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Generalizations of Pauli channels

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Abstract

The Pauli channel acting on 2×2 matrices is generalized to an n-level quantum system. When the full matrix algebra M_n is decomposed into pairwise complementary subalgebras, then trace-preserving linear mappings $M_n \to M_n$ are constructed such that the restriction to the subalgebras are depolarizing channels. The result is the necessary and sufficient condition of complete positivity. The main examples appear on bipartite systems.

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In this paper we consider particular subalgebras of a full matrix algebra $M_n = M_n(\mathbb{C})$. (By a subalgebra we mean *-subalgebra with unit.) An F-subalgebra is a subalgebra isomorphic to a full matrix algebra M_k. ("F" is the abbreviation of "factor", the center of such a subalgebra is minimal, $\mathbb{C}I$.) An M-subalgebra is a maximal Abelian subalgebra, equivalently, it is isomorphic to $\mathbb{C}^n$. ("M" is an abbreviation of "MASA", the center is maximal, it is the whole subalgebra.) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are subalgebras, then they are called quasi-orthogonal or complementary if the subspaces $\mathcal{A}_1 \ominus \mathbb{C}I$ and $\mathcal{A}_2 \ominus \mathbb{C}I$ are orthogonal (with respect to the Hilbert-Schmidt inner product $\langle A, B\rangle = \operatorname{Tr} A^*B$). Concerning complementary subalgebras we refer to [8], see also [6, 7, 9].

Complementary M-subalgebras can be given by mutually unbiased bases. Assume that $\xi_1, \xi_2, \dots, \xi_n$ and $\eta_1, \eta_2, \dots, \eta_n$ are orthonormal bases such that

$$|\langle \xi_i, \eta_j\rangle| = \frac{1}{\sqrt{n}} \qquad (1 \le i, j \le n).$$

If $\mathcal{A}_1$ is the algebra of all operators with diagonal matrix in the first basis and $\mathcal{A}_2$ is defined similarly with respect to the second basis, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are complementary M-subalgebras.

There are examples such that M_n is the linear span of pairwise complementary subalgebras in the case when n is a power of a prime number. If M_n is decomposed into complementary subalgebras, then we construct trace-preserving mappings $M_n \to M_n$ which are completely positive under some conditions.

1 Introduction

If the pairwise complementary subalgebras $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_r$ of M_n are given and they linearly span the whole algebra M_n, then any operator is the sum of the components in the subspaces $\mathcal{A}_i \ominus \mathbb{C}I$ $(1 \le i \le r)$ and $\mathbb{C}I$:

$$A = -\frac{(r-1)\mathrm{Tr}\, A}{n} I + \sum_{i=1}^{r} E_i(A)\,,$$

where $E_i : M_n \to \mathcal{A}_i$ is the trace-preserving conditional expectation (which is nothing else but the orthogonal projection with respect to the Hilbert-Schmidt inner product, see [10] about details). It is easier to formulate things for matrices of trace 0. If $\mathrm{Tr}\, B = 0$, then it has orthogonal decomposition

$$B = \sum_{i=1}^{r} E_i(B)\,.$$

As a generalization of the Pauli channel on a qubit, we define a linear mapping $\alpha : M_n \to M_n$ such that

$$\alpha(B) = \sum_{i=1}^{r} \lambda_i E_i(B)$$

or for an arbitrary A

$$\alpha(A) = \Big(1 - \sum_{i=1}^{r} \lambda_i\Big) \frac{\mathrm{Tr}\, A}{n} I + \sum_{i=1}^{r} \lambda_i E_i(A), \tag{1}$$

where $\lambda_i \in \mathbb{R}$, $1 \le i \le r$. We want to find the condition for complete positivity. The motivation is the following well-known example in which the complementary subalgebras are generated by the Pauli matrices [10].

