

A SIMPLE PROOF OF THE STRONG SUBADDITIVITY INEQUALITY

MICHAEL A. NIELSEN

*School of Physical Sciences, The University of Queensland
Brisbane, Queensland 4072, Australia*

DÉNES PETZ

*Department of Mathematical Analysis, Budapest University of Technology and Economics
H-1521 Budapest XI, Hungary*

Received March 3, 2005

Revised June 8, 2005

Arguably the deepest fact known about the von Neumann entropy, the strong subadditivity inequality is a potent hammer in the quantum information theorist's toolkit. This short tutorial describes a simple proof of strong subadditivity due to Petz [Rep. on Math. Phys. **23** (1), 57–65 (1986)]. It assumes only knowledge of elementary linear algebra and quantum mechanics.

Keywords: entropy, strong subadditivity, monotonicity, relative entropy

Communicated by: H-K Lo & B Terhal

1 Introduction

The von Neumann entropy of a density matrix ρ is defined by $S(\rho) \equiv -\text{tr}(\rho \ln \rho)$. Suppose ρ_{ABC} is a density matrix for a system with three components, A, B and C . The strong subadditivity inequality states that

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}), \quad (1)$$

where notations like ρ_B denote the appropriate reduced density matrices.

The strong subadditivity inequality appears quite mysterious at first sight. Some intuition is gained by reexpressing strong subadditivity in terms of the *conditional entropy* $S(A|B) \equiv S(\rho_{AB}) - S(\rho_B)$. Classically, when the von Neumann entropy is replaced by the Shannon entropy function, the conditional entropy has an interpretation as the average uncertainty about the state of A , given knowledge of the state of B [2]. Although this interpretation is more problematic in the quantum case — for one thing, the quantum conditional entropy can be negative! — it can still be useful for developing intuition and suggesting results. In particular, we see that strong subadditivity may be recast in the equivalent form

$$S(A|BC) \leq S(A|B). \quad (2)$$

That is, strong subadditivity expresses the intuition that our uncertainty about A when B and C are known is not more than when only B is known. This intuition is perhaps best

viewed as a mnemonic, due to the problematic interpretation of the conditional entropy, but may nonetheless be helpful.

Strong subadditivity has many applications in quantum information theory (see, e.g., [8, 9]). Our purpose here is not to discuss these applications, but rather to provide an expository account of a simple proof of strong subadditivity due to Petz [10] (see also [9]). In so doing we hope to help publicise this proof to a wider audience. The reader looking for a more comprehensive account in a similar vein to the present paper should consult [11].

Our proof strategy is to show that strong subadditivity is implied by a related result, the *monotonicity of the relative entropy*, and then to prove this monotonicity result. The *relative entropy* between density matrices ρ and σ is defined as:

$$S(\rho\|\sigma) \equiv \text{tr}(\rho \ln \rho - \rho \ln \sigma). \quad (3)$$

Roughly speaking, the relative entropy is a measure of the distance between ρ and σ . In particular, it can be shown that $S(\rho\|\sigma) \geq 0$, with equality if and only if $\rho = \sigma$. Be warned, however, that it is not symmetric in ρ and σ , and $S(\rho\|\sigma)$ diverges unless the support of ρ is contained within the support of σ . Further background on the relative entropy may be found in [8, 9]. The monotonicity of the relative entropy is the property that discarding part of a composite system AB can only decrease the relative entropy between two density matrices ρ_{AB} and σ_{AB} :

$$S(\rho_A\|\sigma_A) \leq S(\rho_{AB}\|\sigma_{AB}). \quad (4)$$

To see that monotonicity of the relative entropy implies strong subadditivity, we reexpress strong subadditivity in terms of the relative entropy, using the identity:

$$S(B|A) = \ln d_B - S\left(\rho_{AB}\|\rho_A \otimes \frac{I_B}{d_b}\right). \quad (5)$$

Proving this identity is a straightforward application of the definitions. Using this identity we may recast the conditional entropic form of strong subadditivity, Eq. (2), as an equivalent inequality between relative entropies:

$$S\left(\rho_{AB}\|\frac{I_A}{d_A} \otimes \rho_B\right) \leq S\left(\rho_{ABC}\|\frac{I_A}{d_A} \otimes \rho_{BC}\right) \quad (6)$$

This inequality obviously follows from the monotonicity of the relative entropy, and thus strong subadditivity also follows from the monotonicity of the relative entropy.

Strong subadditivity and the monotonicity of the relative entropy have an interesting and lengthy history, and we will merely note a few highlights. The reader interested in a more thorough account should see, e.g., the discussion in [12, 15] and the end notes to Chapter 11 of [8].

