr} \kappa(B) + \operatorname{Tr} \kappa(C). \tag{13}$$

The equality holds if and only if

$$tA_{13} = tA_{12}(tA_{22} + I)^{-1} tA_{23}$$

for almost every $t > 1$. The continuity gives that actually for every $t > 1$ we have

$$A_{13} = A_{12}(A_{22} + t^{-1}I)^{-1}A_{23}.$$

The right-hand-side, $A_{12}(A_{22}+zI)^{-1}A_{23}$, is an analytic function on $\{z \in \mathbb{C} : \mathrm{Re} z > 0\}$, therefore we have

$$A_{13} = 0 = A_{12}(A_{22} + sI)^{-1}A_{23} \qquad (s \in \mathbb{R}^+),$$

as the $s \to \infty$ case shows. Since $A_{12}s(A_{22} + sI)^{-1}A_{23} \to A_{12}A_{23}$ as $s \to \infty$, we have also $0 = A_{12}A_{23}$. The latter condition means that $\operatorname{Rng} A_{23} \subset \operatorname{Ker} A_{12}$, or equivalently $(\operatorname{Ker} A_{12})^\perp \subset \operatorname{Ker} A_{23}^*$.

The linear combinations of the functions $x \mapsto 1/(s+x)$ form an algebra and due to the Stone-Weierstrass theorem $A_{12}g(A_{22})A_{23} = 0$ for any continuous function g.

We want to show that the equality implies the structure (9) of the operator A. We have $A_{23} : \mathcal{H}_3 \to \mathcal{H}_2$ and $A_{12} : \mathcal{H}_2 \to \mathcal{H}_1$. To show the structure (9), we have to find a subspace $H \subset \mathcal{H}_2$ such that

$$A_{22}H \subset H, \quad H^\perp \subset \operatorname{Ker} A_{12}, \quad H \subset \operatorname{Ker} A_{23}^*,$$

or alternatively $(H^\perp =)K \subset \mathcal{H}_2$ should be an invariant subspace of A_{22} such that

$$\operatorname{Rng} A_{23} \subset K \subset \operatorname{Ker} A_{12}.$$

Let

$$K := \Big\{ \sum_i A_{22}^{n_i} A_{23} x_i \,:\, x_i \in \mathcal{H}_3, n_i \in \mathbb{Z}^+ \Big\}$$

be a set of finite sums. It is a subspace of $\mathcal{H}_2$. The property $\operatorname{Rng} A_{23} \subset K$ and the invariance under A_{22} are obvious. Since

$$A_{12}A_{22}^n A_{23} x = 0,$$

$K \subset \operatorname{Ker} A_{12}$ also follows. The proof is complete. □

2.2 Comparisons

We can compare the structure of a Markov state on the CCR-algebra with the tensor product of full matrix algebras [10]. In the case $M_k(\mathbb{C}) \otimes M_k(\mathbb{C}) \otimes M_k(\mathbb{C})$, the middle factor contains a decomposition

$$\oplus_i (\mathcal{B}_i^L \otimes \mathcal{B}_i^R) \tag{14}$$

and the Markov state has the form $\sum_i p_i \psi_i \otimes \varphi_i$, where ψ_i is a state of $M_k(\mathbb{C}) \otimes \mathcal{B}_i^L$ and φ_i is a state of $\mathcal{B}_i^R \otimes M_k(\mathbb{C})$ [10]. The CCR situation is similar, but we do not have direct sum as (14), but only tensor product decomposition.

We want to compare the classical Gaussian situation with the CCR setting. For the sake of simplicity in this subsection we assume that the Hilbert space $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ are all k-dimensional.

Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ be vector-valued random variables with Gaussian joint probability distribution

$$\sqrt{\frac{\operatorname{Det} M}{(2\pi)^{3k}}} \exp\Big(- \tfrac{1}{2} \langle \mathbf{x}, M\mathbf{x} \rangle \Big), \tag{15}$$

where $M \in M_{3k}(\mathbb{C})$ is positive definite matrix. The triplet $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ has the Markov property if and only if the covariance matrix $S = M^{-1}$ of $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ is of the form

$$\begin{bmatrix} S_{11} & S_{12} & S_{12}S_{22}^{-1}S_{23} \\ S_{12}^* & S_{22} & S_{23} \\ S_{23}^* S_{22}^{-1} S_{12}^* & S_{23}^* & S_{33} \end{bmatrix}, \tag{16}$$

that is

$$S_{13} = S_{12} S_{22}^{-1} S_{23}, \tag{17}$$

see [3]. To show some analogy between the classical Gaussian and the CCR Gaussian case, we formulate a somewhat similar description to (17) in the CCR setting.

