International Journal of Computational Geometry & Applications
© World Scientific Publishing Company

PARTITIONING COLORED POINT SETS INTO MONOCHROMATIC PARTS *

ADRIAN DUMITRESCU
Electrical Engineering and Computer Science, University of Wisconsin – Milwaukee
Milwaukee, WI 53211, USA
E-mail: ad@cs.uwm.edu

and

JÁNOS PACH
Courant Institute of Mathematical Sciences, 251 Mercer Street
New York, NY 10012-1185, USA
E-mail: pach@cims.nyu.edu

Received (received date)
Revised (revised date)
Communicated by (Name of Editor)

ABSTRACT

It is shown that any two-colored set of n points in general position in the plane can be partitioned into at most $\lceil \frac{n+1}{2} \rceil$ monochromatic subsets, whose convex hulls are pairwise disjoint. This bound cannot be improved in general. We present an $O(n \log n)$ time algorithm for constructing a partition into fewer parts, if the coloring is unbalanced, i.e., the sizes of the two color classes differ by more than one. The analogous question for k-colored point sets ($k > 2$) and its higher dimensional variant are also considered.

Keywords: Colored point sets, partitioning algorithm, matching algorithm, Davenport-Schinzel sequence.

1. Introduction

A set of points in the plane is said to be in *general position*, if no three of its elements are collinear. Given a two-colored set S of n points in general position in the plane, let $p(S)$ be the minimum number of monochromatic subsets S can be partitioned into, such that their convex hulls are pairwise disjoint. Let

$$p(n) = \max\{p(S) : S \subset \mathbb{R}^2 \text{ is in general position }, |S| = n\}.$$

*A preliminary version of this paper appeared in the Proceedings of the 7th International Workshop, WADS 2001, Providence, RI, USA, August 2001, 264–275.

The following related quantities were introduced by Ron Aharoni and Michael Saks [8]. Given a set S of w white and b black points in general position in the plane, let $g(S)$ denote the number of edges plus the number of unmatched points in a largest non-crossing matching of S, where every edge is a straight-line segment connecting two points of the same color. Let

$$g(n) = \min\{g(S) : S \subset \mathbb{R}^2 \text{ is in general position}, |S| = n\}.$$

Aharoni and Saks asked if it is always possible to match all but a constant number of points, i.e., if $g(n) \leq n/2 + O(1)$ holds. It was shown in [8] that the answer to this question is in the negative. However, according to our next result (proved in Section 2), this inequality holds for $p(n)$. In other words: one can partition a set of n points in the plane into $n/2 + O(1)$ monochromatic parts whose convex hulls are disjoint; however if the size of each part is restricted to at most two points, this is not possible (there are configurations which require $(1/2 + \delta)n$ parts, where δ is a positive constant).

Theorem 1 *Let $p(n)$ denote the smallest integer p with the property that every 2-colored set of n points in general position in the plane can be partitioned into p monochromatic subsets whose convex hulls are pairwise disjoint. Then we have*

$$p(n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

For *unbalanced* colorings, i.e., when the sizes of the two color classes differ by more than one, we prove a stronger result. Slightly abusing notation, for any $w \leq b$, we write $p(w, b)$ for the minimum number of monochromatic subsets, into which a set of w white and b black points can be partitioned, so that their convex hulls are pairwise disjoint. Obviously, we have $p(w, b) \leq p(w + b)$.

Theorem 2 *For any $w \leq b$, we have*

$$w + 1 \leq p(w, b) \leq 2 \left\lceil \frac{w}{2} \right\rceil + 1. \tag{1}$$

More precisely,

(i) if $w \leq b \leq w + 1$ or w is even, we have $p(w, b) = w + 1$;

(ii) for all odd $w \geq 1$ and $b \geq 2w$, we have $p(w, b) = w + 2$;

(iii) for all odd $w \geq 3$ and $w + 2 \leq b \leq 2w - 1$, we have $w + 1 \leq p(w, b) \leq w + 2$.

We prove this theorem in Section 3, where we also present an $O(n \log n)$ time algorithm which computes a partition meeting the requirements.

