Representation of planar graphs by segments

H. de Fraysseix[1], P.O. de Mendez[1] and J. Pach[2]

Abstract. Given any bipartite planar graph G, one can assign vertical and horizontal segments to its vertices so that (a) no two of them have an interior point in common, (b) two segments have a point in common if and only if the corresponding vertices are adjacent in G.

[1]Centre de Mathématiques Sociales, EHESS, 54 Boulevard Raspail, 75006 Paris, France
[2]Mathematical Institute of the Hungarian Academy of Sciences, 1364 Budapest P.O.B. 127, Hungary.
Supported by NSF grant CCR–89–01484.

1. Introduction

Let $\mathcal{C}$ be a collection of compact sets in the plane. The *intersection graph (contact graph)* of $\mathcal{C}$ is defined as a graph on the vertex set $\mathcal{C}$, where two members $C_1, C_2 \in \mathcal{C}$ are connected by an edge if and only if they have a pont in common (they touch each other, respectively). The problem of characterizing the intersection and contact graphs of various classes of geometric objects has a vast literature. Many results of this type have direct applications in VLSI design, in the complexity theory of algorithms and in the theory of (nearly) perfect graphs (cf. [MP], [G], [GJ]). A particularly useful and attractive theorem of Koebe [K] states that any *planar* graph is isomorphic to the contact graph of a suitable collection of disks. However, it is not known whether every planar graph can be realized as the intersection graph of a collection of segments.

The aim of this note is to show that any *bipartite* planar graph can be represented as the contact graph of a set of segments. We say that two segments *cross* each other if they have an interior point in common.

Theorem. *The vertices of any bipartite planar graph can be represented by noncrossing vertical and horizontal segments in the plane so that two segments have a point in common if and only if the corresponding vertices are adjacent.*

A *multigraph* is an undirected graph which may have multiple edges. It is called *2–connected* if it does not fall into two or more components by the deletion of a vertex. A *bipolar orientation* of a multigraph from a vertex s to a vertex t is an orientation of the edges with the following properties:

- (a) there are no oriented cycles,
- (b) s and t are the unique source (point of in–degree 0) and the unique sink (point of out–degree 0), respectively.

It is well known and easy to prove that a multigraph has a bipolar orientation from s to t if and only if it becomes 2–connected by the addition of the edge st ([LEC], [Á], [L]).

Furthermore, it is clear that such an orientation exists if and only if the vertices of the multigraph can be linearly ordered so that s and t are the smallest and largest elements, respectively, and any other vertex has at least one smaller and at least one larger neighbor. In the literature an ordering with these properties is usually called an *st–ordering* ([ET], [E], [T], [TT], [RT], [R], [MR]).

2. Proof of Theorem

Let G be a bipartite planar map of n vertices, colored with black and white. By adding $O(n)$ "dummy" edges and vertices, if necessary, one can assume that every face of G is a quadrilateral. Let b, w, b', w' denote the vertices of the external face listed in clockwise order, and suppose that b is colored black.

For each internal face f of G, connect its two black vertices by an edge within f. The collection of these edges is a multigraph G_b on the set of black vertices. Let G_w be defined similarly on the set of white vertices of G. Observe that every edge of G_b is crossing exactly one edge of G_w, and vice versa. (See Fig. 1.)

It is easy to see that G_b becomes 2-connected by the addition of the edge bb', hence G_b has a bipolar orientation $\overrightarrow{G_b}$ from b to b'. This induces a 'dual' orientation $\overrightarrow{G_w}$ of G_w, as follows. For any internal face $b_1 w_1 b_2 w_2$ of G (in clockwise order), let

$$\overrightarrow{w_1 w_2} \in \overrightarrow{G_w} \Longleftrightarrow \overrightarrow{b_1 b_2} \in \overrightarrow{G_b}.$$

Lemma 1. *$\overrightarrow{G_w}$ is a bipolar orientation of G_w from w to w'.*

Proof. First we show that $\overrightarrow{G_w}$ is acyclic. Assume for the sake of contradiction that there is a minimal cycle in $\overrightarrow{G_w}$ with (say) clockwise orientation. Then all edges of $\overrightarrow{G_b}$ would leave the region enclosed by this cycle, implying that it contains b (the only source of $\overrightarrow{G_b}$) in its interior, a contradiction.

