Disjoint Edges in Topological Graphs*

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Abstract. A topological graph G is a graph drawn in the plane so that its edges are represented by Jordan arcs. G is called *simple*, if any two edges have at most one point in common. It is shown that the maximum number of edges of a simple topological graph with n vertices and no k pairwise disjoint edges is $O\left(n \log^{4k-8} n\right)$ edges. The assumption that G is simple cannot be dropped: for every n, there exists a complete topological graph of n vertices, whose any two edges cross at most twice.

1 Introduction

A *topological graph* G is a graph drawn in the plane so that its vertices are represented by points in the plane and its edges by (possibly intersecting) Jordan arcs connecting the corresponding points and not passing through any vertex other than its endpoints. We also assume that no two edges of G "touch" each other, i.e., if two edges share an interior point, then at this point they properly cross. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We will make no notational distinction between the vertices (edges) of the underlying abstract graph, and the points (arcs) representing them in the plane.

A topological graph G is called *simple* if any two edges cross at most once. G is called x*-monotone* if (in a properly chosen (x, y) coordinate system) every line parallel to the y-axis meets every edge at most once. Clearly, every *geometric graph*, i.e., every graph drawn by straight-line edges, is both simple and x-monotone.

The extremal theory of geometric graphs is a fast growing area with many exciting results, open problems, and applications in other areas of mathematics [P99]. Most of the known results easily generalize to simple x-monotone topological graphs. For instance, it was shown by Pach and Törőcsik [PT94] that for any fixed k, the maximum number of edges of a geometric graph with n vertices

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and no k pairwise disjoint edges is $O(n)$. The special cases $k = 2$ and 3 had been settled by Perles and by Alon and Erdős [AE89], respectively. All known proofs readily generalize to simple x-monotone topological graphs. (See [T00] for a more precise statement.)

Of course, here we cannot drop the assumption that G is simple, because one can draw a complete graph so that any pair of its edges cross. However, it is possible that the above statement remains true for all simple topological graphs, i.e., without assuming x-monotonicity. The aim of the present note is to discuss this problem. We will apply some ideas of Kolman and Matoušek [KM03] and Pach, Shahrokhi, and Szegedy [PSS96] to prove the following result.

Theorem 1. *For any $k \geq 2$, the number of edges of every simple topological graph G with n vertices and no k pairwise disjoint edges is at most $Cn \log^{4k-8} n$, where C is an absolute constant.*

As an immediate consequence, we obtain

Corollary. *Every simple complete topological graph with n vertices has $\Omega(\frac{\log n}{\log \log n})$ pairwise disjoint edges.* □

The best previously known lower bound for this quantity, $\Omega\left(\log^{1/6} n\right)$, was established by Pach, Solymosi and Tóth [PST01].

We also prove that Theorem 1 does not remain true if we replace the assumption that G is simple by the slightly weaker condition that any pair of its edges cross at most *twice.*

Theorem 2. *For every n, there exists a complete topological graph of n vertices whose any pair of edges have exactly one or two common points.*

The analogous question, when the excluded configuration consists of k *pairwise crossing* (rather than pairwise disjoint) edges, has also been considered. For $k = 2$, the answer is easy: every crossing-free topological graph with $n > 2$ vertices is *planar*, so its number of edges is at most $3n - 6$. For $k = 3$, it was shown by Agarwal et al. [AAP97] that every geometric graph (in fact, every simple x-monotone topological graph) G with n vertices and no 3 pairwise crossing edges has $O(n)$ edges. This argument was extended to all topological graphs by Pach, Radoičić and Tóth [PRT03a]. It is a major unsolved problem to decide whether, for any fixed $k > 3$, every geometric (or topological) graph of n vertices which contains no k pairwise crossing edges has $O(n)$ edges. It is known, however, that the number of edges cannot exceed n times a polylogarithmic factor [PSS96], [V98], [PRT03a]. Here the assumption that G is *simple* does not seem to play such an central role as, e.g., in Theorem 1.