For k-colored point sets with $k \geq 3$, the functions $p_k(n)$ and $g_k(n)$ can be defined similarly. In Section 4, we prove

Theorem 3 *For any $k, n \geq 3$, let $p_k(n)$ denote the smallest integer p with the property that every k-colored set of n points in general position in the plane can be partitioned into p monochromatic subsets whose convex hulls are pairwise disjoint. Then we have*

$$\left\lceil \left(1 - \frac{1}{k}\right) n + \frac{1}{k} \right\rceil \leq p_k(n) \leq \left(1 - \frac{1}{k + 1/6}\right) n + O(1).$$

We also provide an $O(n \log n)$ time algorithm for computing such a partition.

In Section 5, we discuss the analogous problem for $k = 2$ colors, but in higher dimensions. A set of $n \geq d + 1$ points in d-space is said to be in *general position*, if no $d + 1$ of its elements lie in a hyperplane.

A sequence $a_1 a_2 \ldots$ of integers between 1 and m is called an (m, d)-*Davenport-Schinzel sequence*, if (i) it has no two consecutive elements which are the same, and (ii) it has no *alternating* subsequence of length $d + 2$, i.e., there are no indices $1 \leq i_1 < i_2 < \ldots < i_{d+2}$ such that

$$a_{i_1} = a_{i_3} = a_{i_5} = \ldots = a, \quad a_{i_2} = a_{i_4} = a_{i_6} = \ldots = b,$$

where $a \neq b$. Let $\lambda_d(m)$ denote the maximum length of an (m, d)-Davenport-Schinzel sequence (see [4], [14]). Obviously, we have $\lambda_1(m) = m$, and it is easy to see that $\lambda_2(m) = 2m - 1$, for every m. It was shown by Hart and Sharir [9] that $\lambda_3(m) = O(n\alpha(n))$, where $\alpha(n)$ is the extremely slowly growing functional inverse of Ackermann's function, and that this estimate is asymptotically tight. They also proved that $\lambda_d(m)$ is only slightly superlinear in m, for every fixed $d > 3$. (For the best currently known bounds of this type, see [1].)

For any fixed d, let

$$\mu_d(n) = \min\{m : \lambda_d(m) \geq n\}.$$

Thus, we have that

$$\mu_2(n) = \left\lceil \frac{n+1}{2} \right\rceil,$$

and $\mu_d(n)$ is only very slightly sublinear in n, for any $d \geq 3$.

Theorem 4 *For any $n > d \geq 2$, let $p^{(d)}(n)$ denote the smallest integer p with the property that every 2-colored set of n points in general position in d-space can be partitioned into p monochromatic subsets whose convex hulls are pairwise disjoint.*

For a fixed $d \geq 2$, we have

(i) $p^{(d)}(n) \leq \frac{n}{d} + O(1)$;

(ii) $p^{(d)}(n) \geq \mu_d(n)$.

2. Proof of Theorem 1

We prove the lower bound by induction on n. Clearly, we have $p(1) \geq 1$, $p(2) \geq 2$. Assume the inequality holds for all values smaller than n. Consider a set S_a of n points placed on a circle, and having alternating colors, white and black (if n is odd, there will be two adjacent points of the same color, say, white). We call this configuration *alternating*. Denote by $h(n) = p(S_a)$ the minimum number of parts necessary to partition an alternating configuration of n points. Clearly, we have $p(n) \geq h(n)$.

Assume first that n is even, and fix a partition of S_a. We may suppose without loss of generality that this partition has a monochromatic (say, white) part P of size $l \geq 2$, otherwise the number of parts is n. The set $S_a \setminus P$ falls into l contiguous alternating subsets, each consisting of an odd number, $n_1, n_2, \ldots, n_l$, of elements, such that

$$\sum_{i=1}^{i=l} n_i + l = n, \quad \text{or} \quad \sum_{i=1}^{i=l} (n_i + 1) = n.$$

Hence, by induction,

$$h(n) \geq 1 + \sum_{i=1}^{i=l} h(n_i) \geq 1 + \sum_{i=1}^{i=l} \frac{n_i + 1}{2} = 1 + \frac{n}{2} = \left\lceil \frac{n+1}{2} \right\rceil,$$

as required. If n is odd, let P denote one of the two adjacent white points of S_a. Then

$$h(n) = h(S_a) \geq h(S_a - \{P\}) = h(n-1) \geq \frac{n-1}{2} + 1 = \frac{n+1}{2} = \left\lceil \frac{n+1}{2} \right\rceil.$$