Suppose next that $\overrightarrow{G_w}$ has a source $w_0 \neq w, w'$. Let $\overrightarrow{w_0 w_1}$, $\overrightarrow{w_0 w_2}$, $\ldots, \overrightarrow{w_0 w_k}$ be the edges incident to w_0, listed in clockwise order. (Note that the vertices w_i are not necessarily distinct.) Each $\overrightarrow{w_0 w_i}$ belongs to a face of G, and the other diagonals of these faces form a cycle in $\overrightarrow{G_b}$, which is impossible. Similarly, $\overrightarrow{G_w}$ cannot have a sink different from w and w'.

In order to exclude that w' is the source and w is the sink of $\overrightarrow{G_w}$, consider the internal face f of G sitting on the edge bw. Since b is the cource of $\overrightarrow{G_b}$, the black diagonal of f cannot be oriented towards b. By the definition of $\overrightarrow{G_w}$, this implies that the white diagonal of f is not oriented towards w. Thus w cannot be a sink. $\square$

Let $b_0 = b < b_1 < b_2 < \cdots < b_k = b'$, $w_0 = w < w_1 < w_2 < \cdots < w_e = w'$ be st–orderings of the black and white vertices of G consistent to the bipolar orientations $\overrightarrow{G_b}$ and $\overrightarrow{G_w}$, i.e.,

$$\overrightarrow{b_i b_j} \in \overrightarrow{G_b} \Longrightarrow i < j \quad \text{and} \quad \overrightarrow{w_i w_j} \in \overrightarrow{G_w} \Longrightarrow i < j.$$

Let x and y be real functions on the set of black and white vertices, respectively, such that

$$x(b_0) < x(b_1) < \cdots < x(b_k), \qquad y(w_0) < y(w_1) < \cdots < y(w_l).$$

Assign to any black vertex b_i $(0 \le i \le k)$ a vertical segment V_i with endpoints

$$\left(x(b_i), \min_{b_i w_j \in G} y(w_j)\right) \quad \text{and} \quad \left(x(b_i), \max_{b_i w_j \in G} y(w_j)\right).$$

Similarly, for any white vertex w_j $(0 \le j \le l)$, let H_j denote the horizontal segment whose endpoints are

$$\left(\min_{b_i w_j \in G} x(b_i), y(w_j)\right) \quad \text{and} \quad \left(\max_{b_i w_j \in G} x(b_i), y(w_j)\right).$$

Note that the combinatorial structure of the arrangements of the segments V_i and H_j depends only on the st–orderings of the black and white vertices but not on the actual values of the functions x and y. It is also clear by the above definitions that, if b_i and w_j are adjacent, then $(x(b_i), y(w_j)) \in V_i \cap H_j$.

Lemma 2. *No two segments V_i and H_j have an interior point in common. Furthermore, the segments V_i $(0 < i < k), H_j$ $(0 < j < l)$ subdivide the rectangle bounded by V_0, H_0, V_k, H_l into smaller rectangles.*

PROOF. By induction on the number vertices (segments) $n = k + l + 2$.

If G has 4 vertices $(b_0 = b, w_0 = w, b_1 = b', w_1 = w')$ then there is nothing to prove. Assume that $n \geq 5$, and that we have already established the lemma for all bipartite planar maps of at most $n - 1$ vertices, whose faces are quadrilaterials.