The original proof of strong subadditivity was by Lieb and Ruskai [5], based on the beautiful concavity results of Lieb [4]. Ruskai [13] has recently given an elegant exposition along the lines of this original proof. Monotonicity of the relative entropy was actually proved after strong subadditivity, by Lindblad [6] (see also [14]). As already noted, our approach to strong subadditivity and monotonicity is due to Petz [10]. Independently of Petz, Narnhofer and Thirring [7] developed a related approach, based on similar broad ideas, but differing substantially in the details.

2 Background on operator convex functions

The only background required for our proofs is a few simple facts from the theory of operator convex functions. The reader is referred to Chapter 5 of [1] for an introduction to this beautiful theory.

Suppose $I \subseteq \mathbf{R}$ is an interval in the real line, and $f : I \rightarrow \mathbf{R}$ is a real-valued function on I . We will define a corresponding map $f : M_n \rightarrow M_n$, where M_n is the space of $n \times n$ Hermitian matrices whose spectra lie in I . To define such a map, suppose D is an $n \times n$ diagonal matrix with real diagonal entries $d_1, \dots, d_n \in I$. We define $f(D)$ to be the $n \times n$ diagonal matrix with diagonal entries $f(d_1), \dots, f(d_n)$. Generalizing this definition, if X is any element of M_n then we can write $X = UDU^\dagger$ for some unitary U and diagonal matrix D . We define the induced map $f : M_n \rightarrow M_n$ by $f(UDU^\dagger) \equiv Uf(D)U^\dagger$. More informally, we work in a basis in which X is diagonal, and apply f to each of the diagonal entries. In cases where X can be decomposed in many different ways as $X = UDU^\dagger$ it is an easy exercise to show that $f(X)$ does not depend upon the decomposition chosen.

To define operator convexity, we first introduce a partial order on Hermitian matrices. Given Hermitian matrices $X, Y \in M_n$ we define $X \leq Y$ if $Y - X$ is a positive matrix. We say a function $f : I \rightarrow \mathbf{R}$ is *operator convex* if for all n , for all $X, Y \in M_n$, and for all $p \in [0, 1]$ we have $f(pX + (1 - p)Y) \leq pf(X) + (1 - p)f(Y)$.

Our later proofs use two simple lemmas about operator convexity, which we state at the end of this paragraph. We defer proofs of these lemmas until after the proof of the monotonicity of relative entropy, so as to not obscure the simplicity of the ideas used in that proof.

Lemma 1: The function $f(x) = -\ln(x)$ is operator convex.

Lemma 2: If f is operator convex, and $U : V \rightarrow W$ is an isometry (where $\dim(V) \leq \dim(W)$), then $f(U^\dagger XU) \leq U^\dagger f(X)U$ for all^a X .

3 Proof of the monotonicity of the relative entropy

To appreciate the ideas used in proving monotonicity, it is helpful to look at the proof of the analogous classical result. This states that for probability distributions r_{jk} and s_{jk} in two variables we have $\sum_j r_j (\ln r_j - \ln s_j) \leq \sum_{jk} r_{jk} (\ln r_{jk} - \ln s_{jk})$, where $r_j \equiv \sum_k r_{jk}$ and $s_j \equiv \sum_k s_{jk}$ are the marginal probability distributions. This is easily seen to be equivalent to the inequality $\sum_{jk} r_{jk} \ln \frac{r_j s_{jk}}{r_{jk} s_j} \leq 0$, which may be proved by applying the calculus result $\ln x \leq x - 1$ to the left-hand side, and showing that the resulting expression vanishes.

The difficulty in the quantum case is that the density matrices involved may not commute, and this prevents them from being combined in a single logarithm. To overcome this difficulty we reexpress the relative entropy $S(\rho||\sigma)$ using a linear map on matrices known as the *relative modular operator*. In defining this operator we will assume that ρ and σ are invertible; as a result, our proof of monotonicity of the relative entropy and of strong subadditivity only applies directly for invertible density matrices. The general results follow via a straightforward continuity argument, which we omit.

To define the relative modular operator, we fix ρ and σ and define linear maps on matrices \mathcal{L} and \mathcal{R} by $\mathcal{L}(X) \equiv \sigma X$ and $\mathcal{R}(X) \equiv X\rho^{-1}$, i.e., left multiplication by σ , and right

^aWe will follow the physicists' convention in often expecting the reader to work out from context the domain and range of mappings. Thus, in this example X is a Hermitian matrix on the space W , and with a spectrum lying within I , the domain of f .

multiplication by ρ^{-1} . The relative modular operator is defined to be the product of these linear maps under composition, $\Delta \equiv \mathcal{L}\mathcal{R}$. Note that \mathcal{L} and \mathcal{R} commute, so we could equally well have written $\Delta = \mathcal{R}\mathcal{L}$.