Theorem 2.2 *The block matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

gives a quasi-free state with the Markov property if and only if

$$A_{13} = A_{12} f(A_{22}) A_{23} \tag{18}$$

for any continuous function $f : \mathbb{R} \to \mathbb{R}$.

Proof. If the Markov property holds, then A has the form of (9) and we have

$$A_{12} f(A_{22}) A_{23} = [\,a \quad 0\,] \begin{bmatrix} f(c) & 0 \\ 0 & f(d) \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = 0 = A_{13}.$$

The converse is part of the proof of Theorem 2.1. □

We choose unit vectors e_j, $1 \le j \le 3k$ such that

$$e_{(i-1)k+r} \in \mathcal{H}_i, \qquad 1 \le i \le 3, \quad 0 \le r \le k-1 \tag{19}$$

and

$$\langle e_t, e_u \rangle \text{ is real for any } 1 \le t, u \le 3k. \tag{20}$$

In the Fock representation the Weyl unitaries $W(te_j) = \exp(ti B(e_j))$ commute and give the (unbounded) field operators $B(e_j)$. It follows from [3, 13] that the classical (multi-valued) Gaussian triplet

$$(B(e_1), \dots, B(e_k)), \quad (B(e_{k+1}), \dots, B(e_{2k})), \quad (B(e_{2k+1}), \dots, B(e_{3k})) \tag{21}$$

is Markovian if and only if

$$(I + 2A)_{13} = (I + 2A)_{12}(I + 2A)_{22}^{-1}(I + 2A)_{23} \tag{22}$$

which means that $(1, 3)$ element of $(I + 2A)^{-1}$ is 0. If the quasi-free state induced by A gives a Markov triplet, then (22) is true and the classical Markov property of (21) follows. The converse is not true. However, if for every λA ($\lambda > 0$) the classical Markov property is true, then from (22) we have

$$A_{13} = A_{12}(I/(2\lambda) + A)_{22}^{-1} A_{23}$$

and the Markovianity of the quasi-free state follows.

2.3 Minimizing relative entropy

Recall the notation $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ and $\mathrm{CCR}(\mathcal{H}) = \mathrm{CCR}(\mathcal{H}_1) \otimes \mathrm{CCR}(\mathcal{H}_2) \otimes \mathrm{CCR}(\mathcal{H}_3)$. The next theorem is the analogue of [4].

Theorem 2.3 *Let $\omega \equiv \omega_A$ be a Markovian a Gaussian state on the CCR-algebra $\mathrm{CCR}(\mathcal{H})$ and let ψ_1 be a state of $\mathrm{CCR}(\mathcal{H}_1)$ with a 2-point function. If ψ is the state minimizing the relative entropy $S(\psi||\omega_A)$ under the constraint that $\psi|\mathrm{CCR}(\mathcal{H}_1) = \psi_1$ is fixed, then ψ is a Markov state.*

Proof. We have the tensor product structure $\omega = \omega_L \otimes \omega_R$ on (10). Due to the monotonicity of the relative entropy

$$S(\psi||\omega) \ge S(\psi\,|\,\mathrm{CCR}(\mathcal{H}_1 \oplus \mathcal{H}_2^L)\;||\;\omega|\mathrm{CCR}(\mathcal{H}_1 \oplus \mathcal{H}_2^L)),$$

holds and it is enough to minimize the right-hand-side. The right-hand-side can be finite, for example if ψ is Gaussian, therefore, the minimizer is uniquely exists. If the state ψ' is the minimizer, then $\psi = \psi' \otimes \omega_R$ is the minimizer on $\mathrm{CCR}(\mathcal{H})$ due to the conditional expectation property, see Chapter 2 in [18]. From the product structure the Markov property follows. □

Note that the minimizer Markovian state ψ has the same conditional expectation than the given state ω. In the probabilistic case the similar statement is well-known, see [4], for example.