To prove the upper bound, we also use induction on n. Clearly, we have $p(1) \leq 1$, $p(2) \leq 2$. Assume the inequality holds for all values smaller than n. If n is even,

$$p(n) \leq p(1) + p(n-1) \leq 1 + \frac{(n-1)+1}{2} = 1 + \frac{n}{2} = \frac{n+2}{2} = \left\lceil \frac{n+1}{2} \right\rceil.$$

Let n be odd. Consider a set S of n points, $n = w + b$, where w and b denote the number of white and black points respectively. Suppose, without loss of generality, that $w \geq b + 1$. If $conv(S)$, the convex hull of S, has two adjacent vertices of the same color, we can take them as a monochromatic part and apply the induction hypothesis to the remaining $n - 2$ points. Therefore, we may assume that $conv(S)$ has an even number of vertices, and they are colored white and black, alternately. Let x be a white vertex of $conv(S)$, and consider the points of $S - \{x\}$ in clockwise order of visibility from x. By writing a 0 for each white point that we encounter and a 1 for each black point, we construct a $\{0, 1\}$-sequence of length $n - 1$. Obviously, this sequence starts and ends with a 1. Removing the first and the last elements, we obtain a sequence $T = a_0 a_1 \ldots a_{n-4}$ of (even) length $n - 3$.

The sequence T consists of $k = w - 1$ zeroes and $l = b - 2$ ones, where $k - l = (w-1) - (b-2) \geq 2$, $k + l =$ even. According to Claim 1 below, T has a 00 contiguous subsequence (of two adjacent zeroes) starting at an even index $(0, 2, 4, ...)$. Let y and z denote the two white points corresponding to these two consecutive zeroes. Partition the set S into the white triple $\{x, y, z\}$ and two sets of *odd* sizes, n_1 and n_2 (with $n_1 + n_2 = n - 3$), consisting of all points preceding and following $\{y, z\}$ in the clockwise order, respectively. It follows by induction that

$$p(n) \leq 1 + \frac{n_1 + 1}{2} + \frac{n_2 + 1}{2} = \frac{n + 1}{2} = \left\lceil \frac{n + 1}{2} \right\rceil .$$

It remains to establish

Claim 1 *Let $T = a_0 a_1 \ldots a_{k+l-1}$ be a $\{0, 1\}$-sequence of even length, consisting of k zeroes and l ones, where $k - l \geq 2$. Then there is an even index $0 \leq i \leq k + l - 2$ such that $a_i = a_{i+1} = 0$.*

Proof. We prove the claim by induction on $k + l$, the length of the string. The base case $k + l = 2$ is clear, because then we have $T = 00$. For the induction step, distinguish four cases, according to what the first two elements of T are: $00, 01, 10$, or 11. In the first case we are done. In the other three cases, the assertion is true, because the inequality $k' - l' \geq 2$ holds for the sequence T' obtained from T by deleting its first two elements. (Here k' and l' denote the number of zeroes and ones in T', respectively.) □

3. An Algorithmic Proof of Theorem 2

The functions $p(\cdot)$ and $p(\cdot, \cdot)$ are monotone increasing: if $i \leq j$ then $p(i) \leq p(j)$, and if $i \leq k$ and $j \leq l$ then $p(i, j) \leq p(k, l)$. Thus, we have the lower bound in (1):

$$p(w, b) \geq p(w, w) \geq h(2w) \geq \left\lceil \frac{2w + 1}{2} \right\rceil = w + 1.$$

Next we prove the upper bound in (1). For $w = 1$, the inequality holds, so assume $w \geq 2$. Take an edge xy of the convex hull of the w white points, and let l denote its supporting line. Take $\{x, y\}$ as a monochromatic (white) part of size 2, and another (black, possibly empty) monochromatic part by selecting all black points lying in the open half-plane bounded by l, which contains no white points. Continue this procedure in the other open half-plane bounded by l as long as it contains at least two white points. If there is one (resp., no) white point left, then the remaining points can be partitioned into at most three (resp., one) monochromatic parts, whose convex hulls are pairwise disjoint. It is easy to verify that the number of parts in the resulting partition does not exceed $2 \left\lceil \frac{w}{2} \right\rceil + 1$. It is also true that the white parts of this partition form a perfect matching (of the white points) if w is even, and an *almost perfect* one (with one isolated point) if w is odd.