Suppose without loss of generality that G has at least 3 black vertices, i.e., $k \geq 2$. Obviously $\overrightarrow{b_{k-1}b_k}$ is a (possibily multiple) edge of $\overrightarrow{G_b}$. Let $w_{j_1} < \cdots < w_{j_t}$ $(t \geq 2)$ denote the common neighbors of b_{k-1} and b_k in G. For any $1 \leq r < t$, $b_{k-1}w_{j_r}b_kw_{j_{r+1}}$ is a face of G containing the edges $\overrightarrow{b_{k-1}b_k} \in \overrightarrow{G_b}$ and $\overrightarrow{w_{j_r}w_{_{r+1}}} \in \overrightarrow{G_w}$.

Let G^* denote the bipartite planar map obtained from G by identifying b_k with b_{k-1}, and deleting w_{j_r} for every $1 < r < t$. Thus, for any face of G except of $b_{k-1}w_{j_r}b_kw_{j_{r+1}}$ $(1 \leq r < t)$, there will be a corresponding face in G^*. According to this rule, we obtain the following oriented multigraphs from $\overrightarrow{G_b}$ and $\overrightarrow{G_w}$:

$$V\left(\overrightarrow{G_b^*}\right) = \{b_0, \ldots, b_{k-1}\}, \overrightarrow{G_b^*} = \left(\overrightarrow{G_b} - \left\{\overrightarrow{b_ib_k} \in \overrightarrow{G_b}\right\}\right) \cup \left\{\overrightarrow{b_ib_{k-1}} : \overrightarrow{b_ib_k} \in G \text{ and } i \neq k-1\right\},$$
$$V\left(\overrightarrow{G_w^*}\right) = V(G_w) - \{w_{j_r} : 1 < r < t\}, \overrightarrow{G_w^*} = \overrightarrow{G_w} - \left\{\overrightarrow{w_{j_r}w_{j_{r+1}}} : 1 \leq r < t\right\}.$$

Obviously, $\overrightarrow{G_b^*}$ and $\overrightarrow{G_w^*}$ are consistent with the orderings of the black and white vertices of G (restricted to $V(\overrightarrow{G_b^*})$ and $V(\overrightarrow{G_w^*})$, respectively), hence they are bipolar orientations. Moreover, they are dual to each other (in the sense specified before Lemma 1).

Applying the induction hypothesis to G^*, we obtain that the corresponding segments V_i^* and H_j^* $(1 \leq k-1, j \neq j_r$ for $1 < r < t)$ have no interior points in common, and they define a tiling of the rectangle bounded by $V_0^*, H_0^*, V_{k-1}^*, H_l^*$ with smaller rectangles. Observe that

$$V_i^* = V_i \quad \text{for every} \quad 0 \leq i < k-1,$$

$$H_j^* = H_j \quad \text{for every } j \text{ such that} \quad w_jb_k \notin G$$

Consider now the set of white vertices adjacent to b_{k-1} in G^*. They induce an oriented path $P = (w_{h_0} = w_0, w_{h_1}, \ldots, w_{h_s} = w_l)$ in both $\overrightarrow{G_w}$ and $\overrightarrow{G_w^*}$, which passes through w_{j_1} and w_{j_t}. Furthermore, in G

$$w_{h_r} \text{ is adjacent} \begin{cases} \text{to both } b_{k-1} \text{ and } b_k & \text{if } h_r = j_1 \text{ or } h_r = j_t, \\ \text{only to } b_{k-1} & \text{if } j_1 < h_r < j_t, \\ \text{only to } b_k & \text{if } h_r < j_1 \text{ or } h_r > j_t. \end{cases}$$

Let us modify the arrangement of segments V_i^*, H_j^*, as follows.

(i) Replace the rightmost vertical segment $V_{k-1}^* = [(x(b_{k-1}, y(w_0)), (x(b_{k-1}, y(w_l))]$ by its subsegment $V_{k-1} = [(x(b_{k-1}, y(w_{j_1})), (x(b_{k-1}), y(w_{j_t}))]$.

(ii) Add to the arrangement $V_k = [(x(b_k), y(w_0)), (x(b_k), y(w_l))]$.

