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STRONGLY SUBADDITIVE FUNCTIONS

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Abstract. Let $f : \mathbb{R}^+ \to \mathbb{R}$. The subject is the trace inequality $\mathrm{Tr}\, f(A) + \mathrm{Tr}\, f(P_2AP_2) \leqq \mathrm{Tr}\, f(P_{12}AP_{12}) + \mathrm{Tr}\, f(P_{23}AP_{23})$, where A is a positive operator, P_1, P_2, P_3 are orthogonal projections such that $P_1 + P_2 + P_3 = I$, $P_{12} = P_1 + P_2$ and $P_{23} = P_2 + P_3$. There are several examples of functions f satisfying the inequality (called (SSA)) and the case of equality is described.

1. Introduction

Matrix monotone and matrix concave functions play important roles in several applications. Assume that $f : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function. It is *matrix monotone* if $0 \leqq A \leqq B$ implies $f(A) \leqq f(B)$ for every matrix A and B. The function f is called *matrix concave* if one of the following two equivalent conditions holds:

$$f\big(\lambda A + (1-\lambda)B\big) \geqq \lambda f(A) + (1-\lambda) f(B) \tag{1}$$

for every number $0 < \lambda < 1$ and for positive definite square matrices A and B (of the same size). In the other condition the number λ is (heuristically) replaced by a matrix:

$$f(CAC^* + DBD^*) \geqq Cf(A)C^* + Df(B)D^* \tag{2}$$

if $CC^* + DD^* = I$, see the books [3, 8] about the details. It is surprising that a matrix monotone function is matrix concave.

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Motivated by some applications we study the functions f which are strongly subadditivite in the following sense. Let P_1, P_2, P_3 be orthogonal projections such that $P_1+P_2+P_3=I$. Then

$$\operatorname{Tr} f(A)+\operatorname{Tr} f(P_2AP_2) \leqq \operatorname{Tr} f(P_{12}AP_{12})+\operatorname{Tr} f(P_{23}AP_{23}), \tag{3}$$

where $P_{12}:=P_1+P_2$ and $P_{23}:=P_2+P_3$. The special case when $P_2=0$ could be called *subadditivity*. This holds for any concave function [7, Theorem 2.4].

The first example $f(x)=\log x$ appeared already [2]; here we have several other examples and a sufficient condition. The strongly subadditive functions are concave in the sense of real variable and all known examples are matrix concave.

2. Motivation

The *second quantization* in quantum theory is mathematically a procedure which associates an operator on the Fock space $\mathcal{F}(\mathcal{H})$ to an operator on the Hilbert space $\mathcal{H}$ [4]. The simplest example is $\mathcal{H}=\mathbb{C}$, then $\mathcal{F}(\mathcal{H})$ is $\ell^2(\mathbb{Z}^+)$. To the number $\mu>0$ (considered as a positive operator) we associate $\Gamma(\mu)$ defined as

$$\Gamma(\mu)\delta_n=\mu^n\delta_n \qquad (n=0,1,2,\dots),$$

where δ_n are the standard basis vectors. Γ can be extended to arbitrary finite dimension by the formula

$$\Gamma(H_1\oplus H_2)=\Gamma(H_1)\otimes\Gamma(H_2).$$

In this way to any positive operator $H\in B(\mathcal{H})$ we have a positive operator $\Gamma(H)\in B\big(\mathcal{F}(\mathcal{H})\big)$. The construction of a statistical operator, analogue of the Gaussian distribution, is slightly more complicated. For a positive operator A set

$$\alpha(A)=\frac{\Gamma(H)}{\operatorname{Tr}\Gamma(H)}, \quad \text{where} \quad H=A(I+A)^{-1}.$$

In particular, if $A=\lambda$, then

$$\alpha(\lambda)\delta_n=\frac{1}{1+\lambda}\left(\frac{\lambda}{1+\lambda}\right)^n\delta_n.$$

The von Neumann entropy

$$S\big(\alpha(A)\big):=-\operatorname{Tr}\alpha(A)\log\alpha(A)$$

equals $\operatorname{Tr}\kappa(A)$, where $\kappa(x) := -x\log x + (x+1)\log(x+1)$ [4, 5].