Example 1 Let $\sigma_0 = I$ and $\sigma_1, \sigma_2, \sigma_3$ be Pauli matrices, i.e.,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and let $\mathcal{E} : M_2 \to M_2$ be defined as

$$\mathcal{E}\left(w_0\sigma_0 + (w_1, w_2, w_3)\cdot\sigma\right) = w_0\sigma_0 + (\lambda_1 w_1, \lambda_2 w_2, \lambda_3 w_3)\cdot\sigma \tag{2}$$

for $\omega_i \in \mathbb{C}$, where $\lambda_i \in \mathbb{R}$ and

$$(w_1, w_2, w_3)\cdot\sigma = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3.$$

Density matrices are sent to density matrices if and only if

$$-1 \le \lambda_i \le 1.$$

It is not difficult to compute the representing block matrix $X := \sum_{i,j} \mathcal{E}(E_{ij}) \otimes E_{ij}$, we have

$$X = \begin{bmatrix} \frac{1+\lambda_3}{2} & 0 & 0 & \frac{\lambda_1+\lambda_2}{2} \\ 0 & \frac{1-\lambda_3}{2} & \frac{\lambda_1-\lambda_2}{2} & 0 \\ 0 & \frac{\lambda_1-\lambda_2}{2} & \frac{1-\lambda_3}{2} & 0 \\ \frac{\lambda_1+\lambda_2}{2} & 0 & 0 & \frac{1+\lambda_3}{2} \end{bmatrix}.$$

According to Choi's theorem [2] the positivity of this matrix is equivalent to the complete positivity of $\mathcal{E}$. X is unitarily equivalent to the matrix

$$\begin{bmatrix} \frac{1+\lambda_3}{2} & \frac{\lambda_1+\lambda_2}{2} & 0 & 0 \\ \frac{\lambda_1+\lambda_2}{2} & \frac{1+\lambda_3}{2} & 0 & 0 \\ 0 & 0 & \frac{1-\lambda_3}{2} & \frac{\lambda_1-\lambda_2}{2} \\ 0 & 0 & \frac{\lambda_1-\lambda_2}{2} & \frac{1-\lambda_3}{2} \end{bmatrix}.$$

This matrix is obviously positive if and only if

$$1 \pm \lambda_3 \ge |\lambda_1 \pm \lambda_2|. \tag{3}$$

This is necessary and sufficient condition of complete positivity. □

It is not obvious that condition (3) is symmetric in the three variables $\lambda_1, \lambda_2, \lambda_3$. Condition (3) actually determines the tetrahedron which is the convex hull of the points $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$ and $(-1,-1,1)$.

Now we show the idea leading to the generalization. The mapping $\mathcal{E}$ in (2) has the form

$$\mathcal{E}(\,\cdot\,) = \sum_{i=0}^{3} \mu_i \sigma_i(\,\cdot\,)\sigma_i.$$

From the expansion of $\mathcal{E}(\sigma_j)$ we can get equations and the solution is the following:

$$\begin{aligned} \mu_0 &= \frac{1}{4}(1+\lambda_1+\lambda_2+\lambda_3), \qquad \mu_1 = \frac{1}{4}(1+\lambda_1-\lambda_2-\lambda_3), \\ \mu_2 &= \frac{1}{4}(1-\lambda_1+\lambda_2-\lambda_3), \qquad \mu_3 = \frac{1}{4}(1-\lambda_1-\lambda_2+\lambda_3). \end{aligned}$$

If $\mu_i \geq 0$ for every i, then $\mathcal{E}$ is a completely positive mapping. Therefore,

$$1+\lambda_3 \geq \pm(\lambda_1+\lambda_2), \qquad 1-\lambda_3 \geq \pm(\lambda_1-\lambda_2)$$

or together this is (3). (Actually, this argument gives that (3) is a sufficient condition for the complete positivity.)

Pauli channels form an important and popular subject in quantum information theory [1, 3, 4]. The mappings (1) were studied in the paper [5] in the case when the subalgebras are maximal Abelian and pairwise complementary. Our method is different and we allow non-commutative subalgebras as well.

The mapping (1) restricted to $\mathcal{A}_i$ has the form

$$D \mapsto \lambda_i D + (1-\lambda_i)\frac{I}{n}$$

on density matrices D. If $0 \leq \lambda_i \leq 1$, then we can say that D does not change with probability λ_i and with probability $1-\lambda_i$ it is sent to the tracial state. Such mappings are usually called as depolarizing channels [10].

A simple example including non-commutative subalgebras is the following.