At the end of this section we present an $O(n \log n)$ time algorithm which computes such a partition.

Further, we verify the three cases highlighted in the theorem. For even w, the lower and upper bounds in (1) are both equal to $w+1$. The same holds in case (i), i.e., when $w \le b \le w+1$: $p(w,b) \le p(2w+1) = w+1$. For odd w, the gap between the lower and upper bounds is at most one.

It remains to discuss case (ii), when w is odd and $b \ge 2w$, for which we get a tight bound. By the monotonicity of $p(w,b)$, it is enough to show that $p(w,2w) \ge w+2$. Let Y be a regular w-gon. Let X and Z denote the *inner* and *outer* polygons obtained from Y by slightly shrinking it and blowing it up, respectively, around its center O. To bring $X \cup Y \cup Z$ into general position, slightly rotate Y around its center in the clockwise direction so that, if $x \in X$ and $z \in Z$ are the two vertices corresponding to $y \in Y$, then the points O, x, y, z are almost collinear. Place a white point at each vertex of Y, and a black point at each vertex of X and Z (see Fig. 1).

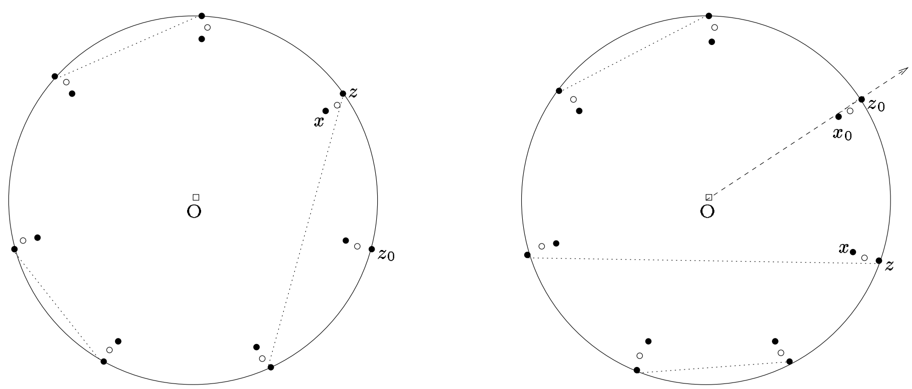

Fig. 1. A point set proving the lower bound in Theorem 2

We say that a *triangle* is *white* (resp. *black*), if all of its vertices are white (resp. black). Notice that the white vertices must be partitioned into at least $(w+1)/2$ parts. Otherwise, there would be a white part of size (at least) 3, which is impossible, because every white triangle Δ contains a black point of X in its interior. (If the center O is inside Δ, there are three black points of X inside Δ. If O is outside Δ, the black point of X corresponding to the "middle" point of Δ lies in Δ.) Similarly, the vertices of Z require at least $(w+1)/2$ parts.

Assume, for contradiction, that S can be partitioned into at most $w+1$ parts. Then the monochromatic parts restricted to the vertices of Z form an almost perfect non-crossing matching using straight-line segments, with exactly one unmatched point, and the same holds for Y. Let $z_0 \in Z$ be the unmatched point of Z. We restrict our attention to the matching M formed by the vertices of Z. These segments partition the circumscribing circle C of Z into $(w+1)/2$ convex regions. The *boundary* $B(R)$ of a region R, is formed by some segments in M and arcs of C. Let Z_0 denote the region containing the center O. Each point of X must be assigned to a black part determined by M. We distinguish two cases.

Case 1: $z_0 \notin Z_0$ (see Fig. 1, left). Pick any segment $s \in M \cap B(Z_0)$, and let z denote its first vertex in the clockwise order. Let x denote the vertex of X belonging to the ray Oz. It is easy to see that x cannot be assigned to any of the black parts determined by M, since the black triangle obtained in this way would contain a white point in its interior, contradiction.

Case 2: $z_0 \in Z_0$ (see Fig. 1, right). Let x_0 be the vertex of X belonging to the ray Oz_0. The point x_0 can only be assigned to the part containing z_0. By adding the segment x_0z_0 to M, we obtain a non-crossing matching $M^{'}$. Let z be the first vertex on $B(Z_0)$ following z_0 in the clockwise order. Let x denote the vertex of X belonging to the ray Oz. Similarly to Case 1, it is easy to see that x cannot be assigned to any of the black parts determined by $M^{'}$, since any black triangle obtained in this way would contain a white point, contradiction.