From the formula

$$\log x = \int_0^\infty \left(\frac{1}{1+t} - \frac{1}{x+t}\right) dt.$$

we get

$$\begin{aligned}\kappa(x) &= -\int_0^\infty \left(\frac{x}{1+t} - \frac{x}{x+t}\right) dt + \int_0^\infty \left(\frac{x+1}{1+t} - \frac{x+1}{x+1+t}\right) dt \\ &= \int_0^1 \left(1 - \frac{x}{1+t} - \frac{t}{x+t}\right) dt + \int_1^\infty \left(\frac{x+1}{t} - \frac{x}{1+t} - \frac{1}{x+t}\right) dt.\end{aligned}$$

Since both integrands are matrix concave, the integrals are matrix concave, too. Now

$$\kappa'(x) = \log\left(1 + \frac{1}{x}\right) > 0$$

and κ is monotone. Hence $\kappa(x) \geqq \kappa(0) = 0$. The positivity together with matrix concavity implies matrix monotonicity, [6].

Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ be a finite dimensional Hilbert space, let

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{bmatrix}$$

be a positive invertible operator and set

$$B = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}, \qquad C = \begin{bmatrix} A_{22} & A_{23} \\ A_{23}^* & A_{33} \end{bmatrix}.$$

The strong subadditivity of the von Neumann entropy

$$S\big(\alpha(A)\big) + S\big(\alpha(A_{22})\big) \leqq S\big(\alpha(B)\big) + S\big(\alpha(C)\big)$$

has the equivalent form

$$\operatorname{Tr}\kappa(A) + \operatorname{Tr}\kappa(A_{22}) \leqq \operatorname{Tr}\kappa(B) + \operatorname{Tr}\kappa(C). \tag{4}$$

The case of equality is studied in the paper [5] and the general properties of entropy are in the book [8].

Proposition 2.1. *The equality*

$$\operatorname{Tr}\kappa(A)+\operatorname{Tr}\kappa(A_{22})=\operatorname{Tr}\kappa(B)+\operatorname{Tr}\kappa(C),$$

in the strong subadditivity holds if and only if A *has the form*

$$A=\begin{bmatrix} A_{11} & [a\ \ 0] & 0 \\ \begin{bmatrix} a^* \\ 0 \end{bmatrix} & \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} & \begin{bmatrix} 0 \\ b \end{bmatrix} \\ 0 & [0\ \ b^*] & A_{33} \end{bmatrix}=\begin{bmatrix} \begin{bmatrix} A_{11} & a \\ a^* & c \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} d & b \\ b^* & A_{33} \end{bmatrix} \end{bmatrix}, \tag{5}$$

where the parameters a*,* b*,* c*,* d *(and* 0*) are matrices.*

Note that the matrix c or d in the theorem can be 0×0.

We are interested in the (differentiable) functions f such that the inequality

$$\operatorname{Tr}f(A)+\operatorname{Tr}f(A_{22})\leqq\operatorname{Tr}f(B)+\operatorname{Tr}f(C) \tag{SSA}$$

holds. We call this *strong subadditivity* for the function f. The strong subadditivity holds for the function κ. Another equivalent formulation of the strong subadditivity is (3).

3. Particular examples

Example 3.1. If

$$A=\begin{bmatrix} a & 0 & d \\ 0 & b & 0 \\ d^* & 0 & c \end{bmatrix} \tag{6}$$

is a numerical matrix, then it is an exercise to show that (SSA) holds for this kind of A if and only if f is a concave function.

Example 3.2. The strong subadditivity does not hold for the function $f(t)=-1/t$. The following counterexample is due to Ando [1]: Let

$$X\equiv A^{-1}:=\begin{bmatrix} 4 & 8 & -2 \\ 8 & 20 & 0 \\ -2 & 0 & 9 \end{bmatrix}.$$

Then

$$A=\begin{bmatrix} \frac{45}{16} & -\frac{9}{8} & \frac{5}{8} \\ -\frac{9}{8} & \frac{1}{2} & -\frac{1}{4} \\ \frac{5}{8} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

We have

$$\operatorname{Tr} A^{-1} = 33, \quad \operatorname{Tr} A_{22}^{-1} = 2, \quad \operatorname{Tr} B^{-1} = \frac{212}{9}, \quad \operatorname{Tr} C^{-1} = 12$$

and (SSA) becomes

$$33 + 2 \geqq \frac{212}{9} + 12$$

and this is not true.