Example 2 Consider $M_4 = M_2 \otimes M_2$ and the complementary F-subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_4$ generated by the following triplets of unitaries:

$$\begin{array}{lll} \sigma_0\otimes\sigma_1 & \sigma_0\otimes\sigma_2 & \sigma_0\otimes\sigma_3, \\ \sigma_1\otimes\sigma_0 & \sigma_2\otimes\sigma_1 & \sigma_3\otimes\sigma_1, \\ \sigma_2\otimes\sigma_0 & \sigma_3\otimes\sigma_2 & \sigma_1\otimes\sigma_2, \\ \sigma_3\otimes\sigma_0 & \sigma_1\otimes\sigma_3 & \sigma_2\otimes\sigma_3. \end{array}$$

We take also the M-subalgebra $\mathcal{A}_5$ generated by $\sigma_1\otimes\sigma_1, \sigma_2\otimes\sigma_2, \sigma_3\otimes\sigma_3$. The conditional expectations $E_j : M_4 \to \mathcal{A}_j$ are convex combinations of automorphisms

$$E_j(A) = \frac{1}{4}\sum_{i=1}^{4} U_{ji}^* A U_{ji}, \tag{4}$$

where $U_{j1} = I$ and U_{ji}'s are orthogonal unitaries from $\mathcal{A}_j'$. Since $\mathcal{A}_5$ is an M-subalgebra, $\mathcal{A}_5' = \mathcal{A}_5$. The subalgebras $\mathcal{A}_1', \dots, \mathcal{A}_4'$ are F-subalgebras generated by the following unitaries:

$$\begin{array}{lll} \sigma_1\otimes\sigma_0 & \sigma_2\otimes\sigma_0 & \sigma_3\otimes\sigma_0, \\ \sigma_0\otimes\sigma_1 & \sigma_1\otimes\sigma_2 & \sigma_1\otimes\sigma_3, \\ \sigma_2\otimes\sigma_1 & \sigma_0\otimes\sigma_2 & \sigma_2\otimes\sigma_3, \\ \sigma_3\otimes\sigma_1 & \sigma_3\otimes\sigma_2 & \sigma_0\otimes\sigma_3. \end{array}$$

(The above triplets generating $\mathcal{A}_j$ and $\mathcal{A}_j'$ $(1 \leq j \leq 4)$ are Pauli triplets, see [7] for details.) Moreover,

$$(\mathrm{Tr}\, A)I = \frac{1}{4}\Big(A + \sum_{j=1}^{5}\sum_{k=2}^{4} U_{jk}^* A U_{jk}\Big).$$

The linear mapping (1) has the concrete form

$$\alpha(A) = \Big(1 - \sum_{i=1}^{5} \lambda_i\Big) \frac{\mathrm{Tr}\, A}{4} I + \sum_{i=1}^{5} \lambda_i E_i(A),$$

where the conditional expectations E_j is expressed by the commutant, see (4). (The condition for complete positivity of α is in Theorem 4.) $\square$

Our main result is the necessary and sufficient condition for the complete positivity of mappings like (1) which can be called generalized Pauli channel.

2 Generalized Pauli channels

Let $\mathcal{A}$ be a (unital *-) subalgebra of M_n. Our aim is to describe the conditional expectation onto $\mathcal{A}$ by means of an orthogonal system in the commutant.

Up to unitary equivalence, a subalgebra $\mathcal{A}$ of M_n can be written as

$$\mathcal{A} = \bigoplus_{i=1}^{k} M_{n_i} \otimes I_{m_i}.$$

The commutant $\mathcal{A}'$ in M_n is

$$\mathcal{A}' = \bigoplus_{i=1}^{k} I_{n_i} \otimes M_{m_i}.$$

Let $N = \sum_{i=1}^{k} n_i^2$ and let P_i be a minimal central projection of $\mathcal{A}$, that is, $P_i = I_{n_i} \otimes I_{m_i}$.

Proposition 1 *Let $\{U_i\}_{i=1}^{N}$ be an orthonormal basis of $\mathcal{A}$. Then the completely positive map F from M_n onto $\mathcal{A}'$ given by*

$$F(X) = \sum_{i=1}^{N} U_i^* X U_i \qquad (X \in M_n)$$

is equal to

$$F(X) = \bigoplus_{i=1}^{k} \frac{n_i}{m_i} \mathrm{Tr}_{\,n_i}(P_i X P_i),$$

where $\mathrm{Tr}_{\,n_i}$ is a partial trace from $M_{n_i} \otimes M_{m_i}$ onto M_{m_i}.

In particular, if all n_i/m_i are equal, then $\frac{n}{\dim \mathcal{A}} F$ is the trace-preserving conditional expectation from M_n onto $\mathcal{A}'$.

Proof: If all n_i/m_i are equal, then their ratio is equal to $\frac{\dim\mathcal{A}}{n}$. Therefore it is sufficient to prove the first assertion.