(iii) For every $h_r \leq j_1$ or $h_r \geq j_t$, extend $H_{h_r}^*$ to the right until it hits V_k. The resulting segment is clearly H_{h_r}.

(iv) Add to arrangement $t-2$ new horizontal segments $H_{j_2}, \ldots, H_{j_{t-1}}$ connecting V_{k-1} and V_k.

The final configuration is clearly isomorphic to the arrangement of segments V_i, H_j, and forms a tiling of the rectangle bounded by V_0, H_0, V_k, H_l with smaller rectangles. □

To complete the proof of the theorem, it remains to show that if $V_i \cap H_j \neq \emptyset$ then $b_i w_j \in G$. By Lemma 2, V_i and H_j cannot have an interior point in common. Suppose without loss of generality that the upper endpoint of V_i belongs to H_j, i.e., $\max_{b_i w_h \in G} y(w_h) = y(w_j)$. Since all values of y are distinct, we conclude that $b_i w_j \in G$, as required.

3. Concluding remarks

Our theorem immediately implies the following result of Hartman, Newman and Ziv [HNZ].

Corollary.. *The vertices of any bipartite planar graph can be represented by vertical and horizontal segments in the plane so that two segments cross each other if and only if the* *corresponding vertices are adjacent.*

Notice that there is an unusual kind of duality between the bipartite planar map G and the tiling described in Lemma 2. There is a one–to–one correspondence between the faces of G and the rectangles of this tiling such that if $b_g w_h b_i w_j$ is a face of G then the sides of the corresponding rectangle belong to the segments V_g, H_h, V_i and H_j.

References

[Á] A. Ádám: A problem on the Schweitzer competition, Matematikai Lapok **22**(1971), 34.

[E] J. Ebert: *st*–ordering of the vertices of biconnected graphs, Computing **30**(1983), 19–33.

[ET] S. Even and R.E. Tarjan: Computing an *st*–numbering, Theoret. Comput. Sci. **2**(1976), 339–344.

[G] M.C. Golumbic: Algorithmic Graph Theory and Perfect Graphs, Acad. Press, New York, 1980.

[GJ] M.R. Garey and D.S. Johnson: Computers and Intractability, A Guide to the Theory of NP–completeness, W.H. Freeman and Co., San Francisco, 1979.

[HNZ] I. Hartman, Newman and Ziv: ?, Discrete Math.**87**(1991), 41-52.

[K] P. Koebe: Kontaktprobleme der konformen Abbildung, Berichte über die Verhandlungen der Sächsischen Akad. d. Wiss., Math.–Physische Klasse **88**(1936), 141–164.

[L] L. Lovász: Combinatorial Problems and Excercises, North–Holland, Amsterdam, 1979 (p. 43).

[LEC] A. Lempel, S. Even and I. Cederbaum: An algorithm for planarity testing of graphs, in: Theory of Graphs (International Symposium, Rome, July 1966, P. Rosenfield, ed.), Gordon and Breach, New York, 1967, 215–232.

[MP] W.O.J. Moser and J. Pach: Recent developments in combinatorial geometry, in: New Trends in Discrete and Computational Geometry (J. Pach, ed.), Springer–Verlag, (to appear)

[MR] P.O. de Mendez and P. Rosenstiehl: Depth indices: divide and conquer!, (to appear)

[R] P. Rosenstiehl: Embedding in the plane with orientation constraints: The angle graph, Ann. N.Y. Acad. Sci., 340–346.

[RT] P. Rosentiehl and R.E. Tarjan: Rectilinear planar layouts and bipolar orientations of planar graphs, Discrete Comput. Geom. **1**(1986), 343–353.

[T] R.E. Tarjan: Two streamlined depth–first search algorithms, Fund. Inform. **9**(1986), 85–94.

[TT] R. Tamassia and I.G. Tollis: A unified approach to visibility representations of planar graphs, Discrete Comput. Geom. **1**(1986), 321– 241.