Our next step is to define a function \ln on linear maps on matrices, i.e., to define $\ln(\mathcal{E})$, where \mathcal{E} is a linear map on matrices that is strictly positive with respect to the Hilbert-Schmidt inner product $\langle X, Y \rangle \equiv \text{tr}(X^\dagger Y)$. To do this we follow the same approach as described earlier in the section on operator convex functions, expanding \mathcal{E} in a diagonal basis as $\mathcal{E} = \sum_j \lambda_j \mathcal{E}_j$, and defining $\ln(\mathcal{E}) \equiv \sum_j \ln(\lambda_j) \mathcal{E}_j$.

With this definition, $\ln(\mathcal{L}), \ln(\mathcal{R}),$ and $\ln(\Delta)$ are all defined, i.e., $\mathcal{L}, \mathcal{R},$ and Δ are all strictly positive with respect to the Hilbert-Schmidt inner product. To see that \mathcal{L} is strictly positive observe that $\langle X, \mathcal{L}(X) \rangle = \text{tr}(X^\dagger \sigma X) > 0$ for all non-zero X . The proof that \mathcal{R} is strictly positive follows similar lines. Finally, since Δ is a product of strictly positive and commuting linear maps on matrices, it follows that Δ is strictly positive.

A little thought shows that $\ln(\mathcal{L})(X) = \ln(\sigma)X$ and $\ln(\mathcal{R})(X) = -X \ln(\rho)$. Whatsmore, since \mathcal{L} and \mathcal{R} commute, we obtain the beautiful relationship $\ln(\Delta) = \ln(\mathcal{L}) + \ln(\mathcal{R})$. Some algebra shows that

$$S(\rho \parallel \sigma) = \langle \rho^{1/2}, -\ln(\Delta)(\rho^{1/2}) \rangle. \quad (7)$$

That is, the relative modular operator has enabled us to combine the logarithms in the definition of the relative entropy into a single logarithm, which greatly simplifies analysis. Using Eq. (7) we may rewrite the monotonicity of the relative entropy in the equivalent form

$$\langle \rho_A^{1/2}, -\ln(\Delta_A)(\rho_A^{1/2}) \rangle \leq \langle \rho_{AB}^{1/2}, -\ln(\Delta_{AB})(\rho_{AB}^{1/2}) \rangle, \quad (8)$$

where the first inner product $\langle \cdot, \cdot \rangle$ is on the space $M(A)$ of matrices acting on A , the second inner product is on the space $M(AB)$ of matrices acting on AB , and $\Delta_A(X) \equiv \sigma_A X \rho_A^{-1}$, $\Delta_{AB}(X) \equiv \sigma_{AB} X \rho_{AB}^{-1}$ are the natural relative modular operators on systems A and AB , respectively.

The final step in the proof is to find a linear map on matrices $\mathcal{U} : M(A) \rightarrow M(AB)$ such that: (1) $\mathcal{U}^\dagger \Delta_{AB} \mathcal{U} = \Delta_A$; (2) $\mathcal{U}(\rho_A^{1/2}) = \rho_{AB}^{1/2}$; and (3) \mathcal{U} is an isometry from $M(A)$ to $M(AB)$. It is not obvious such a \mathcal{U} ought to exist. We explicitly construct \mathcal{U} below, but for now we assume \mathcal{U} exists, and investigate the consequences. Using Eq. (8) we rewrite the monotonicity of the relative entropy as:

$$\begin{aligned} & \langle \rho_A^{1/2}, -\ln(\mathcal{U}^\dagger \Delta_{AB} \mathcal{U})(\rho_A^{1/2}) \rangle \\ & \leq \langle \rho_{AB}^{1/2}, -\ln(\Delta_{AB})(\rho_{AB}^{1/2}) \rangle. \end{aligned} \quad (9)$$