Let $\{e_{ij}^{(l)}\}_{i,j=1}^{n_l}$ and $\{f_{ij}^{(l)}\}_{i,j=1}^{m_l}$ be matrix units of M_{n_l} and M_{m_l}, respectively. Then U_i is written by

$$U_i = \sum_{l=1}^{k} \sum_{s,t=1}^{n_l} U_{i,st}^{(l)} e_{st}^{(l)}$$

for some $U_{i,st}^{(l)} \in \mathbb{C}$. The operator $W \in M_N$ given in terms of its matrix entries by the formula

$$W_{i,(l,s,t)} = \sqrt{m_l}\, U_{i,st}^{(l)}$$

for $1 \le i \le N$, $1 \le l \le k$ and $1 \le s,t \le n_l$ is unitary. Indeed, W can be considered as the matrix which takes the orthonormal basis $\{\frac{1}{\sqrt{m_l}} e_{st}^{(l)}\}$ of $\mathcal{A}$ into the orthonormal basis $\{U_i\}$. Hence we have

$$\sum_{i=1}^{N} \overline{W_{i,(l,s,t)}} W_{i,(l',s',t')} = \delta_{ll'}\delta_{ss'}\delta_{tt'}.$$

and therefore

$$\sum_{i=1}^{N} \overline{U_{i,st}^{(l)}} U_{i,s't'}^{(l')} = \delta_{ll'}\delta_{ss'}\delta_{tt'} \frac{1}{m_l}. \tag{5}$$

Let T be a partial isometry with $T^*T = e_{s_1s_1}^{(l_1)} \otimes f_{t_1t_1}^{(l_1)}$ and $TT^* = e_{s_2s_2}^{(l_2)} \otimes f_{t_2t_2}^{(l_2)}$. Then we obtain

$$\begin{aligned} F(T) &= \sum_{i=1}^{N} U_i^* T U_i = \sum_{i=1}^{N} \sum_{p=1}^{n_{l_2}} \sum_{q=1}^{n_{l_1}} \overline{U_{i,s_2p}^{(l_2)}} U_{i,s_1q}^{(l_1)} e_{ps_2}^{(l_2)} T e_{s_1q}^{(l_1)} \\ &= \delta_{l_1l_2}\delta_{s_1s_2}\delta_{pq} \sum_{p=1}^{n_{l_1}} \frac{1}{m_{l_1}} e_{ps_1}^{(l_1)} T e_{s_1p}^{(l_1)} \end{aligned}$$

by (5) so that F maps the off-diagonal part to 0, that is, if $l_1 \neq l_2$ then $F(T) = 0$.

Now let $T = e_{s_2s_1}^{(l)} \otimes f_{t_2t_1}^{(l)}$. Then we obtain

$$\begin{aligned} F(T) &= \delta_{s_1s_2} \sum_{p=1}^{n_l} \frac{1}{m_l} e_{pp}^{(l)} \otimes f_{t_2t_1}^{(l)} = \delta_{s_1s_2} \frac{1}{m_l} I_{n_l} \otimes f_{t_2t_1} \\ &= \frac{n_l}{m_l} \mathrm{Tr}_{n_l}(T) \end{aligned}$$

which shows the first assertion. □

The commutant of M-subalgebras and F-subalgebras are again M-subalgebras and F-subalgebras, and in both types it is possible to choose an orthogonal basis consisting of unitaries, only. Thus by an application of the previous proposition, for such a subalgebra $\mathcal{A}$, the trace-preserving conditional expectation is the convex combination of automorphisms:

$$X \mapsto \frac{1}{\dim \mathcal{A}'} \sum_{i=1}^{m} U_i'^* X U_i' \qquad (X \in M_n),$$

where $\{U_i'\}$ is an orthogonal basis of $\mathcal{A}'$ consisting of unitaries. Bases consisting of unitaries are important also in quantum state teleportation [13].

Theorem 1 *Let $\{U_i : 1 \le i \le m\}$ be an orthonormal system in M_n. Then the linear mapping*

$$\alpha(A) = \sum_{i=1} \mu_i U_i^* A U_i$$

is completely positive if and only if $\mu_i \ge 0$ for every $1 \le i \le m$.