Thus, any partition of S requires at least $w+2$ parts, which completes the proof of the theorem.

Algorithm outline. We follow the general scheme described in the proof of the upper bound in Theorem 2, but allow some variations. Let W and B denote the set of white and black points of S, respectively.

1. This step ignores the black points. First, compute the convex layer decomposition of W. Based on this decomposition, find a perfect (or almost perfect) non-crossing matching M, using straight-line segments of W. At the same time, construct a planar subdivision D of the plane into at most $\lceil \frac{w}{2} \rceil + 1$ convex regions by extending the segments of M along their supporting lines. (If w is odd, draw an arbitrary line through the point which is left unmatched by M, and consider this point a part of M.) See Claim 2 below for a more precise description of the partitioning procedure using the extension of segments.
2. Here the black points come into play. Preprocess D for point location and perform point location in D for each black point.
3. Output the partition consisting of the pairs of points in M (the unmatched point as a part, if w is odd) for the white points, and the groups of points in the same region for the black points.

The following statement is an easy consequence of Euler's polyhedral formula (see also [13], page 259).

Claim 2 *Let $S = \{s_1, s_2, \ldots, s_m\}$ be a family of m pairwise disjoint segments in the plane, whose endpoints are in general position. In the given order, extend each segment in both directions until it hits another segment or the extension of a segment, or to infinity. After completing this process, the plane will be partitioned into $m+1$ convex regions.*

Algorithm details and analysis. The algorithm outputs at most $2\lceil \frac{w}{2} \rceil + 1$ monochromatic parts, $\lceil \frac{w}{2} \rceil$ of which are white and at most $\lceil \frac{w}{2} \rceil + 1$ are black. The decomposition into convex layers takes $O(n \log n)$ time [3]. The matching M is constructed starting with the outer layer and continues with the next layer, etc. If, for example, the size of the outer layer is even, every other segment in the layer is included in M. If the size of the outer layer is odd, we proceed in the same way, but at the end we compute the tangent from the last unmatched point to the convex polygon of the next layer, include this segment in M, and proceed to the next layer. The algorithm maintains at each step a current convex (possibly unbounded) polygonal region P containing all unmatched points. It also maintains the current planar subdivision D computed up to this step. The next segment s to be included in M lies inside P. Let s be oriented so that all unmatched (white) points are to the right of it. After extending s, P will fall into two convex polygonal regions, P' and P'', containing all the remaining unmatched points, and none of the points, respectively. At this point, P' becomes the new P and P'' is included in the polygonal subdivision D, which is thus refined. An example of step 1 is shown in Fig. 2, with the segments numbered in their order of inclusion in M.

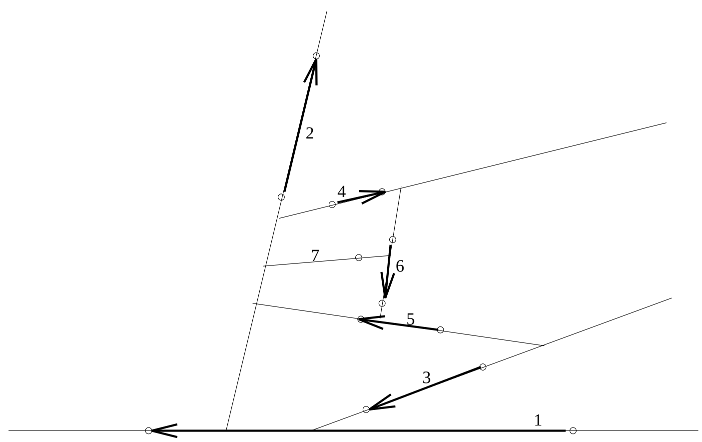

Fig. 2. Illustration of Step 1 of the algorithm

For each convex layer, we compute a balanced hierarchical representation (see Lemma 1 below). Since a tangent to a convex polygon of size at most n from an exterior point can be computed in $O(\log n)$ time, having such a representation, the time to compute all tangents is $O(n \log n)$. By dynamically maintaining the same (balanced hierarchical) representation for the current polygonal region P, we can compute the (at most two) intersection points between P and the supporting line of s in $O(\log n)$ time. The reader is referred to [11], pages 84–88 for this representation and the for two statements below.