EXAMPLE 3.3. It is elementary that the strong subadditivity holds for the function $f(t) = -t^2$. Equality holds if and only $A_{13} = 0$. □

EXAMPLE 3.4. It was proved in [2] that the strong subadditivity holds for the function $f(t) = \log t$ and equality holds if and only if $A_{13} = A_{12} A_{22}^{-1} A_{23}$.

We present an alternative approach. Now (SSA) is equivalent to

$$\operatorname{Det} A \cdot \operatorname{Det} A_{22} \leqq \operatorname{Det} B \cdot \operatorname{Det} C.$$

Let

$$\hat{A} := \operatorname{Diag}\big(A_{11}^{-1/2}, A_{22}^{-1/2}, A_{33}^{-1/2}\big) A \operatorname{Diag}\big(A_{11}^{-1/2}, A_{22}^{-1/2}, A_{33}^{-1/2}\big).$$

Then (SSA) is equivalent to

$$\operatorname{Det} \hat{A} \leqq \operatorname{Det} \hat{B} \cdot \operatorname{Det} \hat{C}.$$

In other words, we may assume that the diagonal of A consists of I's. Since

$$\begin{bmatrix} I & -\hat{A}_{12} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & \hat{A}_{12} & \hat{A}_{13} \\ \hat{A}_{12}^* & I & \hat{A}_{23}^* \\ \hat{A}_{13}^* & \hat{A}_{23}^* & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & -\hat{A}_{23} \\ 0 & 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} I - \hat{A}_{12}\hat{A}_{12}^* & 0 & \hat{A}_{13} - \hat{A}_{12}\hat{A}_{23} \\ \hat{A}_{12}^* & I & 0 \\ \hat{A}_{13}^* & \hat{A}_{23}^* & I - \hat{A}_{23}^*\hat{A}_{23} \end{bmatrix},$$

equality holds in (SSA) if $\hat{A}_{13} = \hat{A}_{12}\hat{A}_{23}$, equivalently $A_{13} = A_{12} A_{22}^{-1} A_{23}$. This condition is sufficient for the equality. □

EXAMPLE 3.5. Since

$$\frac{1}{2}\big(A + \operatorname{Diag}(1,1,-1) A \operatorname{Diag}(1,1,-1)\big) = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12}^* & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$$

we get a majorization

$$A \succ \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12}^* & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix},$$

that is, the eigenvalue vector $\vec{\lambda}(A)$ majorizes that of

$$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12}^* & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}.$$

For any concave function f, this implies that $f \circ \vec{\lambda}(A)$ is weakly majorized by the $f \circ \lambda$ of

$$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12}^* & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$$

so that

$$\operatorname{Tr} f(A) \leqq \operatorname{Tr} f\left(\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12}^* & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}\right) = \operatorname{Tr} f(B) + \operatorname{Tr} f(A_{33}). \tag{7}$$

Hence

$$\operatorname{Tr} f(A) + \operatorname{Tr} f(A_{22}) \leqq \operatorname{Tr} f(B) + \operatorname{Tr} f(A_{22}) + \operatorname{Tr} f(A_{33}).$$

This says that if $A_{23} = 0$ (or $A_{12} = 0$), then (SSA) holds for every concave function f.

Note that inequality (7) is written as

$$\operatorname{Tr} f(A) = \operatorname{Tr} Pf(A)P + \operatorname{Tr} Qf(A)Q \leqq \operatorname{Tr} f(PAP + QAQ)$$

when P and Q are orthogonal projections and $P + Q = I$. This is a special case of Jensen's trace inequality for concave functions [7, Theorem 2.4]. □

EXAMPLE 3.6. The representation

$$y^t = \frac{\sin \pi t}{\pi} \int_0^\infty \frac{\lambda^{t-1} y}{\lambda + y}\, d\lambda \tag{8}$$

is used to show that $f(x) = x^t$ is operator monotone when $0 < t < 1$. From this we obtain

$$\int_0^x y^{t-1}\, dy = \frac{\sin \pi t}{\pi} \int_0^x \int_0^\infty \frac{\lambda^{t-1}}{\lambda + y}\, d\lambda\, dy$$

which gives

$$x^t = \frac{t \sin \pi t}{\pi} \int_0^\infty \lambda^{t-1} (\log(x+\lambda) - \log \lambda)\, d\lambda.$$

So we have a similar formula to (8):

$$x^t = \frac{t \sin \pi t}{\pi} \int_0^\infty \lambda^{t-1} \log\left(1 + \frac{x}{\lambda}\right) d\lambda. \tag{9}$$

Since inequality (SSA) is true for the functions $f_\lambda(x) := \log\left(1 + \frac{x}{\lambda}\right)$, by integration it follows for x^t when $0 < t < 1$.