Proof: If $\mu_i \ge 0$ for every $1 \le i \le m$, it is clear that α is completely positive. To prove the converse, we first show that

$$\sum_{i,j} W^* E_{ij} W \otimes E_{ij}$$

is a projection if $\operatorname{Tr} WW^* = 1$. This is obviously self-adjoint and we can compute that it is idempotent:

$$\begin{aligned} &\Big(\sum_{i,j} W^* E_{ij} W \otimes E_{ij}\Big)\Big(\sum_{k,l} W^* E_{kl} W \otimes E_{kl}\Big) \\ =& \sum_{i,j,l} W^* E_{ij} W W^* E_{jl} W \otimes E_{il} \\ =& \operatorname{Tr} WW^* \left(\sum_{i,l} W^* E_{il} W \otimes E_{il}\right). \end{aligned}$$

It follows that

$$P_k := \sum_{i,j} U_k^* E_{ij} U_k \otimes E_{ij}$$

is a projection for every $1 \le k \le m$. To show that they are pairwise orthogonal, we compute the trace of $P_k P_l$:

$$\begin{aligned} \operatorname{Tr} P_k P_l &= \operatorname{Tr} \sum_{i,j,u,v} U_k^* E_{ij} U_k U_l^* E_{uv} U_l \otimes E_{ij} E_{uv} \\ &= \sum_{i,j} \operatorname{Tr} U_k^* E_{ij} U_k U_l^* E_{ji} U_l = \sum_{i,j} \operatorname{Tr} E_{ij} U_k U_l^* E_{ji} U_l U_k^*. \end{aligned}$$

Due to the Lemma 1 below this equals $\operatorname{Tr} U_k U_l^* \operatorname{Tr} U_l U_k^* = 0$ when $k \ne l$.

The complete positivity implies that

$$\sum_{i,j} \Big(\sum_k \mu_k U_k^* E_{ij} U_k \otimes E_{ij}\Big) = \sum_k \mu_k \Big(\sum_{i,j} U_k^* E_{ij} U_k \otimes E_{ij}\Big) = \sum_k \mu_k P_k$$

is positive, therefore $\mu_k \ge 0$. □

Lemma 1

$$\sum_{i,j} \mathrm{Tr}\, E_{ij} X E_{ji} Y = (\mathrm{Tr}\, X)(\mathrm{Tr}\, Y).$$

Proof: Since both sides are bilinear in the variables X and Y, it is enough to check the case $X = E_{ab}$ and $Y = E_{cd}$. Simple computation gives that left-hand-side is $\delta_{ab}\delta_{cd}$. A physicist might make a different proof of the lemma:

$$\sum_{i,j} \mathrm{Tr}\, E_{ij} X E_{ji} Y = \sum_{i,j} \mathrm{Tr}\, |e_i\rangle\langle e_j|X|e_j\rangle\langle e_i|Y = \sum_{i,j} \langle e_j|X|e_j\rangle\langle e_i|Y|e_i\rangle$$

and the right-hand-side is $(\mathrm{Tr}\, X)(\mathrm{Tr}\, Y)$. □We also need the next lemma; the proof can be found in [13].

Lemma 2 *Let $V_1, V_2, \dots, V_{n^2}$ be matrices in M_n. Then the following conditions are equivalent:*

1. $\mathrm{Tr}\, V_i^* V_j = \delta_{ij} \qquad (1 \le i, j \le n^2)$,
2. $\sum_{i=1}^{n^2} V_i^* A V_i = (\mathrm{Tr}\, A) I$ *for every* $A \in M_n$.

The next result includes important particular cases which are formulated afterwards.

Theorem 2 *Let $\mathcal{A}_1, \dots, \mathcal{A}_r$ be pairwise complementary subalgebras of M_n such that their commutants $\mathcal{A}_1', \dots, \mathcal{A}_r'$ are pairwise complementary as well. Then the trace-preserving conditional expectations $E_j : M_n \to \mathcal{A}_j$ can be expressed by the orthonormal bases $U_{j1}, U_{j2}, \dots, U_{jn(j)} \in \mathcal{A}_j'$, where $U_{j1} = \frac{1}{\sqrt{n}} I$, via the formula*

$$E_j(A) = \frac{n}{\dim \mathcal{A}_j'} \sum_{i=1}^{n(j)} U_{ji}^* A U_{ji} \tag{6}$$

and the generalized Pauli channel (1) is completely positive if and only if

$$1 + \frac{n^2 \lambda_i}{\dim \mathcal{A}_i'} \ge \sum_j \lambda_j$$

for every $1 \le i \le r$ and

$$\sum_j \lambda_j \Big(\frac{n^2}{\dim \mathcal{A}_j'} - 1 \Big) \ge -1.$$

To prove this theorem we prepare the following proposition.