Lemma 1 ([11], page 85) *A balanced hierarchical representation of a convex polygon on n points can be computed in $O(n)$ time.*

Lemma 2 ([11], page 87) *Given a balanced hierarchical representation of a convex polygon on n points and a line l, $P \cap l$ can be computed in $O(\log n)$ time.*

One can obtain a balanced hierarchical representation of a convex polygon in a natural way, by storing the sequence of edges in the leaves of a balanced tree.

Furthermore, selecting (2-3)-trees for this representation, allows us to get the balanced hierarchical representation of P' from that of P in $O(\log n)$ time by using the SPLIT operation on concatenable queues (see [2], pages 155–157). To get P' from P calls for at most two INSERT and at most two SPLIT operations, each taking $O(\log n)$ time. The update of D with P'' takes $O(|P''|)$ time. Since $\sum |P''|$ over the execution of the algorithm is $O(n)$, the total time required by step 1 is $O(n \log n)$.

We recall the following well known fact (e.g. see [12], page 77).

Lemma 3 *For any planar graph with n vertices, one can build a point location data structure of $O(n)$ size in $O(n \log n)$ time, guaranteeing $O(\log n)$ query time.*

The point location is performed for $b \leq n$ points, so the total time of step 2 is also $O(n \log n)$. The complexity of the last step is linear, thus the total time complexity of the algorithm is $O(n \log n)$. The space requirement is $O(n)$.

Another approach. The next algorithm closely mimics the proof of the upper bound in Theorem 2. Using the method of [10] (or of [3]) for convex hull maintenance under deletions of points, a sequence of n deletions performed on an n-point set takes $O(n \log n)$ time. Specifically, maintain the convex hull of the white points and that of the black points. Take an edge e of the white hull. Repeatedly, query the black hull to find a black point in the halfplane determined by e containing none of the white points. Such points (if any) are deleted one by one, until no other is found, and their collection is output as a black part. Then the two endpoints of e are deleted and this pair is output as a white part. The above step is repeated until the white points are exhausted and the partition is complete. The $O(n)$ queries and deletions take $O(n \log n)$ time.

4. Coloring with More Colors – Proof of Theorem 3

We prove the lower bound by induction on n. Clearly, we have $p_k(1) \geq 1$, $p_k(2) \geq 2, \ldots,\ p_k(k) \geq k$. Assume the inequality holds for all values smaller than n. Consider a set S of n points placed on a circle, colored with k colors, $1, 2, \ldots, k$, in a periodic fashion:

$$1, 2, \ldots, k, 1, 2, \ldots, k, \ldots, 1, 2, \ldots, r,$$

where $n = mk + r$, $0 \leq r \leq k-1$. We call such a configuration *periodic*. Let $h_k(n) := p_k(S)$, the smallest number of monochromatic parts a periodic configuration S can be partitioned into so that their convex hulls are pairwise disjoint. Clearly, we have $p_k(n) \geq h_k(n)$. Fix such a partition.

Case 1: $r = 0$. If every monochromatic part of the partition is a singleton, then we are done. Consider a monochromatic part P of size $l \geq 2$, and assume without loss of generality that the color of P is k. $S - P$ splits into smaller periodic subsets of sizes $n_1, n_2, \ldots, n_l$, with $n_i \equiv -1 \pmod{k}$ for every i, such that

$$\sum_{i=1}^{i=l} n_i + l = n, \quad \text{or} \quad \sum_{i=1}^{i=l} (n_i + 1) = n. \tag{2}$$

Hence, by induction,

$$h_k(n) \geq 1 + \sum_{i=1}^{i=l} h_k(n_i) \geq 1 + \sum_{i=1}^{i=l} \left\lceil \frac{(k-1)n_i + 1}{k} \right\rceil = 1 + \sum_{i=1}^{i=l} \frac{(k-1)n_i + (k-1)}{k}$$

$$= 1 + \sum_{i=1}^{i=l} \frac{(k-1)(n_i+1)}{k} = 1 + \frac{(k-1)n}{k} = \frac{(k-1)n + k}{k} = \left\lceil \frac{(k-1)n+1}{k} \right\rceil.$$