2 Auxiliary Results

In this section, after introducing the necessary definitions, we review, modify, and apply some relevant results of Kolman and Matoušek [KM03] and Pach, Shahrokhi, and Szegedy [PSS96].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any partition of $V(G)$ into two non-empty parts, V_1 and V_2, let $E(V_1, V_2)$ denote the set of edges connecting V_1 and V_2. The set $E(V_1, V_2) \subset E(G)$ is said to be a *cut*. The *bisection width* $b(G)$ of G is defined as the minimum size $|E(V_1, V_2)|$ of a cut with $|V_1|, |V_2| \geq |V|/3$. The *edge expansion* of G is

$$\beta(G) = \min_{V_1 \cup V_2 = V(G)} \frac{|E(V_1, V_2)|}{\min \{|V_1|, |V_2|\}},$$

where the first minimum is taken over *all* partitions $V_1 \cup V_2 = V(G)$.

Clearly, we have $\beta(G) \leq 3b(G)/n$. On the other hand, it is possible that $\beta(G)$ is small (even 0) but $b(G)$ is large. However, it is very easy to prove

Lemma 1. [KM03] *Every graph G of n vertices has a subgraph H of at least $2n/3$ vertices such that $\beta(H) \geq b(G)/n$.* □

An *embedding* of a graph H in G is a mapping that takes the vertices of H to distinct vertices of G, and each edge of H to a path of G between the corresponding vertices. The *congestion* of an embedding is the maximum number of paths passing through an edge of G.

As Kolman and Matoušek have noticed, combining a result of Leighton and Rao [LR99] for multicommodity flows with the rounding technique of Raghavan and Thompson [RT87], we obtain the following useful result.

Lemma 2. [KM03] *Let G be any graph of n vertices with edge expansion $\beta(G) = \beta$. There exists an embedding of the complete graph K_n in G with congestion $O(\frac{n \log n}{\beta})$.* □

The *crossing number* $\text{CR}(G)$ of a graph G is the minimum number of crossing points in any drawing of G. The *pairwise crossing number* $\text{PAIR-CR}(G)$ and the *odd-crossing number* $\text{ODD-CR}(G)$ of G are defined as the minimum number of pairs of edges that cross, resp., cross an *odd* number of times, over all drawings of G. It follows directly from the definition that for any graph G $\text{CR}(G) \geq \text{PAIR-CR}(G) \geq odd - cr(G)$. For any graph G, let

$$\text{SSQD}(G) = \sum_{v \in V(G)} d^2(v),$$

where $d(v)$ is the degree of the vertex v in G, and SSQD is the shorthand for the "sum of squared degrees."

Next we apply Lemmas 1 and 2 to obtain the following assertion, slightly stronger than the main result of Kolman and Matoušek [KM03], who established a similar inequality for the pairwise crossing number.

Two edges of a graph are called *independent* if they do not share an endpoint.

Lemma 3. *For every graph G, we have*

$$\text{ODD-CR}(G) \geq \Omega\left(\frac{b^2(G)}{\log^2 n}\right) - O\left(\text{SSQD}(G)\right).$$

Moreover, G has at least this many pairs of independent edges that cross an odd number of times.

Proof. Let H be a subgraph of G satisfying the condition in Lemma 1. Using the trivial inequality $\text{ODD-CR}(G) \geq \text{ODD-CR}(H)$, it is sufficient to show that

$$\text{ODD-CR}(H) \geq \Omega\left(\frac{n^2\beta^2(H)}{\log^2 n}\right) - O\left(\text{SSQD}(H)\right).$$

Letting m denote the number of vertices of H, we have $n \geq m \geq 2n/3$.

Fix a drawing of H, in which precisely $\text{ODD-CR}(H)$ pairs of edges cross an odd number of times. For simplicity, this drawing (topological graph) will also be denoted by H. In view of Lemma 2, there exists an embedding of K_m in H with congestion $O(\frac{m \log m}{\beta(H)})$. In a natural way, this embedding gives rise to a drawing of K_m, in which some portions of Jordan arcs representing different edges of K_m may coincide. By a slight perturbation of this drawing, we can obtain another one that has the following properties:

1. any two Jordan arcs cross a finite number of times;
2. all of these crossings are proper;
3. if two Jordan arcs originally shared a portion, then after the perturbation every crossing between the modified portions occurs in a very small neighborhood of some (point representing a) vertex of H.