Proposition 2 *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be complementary subalgebras of M_n. Then $\mathcal{A}_1'$ and A_2' are complementary if and only if $\mathcal{A}_1 \mathcal{A}_2$ linearly spans M_n. Moreover, in this case the trace-preserving conditional expectation $E_1 : M_n \to \mathcal{A}_1'$ can be expressed as*

$$E_1(X) = \frac{n}{\dim \mathcal{A}_1} \sum_i U_i^* X U_i \quad (X \in M_n),$$

where $\{U_i\}$ is an orthonormal basis of $\mathcal{A}_1$.

Proof: Assume $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are complementary. Let $\{U_i'\}$ and $\{V_j'\}$ be orthonormal bases of $\mathcal{A}_1'$ and $\mathcal{A}_2'$, respectively, which consist of scalar multiple of their matrix units. Then the trace-preserving conditional expectations onto $\mathcal{A}_1$ and $\mathcal{A}_2$ are given by the linear combinations of $U_i'^*(\,\cdot\,)U_i'$ and $V_j'^*(\,\cdot\,)V_j'$, respectively, thanks to Proposition 1.

Since $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are complementary subalgebras, $\{V_j'U_i'\}_{i,j}$ is an orthogonal system. Moreover the trace is written by the linear combination of $U_i'^*V_j'^*(\,\cdot\,)V_j'U_i'$, because $\mathcal{A}_1$ and $\mathcal{A}_2$ are complementary subalgebras if and only if the composition of two conditional expectations equals to $\frac{1}{n}\mathrm{Tr}$. But this shows that $\{V_j'U_i'\}_{i,j}$ linearly spans the whole M_n thanks to Lemma 2.

Conversely assume $\mathcal{A}_1\mathcal{A}_2$ linearly spans the whole space M_n. Since $\mathcal{A}_1$ is a subalgebra of M_n, $\mathcal{A}_1$ can be written as

$$\mathcal{A}_1 = \bigotimes_{l=1}^{k} M_{n_l} \otimes I_{m_l}.$$

Let Q be a minimal central projection in $\mathcal{A}_2$ and let $\{U_i^{(s)}\}$ and $\{V_j\}$ are orthonormal bases of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, with the assumption $U_i^{(s)} \in M_{n_s} \otimes I_{m_s}$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are complementary and $\mathrm{span}\{\mathcal{A}_1\mathcal{A}_2\} = M_n$, $\{\sqrt{n}U_i^{(s)}V_j\}$ is an orthonormal basis of M_n. Therefore by Lemma 2 and Proposition 1, we have

$$\sum_{s,i,j} U_i^{(s)*}V_j^*QV_jU_i^{(s)} = \frac{1}{n}\mathrm{Tr}\,Q \cdot I$$

and

$$\sum_j V_j^*QV_j = cQ$$

for some $c > 0$. These equations imply, for $1 \le s \le k$,

$$\sum_i U_i^{(s)*}QU_i^{(s)} = \frac{\mathrm{Tr}\,Q}{cn}P_s,$$

where P_s is a central projection $I_{n_s} \otimes I_{m_s}$. Now we take the trace to the above equation. Then we have

$$\mathrm{Tr}\left(\sum_i^{n_s^2} U_i^{(s)*}QU_i^{(s)}\right) = \sum_i^{n_s^2} \frac{1}{n}\mathrm{Tr}\left(U_i^{(s)}U_i^{(s)*}\right)\mathrm{Tr}\,Q = \frac{n_s^2}{n}\mathrm{Tr}\,Q$$

and

$$\frac{\mathrm{Tr}\,Q}{cn}\mathrm{Tr}\,P_s = \frac{\mathrm{Tr}\,Q}{cn}n_sm_s$$

so that

$$\frac{n_s}{m_s} = \frac{1}{c}.$$

Hence n_s/m_s is equal to $1/c = \frac{\dim \mathcal{A}_1}{n}$ for all $1 \le s \le k$ and so

$$E_1 = \frac{n}{\dim \mathcal{A}_1} \sum_{s,i} U_i^{(s)*}(\,\cdot\,)U_i^{(s)}$$

is the trace-preserving conditional expectation onto $\mathcal{A}_1'$ by Proposition 1. Similarly,