Case 2: $r = 1$. Discard 1 point of color 1 from the last (incomplete) group, and get a periodic configuration with $n-1$ points, as in the previous case. Then, by induction,

$$h_k(n) \geq h_k(n-1) \geq \left\lceil \frac{(k-1)(n-1)+1}{k} \right\rceil$$

$$= \left\lceil \frac{(k-1)(n-1)+k}{k} \right\rceil = \left\lceil \frac{(k-1)n+1}{k} \right\rceil.$$

Case 3: $r \geq 2$. We distinguish two subcases:

Subcase 3.a: There exists a monochromatic part P of size $l \geq 2$ of color 1. The set $S - P$ splits into smaller periodic subsets of sizes $n_1, n_2, \ldots, n_l$ satisfying (2), where $n_i \equiv -1 \pmod{k}$ for all $1 \leq i \leq l-1$, and $n_l \equiv r-1 \pmod{k}$. Then, using the induction hypothesis, we obtain

$$h_k(n) \geq 1 + \sum_{i=1}^{i=l} h_k(n_i) \geq 1 + \sum_{i=1}^{i=l} \left\lceil \frac{(k-1)n_i+1}{k} \right\rceil$$

$$= 1 + \sum_{i=1}^{i=l-1} \frac{(k-1)n_i + (k-1)}{k} + \frac{(k-1)n_l + (r-1)}{k}$$

$$= 1 + \sum_{i=1}^{i=l} \frac{(k-1)(n_i+1)}{k} + \frac{r-k}{k} = \frac{(k-1)n + r}{k} = \left\lceil \frac{(k-1)n + r}{k} \right\rceil$$

$$= \left\lceil \frac{(k-1)n + r - (k-1)}{k} \right\rceil = \left\lceil \frac{(k-1)n+1}{k} \right\rceil.$$

Subcase 3.b: There is no monochromatic part P of size $l \geq 2$ of color 1.

$$h_k(n) \geq \frac{n-r}{k} + h_{k-1}(n - \frac{n-r}{k}) \geq \frac{n-r}{k} + \left\lceil \frac{(k-2)(n - \frac{n-r}{k}) + 1}{k-1} \right\rceil$$

$$= m + \left\lceil \frac{(k-2)((k-1)m + r) + 1}{k-1} \right\rceil = (k-1)m + \left\lceil \frac{(k-2)r+1}{k-1} \right\rceil.$$

The lower bound we want to prove can be written as

$$\frac{(k-1)n+1}{k} = (k-1)m + \left\lceil \frac{(k-1)r+1}{k} \right\rceil.$$

An easy calculation shows that

$$\left\lceil \frac{(k-2)r+1}{k-1} \right\rceil = r + \left\lceil \frac{1-r}{k-1} \right\rceil = r + \left\lceil \frac{1-r}{k} \right\rceil = \left\lceil \frac{(k-1)r+1}{k} \right\rceil .$$

The value of the above expression is 1 for $r = 0$, and r for $r \geq 1$ (only the latter case is of interest in this subcase). This concludes the proof of the lower bound.

Next, we establish the upper bound in Theorem 3.

Proposition 1 $p_k(n) \leq n - g_k(n)$.

Proof. A set of n points contains a non-crossing matching of $g_k(n)$ pairs. Thus, using parts of size 2 for the points in the matching and singletons for the remaining ones, we obtain a partition, in which the number of parts is

$$g_k(n) + (n - 2g_k(n)) = n - g_k(n).$$

□

Proposition 2 [7] *For* $k \geq 3$*, we have* $g_k(6k+1) = 6$.

It follows immediately from Propositions 1 and 2 that

$$p_k(n) \leq n - g_k(n) = \left(1 - \frac{6}{6k+1}\right) n + O(1) = \left(1 - \frac{1}{k+1/6}\right) n + O(1),$$

as required.

In the end, we outline an algorithm. Its main part is the computation of the matching M, which appears in [7]. Let $k \geq 3$ be a fixed integer.

Algorithm. The n points are sorted according to their x-coordinate and divided into consecutive groups of size $6k+1$. In each group, with the possible exception of the last one, one can match at least 12 elements. This can be done by matching 12 out of 19 points in the three largest color classes. The time to process a group is $O(k)$, so the total time is $O(n \log n)$. The number of output parts is bounded as claimed in the theorem.