Let e_1 and e_2 be two edges of K_m, represented by two Jordan arcs, γ_1 and γ_2, respectively. By the above construction, each crossing between γ_1 and γ_2 occurs either in a small neighborhood of a vertex of H or in a small neighborhood of a crossing between two edges of H. Therefore, if γ_1 and γ_2 cross an odd number of times, then either (i) one of their crossings is very close to a vertex of H, or (ii) γ_1 and γ_2 contain two subarcs that run very close to two edges of H that cross an odd number of times. Clearly, the number of pairs (γ_1, γ_2) satisfying conditions (i) and (ii) is at most the square of the congestion of the embedding of K_m in H multiplied by $\text{SSQD}(H)$ and by $\text{ODD-CR}(H)$, respectively. Thus, we have

$$\text{ODD-CR}(K_m) = O\left((\text{SSQD}(H) + \text{ODD-CR}(H))\frac{n^2 \log^2 n}{\beta^2(H)}\right).$$

On the other hand, it is known ([PT00]) that $\text{ODD-CR}(K_m) = \Omega(m^4)$. Comparing these two bounds and taking into account that $m \geq 2n/3$, the lemma follows. □

Theorem 3. *For any $k \geq 2$, every topological graph of n vertices that contains no k independent edges such that every pair of them cross an odd number of times, has at most $Cn\log^{4k-8} n$ edges, for a suitable absolute constant C.*

Proof. We use double induction on n and k. For $k = 2$ and for every n, the statement immediately follows from an old theorem of Hanani [H34], according to which $\text{ODD-CR}(G) = 0$ holds if and only if G is planar.

Assume that we have already proved Theorem 3 for some $k \geq 2$ and for all n. For $n = 1, 2$ the statement is trivial. Let $n > 2$ and suppose that the assertion is also true for $k + 1$ and for all topological graphs having fewer than n vertices.

We prove, by double induction on k and n, that the number of edges of a topological graph G with n vertices, which has no $k+1$ edges that pairwise cross an odd number of times, is at most $Cn\log^{4k-4} n$. Here C is a constant to be specified later. The statement is trivial for $k=1$. Suppose that it holds (1) for $k-1$ and for all n, and (2) for k and for every $n' < n$. For simplicity, the underlying abstract graph is also denoted by G. For any edge $e \in E(G)$, let $G_e \subset G$ denote the topological graph consisting of all edges of G that cross e an odd number of times. Clearly, G_e has no k edges so that any pair of them cross an odd number of times. By part (2) of the induction hypothesis, we have

$$\textsc{odd-cr}(G) \le \frac{1}{2} \sum_{e\in E(G)} |E(G_e)| \le \frac{1}{2}|E(G)|Cn\log^{4k-8} n.$$

Using the fact that $\textsc{ssqd}(G) \le 2|E(G)|n$ holds for every graph G, it follows from Lemma 3 that

$$b(G) \le C_0 \log n \left(|E(G)|n\log^{4k-8} n\right)^{1/2},$$

for a suitable constant C_0. Consider a partition of $V(G)$ into two parts of sizes $n_1, n_2 \le 2n/3$ such that the number of edges running between them is $b(G)$. Neither of the subgraphs induced by these parts has $k+1$ edges, any pair of which cross an odd number of times. Applying part (1) of the induction hypothesis to these subgraphs, we obtain

$$|E(G)| \le Cn_1\log^{4k-4} n_1 + Cn_2\log^{4k-4} n_2 + b(G).$$

Comparing the last two inequalities and setting $C = \max(100, 10C_0)$, the result follows by some calculation. □

3 Proofs of the Main Results

Proof of Theorem 1. Let G be a simple topological graph with no k pairwise disjoint edges. Let G' be a *bipartite* topological subgraph of G, consisting of at least half of the edges of G, and let V_1 and V_2 denote its vertex classes.

Applying a suitable homeomorphism (continuous one-to-one transformation) to the plane, if necessary, we can assume without loss of generality that

1. all vertices in V_1 lie above the line $y=1$;
2. all vertices in V_2 lie below the line $y=0$;
3. each piece of an edge that lies in the strip $0 \le y \le 1$ is a vertical segment.

Replace the part of the drawing of G' that lies above the line $y=1$ by its reflection about the y-axis. Erase the part of the drawing in the strip $0 \le y \le 1$, and re-connect the corresponding pairs of points on the lines $y=0$ and $y=1$ by straight-line segments.