But by Lemma 1 and Lemma 2 on the properties of operator convex functions we have $-\ln(\mathcal{U}^\dagger \Delta_{AB} \mathcal{U}) \leq -\mathcal{U}^\dagger \ln(\Delta_{AB}) \mathcal{U}$, and so

$$\begin{aligned} & \langle \rho_A^{1/2}, -\ln(\mathcal{U}^\dagger \Delta_{AB} \mathcal{U})(\rho_A^{1/2}) \rangle \\ & \leq \langle \rho_A^{1/2}, -\mathcal{U}^\dagger \ln(\Delta_{AB}) \mathcal{U}(\rho_A^{1/2}) \rangle \end{aligned} \quad (10)$$

$$= \langle \mathcal{U}(\rho_A^{1/2}), -\ln(\Delta_{AB}) \mathcal{U}(\rho_A^{1/2}) \rangle \quad (11)$$

$$= \langle \rho_{AB}^{1/2}, -\ln(\Delta_{AB}) \rho_{AB}^{1/2} \rangle, \quad (12)$$

which completes the proof of monotonicity, provided we can find a \mathcal{U} satisfying properties (1)-(3). Based on property (2) a plausible candidate is $\mathcal{U}(X) \equiv (X\rho_A^{-1/2} \otimes I_B)\rho_{AB}^{1/2}$. With this definition, it is not difficult to check that $\mathcal{U}^\dagger(Y) = \text{tr}_B(Y\rho_{AB}^{1/2}(\rho_A^{-1/2} \otimes I_B))$ is the corresponding adjoint operation, i.e., satisfies $\langle \mathcal{U}^\dagger(Y), X \rangle = \langle Y, \mathcal{U}(X) \rangle$ for all $X \in M(A)$ and $Y \in M(AB)$. Direct calculation now shows that $\mathcal{U}^\dagger \Delta_{AB} \mathcal{U} = \Delta_A$ and $\mathcal{U}^\dagger \mathcal{U} = \mathcal{I}_A$, which completes the list of desired properties, and the proof of monotonicity.

This proof of monotonicity highlights the role of the operator convexity of $f(x) = -\ln(x)$. If f is any operator convex function and we define an f -relative entropy by $S_f(\rho||\sigma) \equiv \langle \rho^{1/2}, f(\Delta)(\rho^{1/2}) \rangle$, the same argument shows that we obtain an analogous monotonicity property.

4 Proofs of the operator convexity lemmas

To prove Lemma 1, we begin with a proof that $f(x) = 1/x$ is operator convex on $(0, \infty)$. A key fact used in the proof is that if $X \leq Y$, then $ZXZ^\dagger \leq ZYZ^\dagger$ for any choice of Z , i.e., conjugation preserves matrix inequalities. The proof of this useful fact is a good exercise in applying the definition of \leq .

To prove the operator convexity of $f(x) = 1/x$, let $X \leq Y$ be strictly positive Hermitian matrices. We begin with the special case $X = I$, where the goal is to prove $(pI + (1-p)Y)^{-1} \leq pI + (1-p)Y^{-1}$. Since I and Y commute, this result follows from the ordinary convexity of the real function $1/x$.

To obtain the general operator convexity from the special case $X = I$, make the replacement $Y \rightarrow X^{-1/2}YX^{-1/2}$, which gives

$$\begin{aligned} & \left(pI + (1-p)X^{-1/2}YX^{-1/2} \right)^{-1} \\ & \leq pI + (1-p)(X^{-1/2}YX^{-1/2})^{-1}. \end{aligned} \tag{13}$$

Conjugating by $X^{-1/2}$ and doing a little algebra gives the desired inequality, and concludes the proof that $f(x) = 1/x$ is operator convex.

The operator convexity of $f(x) = -\ln(x)$ is now established using the integral representation $-\ln(x) = \int_0^\infty dt \left(\frac{1}{x+t} - \frac{1}{1+t} \right)$, from which it follows that for a strictly positive matrix X we have

$$-\ln(X) = \int_0^\infty dt ((X+tI)^{-1} - (I+tI)^{-1}). \tag{14}$$

Our goal is to show $-\ln(pX + (1-p)Y) \leq -p\ln(X) - (1-p)\ln(Y)$. From Eq. (14), this follows if we can prove $(pX + (1-p)Y + tI)^{-1} \leq p(X+tI)^{-1} + (1-p)(Y+tI)^{-1}$. Rewriting the left-hand side as $[p(X+tI) + (1-p)(Y+tI)]^{-1}$ and applying the operator convexity of $1/x$ gives the desired result, completing the proof of Lemma 1.

Moving to Lemma 2, note first a simple related result, namely, that when U maps the space V onto W , then directly from the definition of $f(X)$ we obtain $f(U^\dagger XU) = U^\dagger f(X)U$. This holds true regardless of whether f is operator convex or not. Lemma 2 requires a stronger hypothesis (the operator convexity of f), and gives rise to an inequality instead of an equality, but has the advantage that it holds when the range W' of U is a strict subset of W .

Readers familiar with the operator Jensen inequality (see, e.g., [3]) may recognize Lemma 2 as a variant of this result.