5. Higher Dimensions – Proof of Theorem 4

The upper bound (i) can be shown by straightforward generalization of the proof of Theorem 2: if $w \leq b,\ n = w + b,\ |W| = w$, one can iteratively remove a facet of $conv(W)$. The easy details are left to the reader.

To verify part (ii), we need a simple property of the d-dimensional *moment curve*

$$M_d(\tau) = (\tau, \tau^2, \ldots, \tau^d),\ \tau \in \mathbb{R}.$$

For two points, $x = M_d(\tau_1)$ and $y = M_d(\tau_2)$, we say that x *precedes* y and write $x \prec y$ if $\tau_1 < \tau_2$. For simplicity, let $u = \lceil d/2 \rceil$ and $v = \lfloor d/2 \rfloor$.

Lemma 4 (see [5], [6]) *Let $x_1 \prec x_2 \prec \ldots \prec x_{u+1}$ and $y_1 \prec y_2 \prec \ldots \prec y_{v+1}$ be two sequences of distinct points on the d-dimensional moment curve, whose convex hulls are denoted by X and Y, respectively.*

Then X and Y cross each other if and only if the points x_i and y_j interleave, i.e., every arc $x_i x_{i+1}$ of $M_d(\tau)$ contains exactly one y_j.

Let S_a be a sequence of n points on the d-dimensional moment curve, $x_1 \prec x_2 \prec \ldots \prec x_n$, which are colored white and black, alternately. Consider a partition of S_a into m monochromatic parts, labeled by integers between 1 and m, and suppose that the convex hulls of these parts are pairwise disjoint. Replacing every element of the sequence S_a by the the label of the part containing it, we obtain a sequence T of length n, whose elements are integers between 1 and m. Obviously, no two consecutive elements of this sequence coincide, because adjacent points have different colors, and thus belong to different parts in the partition. It follows from the above lemma that T has no alternating subsequence of length $(u+1)+(v+1) = d+2$. That is, T is an (m,d)-Davenport-Schinzel sequence of length n. Therefore, we have $n \leq \lambda_d(m)$, or, equivalently, $m \geq \mu_d(n)$, as required.

References

1. P. K. Agarwal, M. Sharir, and P. Shor, Sharp upper and lower bounds for the length of general Davenport-Schinzel sequences, *Journal of Combinatorial Theory, Ser. A*, **52** (1989), 228–274.
2. V. Aho, J. Hopcroft and J. Ullman, *The Design and analysis of Computer Algorithms*, Addison-Wesley, Reading, 1974.
3. B. Chazelle, On the convex layers of a planar set, *IEEE Transactions on Information Theory*, **31(4)** (1985), 509–517.
4. H. Davenport and A. Schinzel, A combinatorial problem connected with differential equations, *American Journal of Mathematics*, **87** (1965), 684–694.
5. T. K. Dey and H. Edelsbrunner, Counting triangle crossings and halving planes, *Discrete & Computational Geometry*, **12** (1994), 281–289.
6. T. K. Dey and J. Pach, Extremal problems for geometric hypergraphs, *Discrete & Computational Geometry*, **19** (1998), 473–484.
7. A. Dumitrescu and R. Kaye, Matching colored points in the plane: some new results, *Computational Geometry: Theory and Applications*, **19(1)** (2001), 69–85.
8. A. Dumitrescu and W. Steiger, On a matching problem in the plane, *Workshop on Algorithms and Data Structures*, 1999 (WADS '99). Also in: *Discrete Mathematics*, **211** (2000), 183–195.
9. S. Hart and M. Sharir, Nonlinearity of Davenport-Schinzel sequences and of generalized path compression schemes, *Combinatorica*, **6** (1986), 151–177.
10. J. Hershberger and S. Suri, Applications of a semi-dynamic convex hull algorithm, *BIT*, **32** (1992), 249–267.
11. K. Mehlhorn, *Data Structures and Algorithms 3: Multi-dimensional searching and Computational Geometry*, Springer Verlag, Berlin, 1984.
12. K. Mulmuley, *Computational Geometry – An Introduction through Randomized Algorithms*, Prentice Hall, Englewood Cliffs, 1994.
13. J. O'Rourke, *Art Gallery Theorems and Algorithms*, Oxford University Press, New York, 1987.

14. M. Sharir and P. K. Agarwal. *Davenport-Schinzel Sequences and Their Geometric Applications*, Cambridge University Press, Cambridge, 1995.