If in the original drawing two edges, $e_1, e_2 \in E(G')$, have crossed each other an *even* number of times, then after the transformation their number of crossings

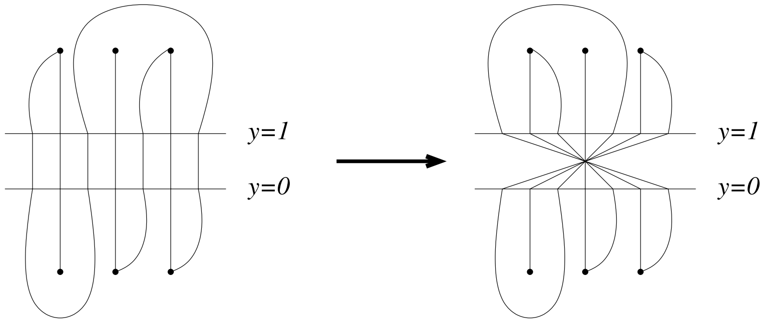

Fig. 1. The redrawing procedure

will be *odd*, and vice versa. Indeed, if originally e_i crossed the strip k_i times, then k_i was odd $(i = 1, 2.)$ After the transformation, we have $k_1 + k_2$ pairwise crossing segments in the strip $0 \leq y \leq 1$. From the $\binom{k_1+k_2}{2}$ crossings between them, $\binom{k_i}{2}$ correspond to self-intersections of e_i. Thus, the number of crossings between e_1 and e_2 in the resulting drawing is equal to their original number of crossings plus

$$\binom{k_1 + k_2}{2} - \binom{k_1}{2} - \binom{k_2}{2}.$$

However, this sum is always odd, provided that k_1 and k_2 are odd. Note that one can easily get rid of the resulting self-intersections of the edges by locally modifying them in small neighborhoods of these crossings.

Suppose that the resulting drawing of G' has k edges, any two of which cross an odd number of times. Then any pair of the corresponding edges in the original drawing must have crossed an even number of times. Since originally G' was a *simple* topological graph, i.e., any two of its edges crossed at most once, we can conclude that the original drawing of G' (and hence the original drawing of G) had k pairwise disjoint edges, contradicting our assumption.

Thus, the new drawing of G' has no k edges that pairwise cross an odd number of times. Now it follows directly from Theorem 3 that $|E(G)| \leq 2|E(G')| \leq 2Cn \log^{4k-8} n$, as required. □

The assumption that G is simple was used only once, at the end of the proof. Another implication of the redrawing procedure is that Theorem 3 hold also if we replace"odd" by "even."

Proof of Theorem 2. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n. For $1 \leq i \leq n$, place v_i at $(i, 0)$. Now, for any $1 \leq i < j \leq n$, represent the edge $v_i v_j$ by a polygon whose vertices are

$$(0, i),\ (0, i - j/n),\ (i - j/n - n, 0),\ (0, i - j/n - n),\ (0, j).$$

It is easy to verify that any two of these polygons cross at most twice. □

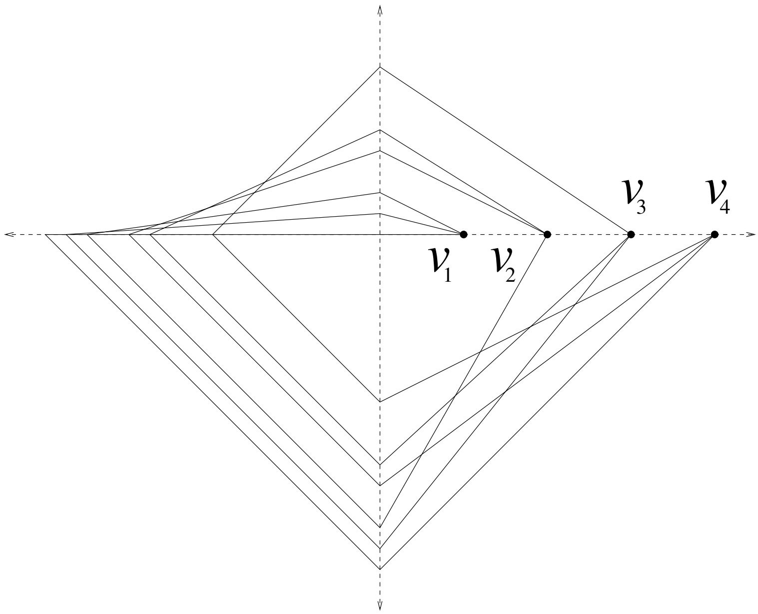

Fig. 2. A DRAWING OF K_4 IN WHICH ANY TWO EDGES HAVE EXACTLY ONE OR TWO COMMON POINTS

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