We analyze the condition for equality and use the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. For f_λ the equality condition is

$$A_{13} = A_{12}(\lambda + A_{22})^{-1} A_{23};$$

see Example 3.4. This holds for every $\lambda > 0$. If $\lambda \to \infty$ in the relation

$$\lambda A_{13} = A_{12}\left[\lambda(\lambda + A_{22})^{-1}\right] A_{23},$$

then we conclude $A_{13} = 0 = A_{12}A_{23}$. The latter condition means that $\operatorname{Rng} A_{23} \subset \operatorname{Ker} A_{12}$, or equivalently $(\operatorname{Ker} A_{12})^\perp \subset \operatorname{Ker} A_{23}^*$.

The linear combinations of the functions $x \mapsto 1/(\lambda + x)$ form an algebra and due to the Stone–Weierstrass theorem $A_{12}g(A_{22})A_{23} = 0$ for any continuous function g.

We want to show that the equality implies the structure (5) of the operator A. We have $A_{23} : \mathcal{H}_3 \to \mathcal{H}_2$ and $A_{12} : \mathcal{H}_2 \to \mathcal{H}_1$. To show the structure (5), we have to find a subspace $H \subset \mathcal{H}_2$ such that

$$A_{22}H \subset H, \quad H^\perp \subset \operatorname{Ker} A_{12}, \quad H \subset \operatorname{Ker} A_{32},$$

or alternatively $(H^\perp =)K \subset \mathcal{H}_2$ should be an invariant subspace of A_{22} such that

$$\operatorname{Rng} A_{23} \subset K \subset \operatorname{Ker} A_{12}.$$

Let

$$K := \left\{ \sum_i A_{22}^{n_i} A_{23} x_i : x_i \in \mathcal{H}_3, n_i \in \mathbb{Z}^+ \right\}$$

be a set of finite sums. It is a subspace of $\mathcal{H}_2$. The property $\operatorname{Rng} A_{23} \subset K$ and the invariance under A_{22} are obvious. Since

$$A_{12} A_{22}^n A_{23} x = 0,$$

$K \subset \operatorname{Ker} A_{12}$ also follows. $\square$

4. Sufficient condition

THEOREM 4.1. *Let $f : (0,+\infty) \to \mathbb{R}$ be a function such that $-f'$ is matrix monotone. Then the inequality* (SSA) *holds.*

PROOF. The idea of the previous example is followed. A matrix monotone function has the representation

$$a + bx + \int_0^\infty \left(\frac{\lambda}{\lambda^2+1} - \frac{1}{\lambda+x} \right) d\mu(\lambda),$$

where $b \geqq 0$, see (V.49) in [3]. Therefore, we have the representation

$$f(t) = c - \int_1^t \left(a + bx + \int_0^\infty \left(\frac{\lambda}{\lambda^2+1} - \frac{1}{\lambda+x} \right) d\mu(\lambda) \right) dx.$$

By integration we have

$$f(t) = d - at - \frac{b}{2}t^2 + \int_0^\infty \left(\frac{\lambda}{\lambda^2+1}(1-t) + \log\left(\frac{\lambda}{\lambda+1} + \frac{t}{\lambda+1} \right) \right) d\mu(\lambda).$$

The first quadratic part satisfies the (SSA) and we have to check the integral. Since $\log x$ is a strongly subadditive function, so is the integrand. The integration keeps the property. □

The previous theorem covers all known examples, but we can get new examples.

EXAMPLE 4.1. By differentiation we can see that

$$f(x) = -(x+t)\log\,(x+t)$$

with $t \geqq 0$ satisfies (SSA). Similarly, $f(x) = -x^t$ satisfies (SSA) if $1 \leqq t \leqq 2$.

In some applications [9] the operator monotone functions

$$f_p(x) = p(1-p)\frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)} \qquad (0 < p < 1)$$

appear.

For $p = 1/2$ this is an (SSA) function. Up to a constant factor, the function is

$$\left(\sqrt{x}+1\right)^2 = x + 2\sqrt{x} + 1$$

and all terms are known to be (SSA). The function $-f'_{1/2}$ is evidently matrix monotone.

Numerical computation shows that $-f'_p$ seems to be matrix monotone. □

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