$$E_2 = \frac{n}{\dim \mathcal{A}_2} \sum_j V_j^*(\,\cdot\,)V_j.$$

is the trace-preserving conditional expectation onto $\mathcal{A}_2'$. Since

$$\sum_{s,i,j} U_i^{(s)*} V_j^*(\,\cdot\,)V_j U_i^{(s)} = \sum_{s,i,j} V_j^* U_i^{(s)*}(\,\cdot\,)U_i^{(s)} V_j$$

is the normalized trace on M_n by Lemma 2, we obtain $\frac{n^2}{\dim \mathcal{A}_1 \dim \mathcal{A}_2} = 1$ and so the composition $E_1 \circ E_2$ equals to $\frac{1}{n}\mathrm{Tr}\,$. □

Proof of Theorem 2. The first assertion is already proven in the above proposition. Due to the Lemma 2, we have

$$(\mathrm{Tr}\, A)I = \frac{A}{n} + \sum_{j=0}^{n} \sum_{k=2}^{n} U_{jk}^* A U_{jk} + \sum_{t=1}^{\ell} W_t^* A W_t,$$

where orthonormal system W_t extend the orthonormal system U_{jk} to a complete system in the linear space M_n. In formula (1) we use this expression for $(\mathrm{Tr}\, A)I$ and the assumed decomposition of the conditional expectations. So in the expansion of $\alpha(A)$ the coefficient of $U_{jk}^* A U_{jk}$ is

$$\frac{1}{n}\Big(1 - \sum_i \lambda_i\Big) + \frac{n\lambda_j}{\dim \mathcal{A}_j'}$$

and the coefficient of $\frac{A}{n} = \left(\frac{1}{\sqrt{n}} I\right) A \left(\frac{1}{\sqrt{n}} I\right)$ is

$$\frac{1}{n}\Big(1 - \sum_j \lambda_j\Big) + \sum_j \frac{n\lambda_j}{\dim \mathcal{A}_j'}.$$

Theorem 1 tells us that completely positivity holds if and only if both are positive. □

Corollary 1 *Assume that M_n contains pairwise complementary M-subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_r$. Then the generalized Pauli channel is completely positive if and only if*

$$1 + n\lambda_i \ge \sum_j \lambda_j \ge -\frac{1}{n-1}$$

for every $1 \le i \le r$.

This result appeared also in [5].

3 Bipartite channels

In this section we consider subalgebras of $M_n \otimes M_n$. A subalgebra isomorphic to M_n will be called F-subalgebra. An M-subalgebra is a maximal Abelian subalgebra. Both kinds of subalgebras are subspaces of dimension n^2.

Theorem 3 *Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are F- or M-subalgebras of $M_n \otimes M_n$. If they are complementary, then the commutants $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are complementary as well.*

Proof: Since both kinds of subalgebras are subspaces of dimension n^2, the dimension of $\mathcal{A}_1\mathcal{A}_2$ is n^4 so that $\mathcal{A}_1\mathcal{A}_2 = M_n \otimes M_n$. Therefore the commutants $\mathcal{A}_1'$ and $\mathcal{A}_2'$ are complementary by Proposition 2. □

Theorem 4 *Assume that $M_n \otimes M_n$ is decomposed to pairwise complementary F- and M-subalgebras $\mathcal{A}_i$ $(1 \le i \le n^2+1)$. The trace-preserving conditional expectation of $M_n \otimes M_n$ onto $\mathcal{A}_i$ is denoted by E_i. The linear trace-preserving mapping acting as*

$$\alpha(B) = \sum_{i=1}^{n^2+1} \lambda_i E_i(B) \qquad (B \in M_n \otimes M_n,\ \mathrm{Tr}\, B = 0)$$

is completely positive if and only if

$$1 + n^2\lambda_i \ge \sum_j \lambda_j \ge -\frac{1}{n^2-1}$$

for every $1 \le i \le n^2+1$.

Proof: Theorem 3 allows to use Theorem 2 and the result follows. □

The theorem can be applied in Example 2. Note that decompositions of $M_2 \otimes M_2$ into F- and M-subalgebras are discussed in [9], while decomposition of $M_n \otimes M_n$ into F-subalgebras is constructed in [6] if $n = p^k$ with a prime number $p > 2$.

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