Theorem 2.4 *Let $\omega \equiv \omega_A$ be a Markovian quasi-free state on the CCR-algebra* $\mathrm{CCR}(\mathcal{H})$. *There exists a state ψ which is minimizing the relative entropy $S(\psi||\omega_A)$ under the constraint that $\psi|\mathcal{A}_1$ has a fixed 2-point operator. Moreover, ψ is a Markov state.*

Proof. Let T be the 2-point operator of $\psi|\mathcal{A}_1 = \varphi$ and assume that ω_A is determined by the matrix (9). Similarly to the proof of the previous theorem, we have to concentrate first to the restrictions of ψ and ω_A to $\mathcal{A}_L := \mathrm{CCR}(\mathcal{H}_1 \oplus \mathcal{H}_2^L)$. Here they have the block matrices

$$\varphi : \begin{bmatrix} T & u \\ u^* & v \end{bmatrix} \quad \text{and} \quad \omega_A|\mathcal{A}_L : \begin{bmatrix} A_{11} & a \\ a^* & c \end{bmatrix}.$$

The unknown entries u and v of the first matrix are uniquely determined by the minimization of $S(\varphi\,\|\ \omega_A|\mathcal{A}_L)$. When φ is obtained, ψ has the matrix

$$\begin{bmatrix} T & u & 0 & 0 \\ u^* & v & 0 & 0 \\ 0 & 0 & d & b \\ 0 & 0 & b^* & A_{33} \end{bmatrix}.$$

So $\psi = \varphi \otimes \omega_R$. From the product structure the Markov property follows. □

Appendix: Relative entropy

The von Neumann entropy and the relative entropy were defined originally for statistical operators:

$$S(D) = -\mathrm{Tr}\, D_1 \log D_1\,, \qquad S(D_1\|D_2) = \mathrm{Tr}\, D_1(\log D_1 - \log D_2).$$

Kosaki's formula can be used to define relative entropy of states of a C*-algebra $\mathcal{A}$:

$$S(\varphi\|\omega) = \sup_n \sup \left\{ \varphi(I)\log n - \int_{1/n}^{\infty} (\varphi(y(t)^* y(t)) + t^{-1}\omega(x(t)x(t)^*))\frac{dt}{t} \right\},$$

where the first sup is taken over all natural numbers n, the second one is over all step functions $x : (1/n, \infty) \to \mathcal{A}$ with finite range and $y(t) = I - x(t)$ [15]. The von Neumann entropy can be defined via the relative entropy:

$$S(\varphi) = \sup \left\{ \sum_i \lambda_i S(\varphi_i\|\varphi) : \sum_i \lambda_i \varphi_i = \varphi \right\}.$$

Here the supremum is over all decompositions of φ into finite (or equivalently countable) convex combinations of other states.

In our situation ω_A is a Gaussian state of the CCR-algebra which has a normal extension $\bar{\omega}_A$ in the Fock representation and so $S(\omega_A) = S(\bar{\omega}_A)$. If the state ψ does not have a normal extension, then $S(\psi \| \omega_A) = +\infty$. When $\bar{\psi}$ is the normal extension, then $S(\psi \| \omega_A) = S(\bar{\psi} \| \bar{\omega}_A)$, see Chapters 5 and 6 in [18] about the details. It is a consequence that we can work in the Fock representation.

We want to compute the relative entropy of a state ψ and a Gaussian state ω_A. The point is the computation of the term $\psi(\log D_A)$, where D_A is the statistical operator of ω_A. Using

$$\log \Gamma(A(I+A)^{-1}) = \sum_i \log \frac{\lambda_i}{1+\lambda_i} a^+(f_i) a(f_i) \tag{23}$$

and

$$\log \mathrm{Tr}\, \Gamma(A(I+A)^{-1}) = \mathrm{Tr}\, \log(I+A). \tag{24}$$

we have

$$\begin{aligned}
\psi(\log D_A) &= \psi(\log \Gamma(A(I+A)^{-1}))) - \log \mathrm{Tr}\, \Gamma(A(I+A)^{-1}) = \\
&= \sum_i \log \frac{\lambda_i}{1+\lambda_i} \psi(a^+(f_i) a(f_i)) - \mathrm{Tr}\, \log(I+A) \\
&= \sum_i \log \frac{\lambda_i}{1+\lambda_i} \langle f_i, T f_i \rangle - \mathrm{Tr}\, \log(I+A) \\
&= \mathrm{Tr}\, T \log A(I+A)^{-1} - \mathrm{Tr}\, \log(I+A),
\end{aligned}$$

where T is the 2-point operator of ψ. Hence

$$S(\psi \| \omega_A) = -S(\psi) - \mathrm{Tr}\, T \log A(I+A)^{-1} + \mathrm{Tr}\, \log(I+A). \tag{25}$$

If $A = T$, then we have

$$-S(\psi) + S(\omega_A) \geq 0,$$

that is the quasi-free state ω_A has the largest entropy among states with 2-point function A. On the other hand, the relative entropy of the Gaussian states ω_B and ω_A is

$$S(\omega_B \| \omega_A) = \mathrm{Tr}\, B(\log B - \log A) - \mathrm{Tr}\,(I+B)(\log(I+B) - \log(I+A)). \tag{26}$$

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