To prove Lemma 2, let P be the projector onto W' , and $Q \equiv I - P$ the projector onto the orthocomplement. As three separate vector spaces are involved, it is useful to introduce the notations f_V, f_W and $f_{W'}$ to denote the different ways f can act, e.g., f_V takes as input a matrix acting on V , and produces as output a matrix acting on V , while f_W takes as input a matrix acting on W , and produces as output a matrix acting on W .

Note that $PU = U$, since P projects onto the range of U . As a result we have $f_V(U^\dagger XU) = f_V(U^\dagger P(PXP)PU)$. Note that PU is an isometry from V onto W' , and since PXP may be regarded as a matrix acting on W' , it follows that $f_V(U^\dagger P(PXP)PU) = U^\dagger P f_{W'}(PXP)PU$. Summing up, we have shown that $f_V(U^\dagger XU) = U^\dagger P f_{W'}(PXP)PU$. To conclude the proof it will suffice to show that $f_{W'}(PXP) \leq P f_W(X)P$. Proving this inequality now becomes our objective.

We observe that

$$f_{W'}(PXP) = P f_W(PXP)P = P f_W(PXP + QXQ)P, \quad (15)$$

since $f_W(PXP + QXQ) = f_W(PXP) + f_W(QXQ)$ and $P f_W(QXQ)P = 0$. Defining a unitary $S \equiv P - Q$ on W , and recalling that $P + Q = I$, we have

$$\frac{X + SXS^\dagger}{2} = \frac{(P + Q)X(P + Q) + (P - Q)X(P - Q)}{2} = PXP + QXQ, \quad (16)$$

for arbitrary X . Applying the operator convexity of f gives $f_W(PXP + QXQ) \leq (f_W(X) + f_W(SXS^\dagger))/2$, and since $f_W(SXS^\dagger) = S f_W(X) S^\dagger$ we obtain $f_W(PXP + QXQ) \leq (f_W(X) + S f_W(X) S^\dagger)/2 = P f_W(X)P + Q f_W(X)Q$. Conjugating by P we obtain $P f_W(PXP + QXQ)P \leq P f_W(X)P$. Combining this inequality with Eq. (15) gives $f_{W'}(PXP) \leq P f_W(X)P$, which, as noted above, is sufficient to establish Lemma 2.

Acknowledgements

Thanks to Elliott Lieb and Beth Ruskai for comments on the history of these inequalities, to Heide Narnhofer for sending us a reprint of [7], and to Runyao Duan for pointing out some typographical errors in an early version of this paper.

References

1. R. Bhatia. *Matrix analysis*. Springer-Verlag, New York, 1997.
2. T. M. Cover and J. A. Thomas. *Elements of Information Theory*. John Wiley and Sons, New York, 1991.
3. F. Hansen and G. K. Pedersen. Jensen's operator inequality, *Bull. London Math. Soc.*, 35:553-564, 2003.
4. E. H. Lieb. Convex trace functions and the Wigner-Yanase-Dyson conjecture. *Advances in Mathematics*, 11:267-288, 1973.
5. E. H. Lieb and M. B. Ruskai. Proof of the strong subadditivity of quantum mechanical entropy. *J. Math. Phys.*, 14:1938-1941, 1973.

6. G. Lindblad. Completely positive maps and entropy inequalities. *Commun. Math. Phys.*, 40:147–151, 1975.
7. H. Narnhofer and W. Thirring. From relative entropy to entropy. *Fizika*, 17(3):257–265, 1985.
8. M. A. Nielsen and I. L. Chuang. *Quantum computation and quantum information*. Cambridge University Press, Cambridge, 2000.
9. M. Ohya and D. Petz. *Quantum entropy and its use*. Springer-Verlag, Berlin, 2nd edition, 2004.
10. D. Petz. Quasi-entropies for finite quantum systems. *Rep. Math. Phys.*, 23(1):57–65, 1986.
11. D. Petz. Monotonicity of quantum relative entropy revisited. *Rev. Math. Phys.*, 15(1):79–91, 2003.
12. M. B. Ruskai. Inequalities for quantum entropy: A review with conditions for equality. *J. Math. Phys.*, 43:4358–4375, 2002. arXiv:quant-ph/0205064.
13. M. B. Ruskai. Lieb’s simple proof of concavity of $\text{Tr} a^p k^* b^{(1-p)} k$ and remarks on related inequalities. *arXiv:quant-ph/0404126*, 2004.
14. A. Uhlmann. Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory. *Commun. Math. Phys.*, 54:21–32, 1977.
15. A. Wehrl. General properties of entropy. *Rev. Mod. Phys.*, 50:221, 1978.