Geometric Intersection Patterns and the Theory of Topological Graphs

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Abstract. The *intersection graph* of a set system $\mathcal{S}$ is a graph on the vertex set $\mathcal{S}$, in which two vertices are connected by an edge if and only if the corresponding sets have nonempty intersection. It was shown by Tietze (1905) that every finite graph is the intersection graph of 3-dimensional convex polytopes. The analogous statement is false in any fixed dimension if the polytopes are allowed to have only a bounded number of faces or are replaced by simple geometric objects that can be described in terms of a bounded number of real parameters. Intersection graphs of various classes of geometric objects, even in the plane, have interesting structural and extremal properties.

We survey problems and results on geometric intersection graphs and, more generally, intersection patterns. Many of the questions discussed were originally raised by Berge, Erdős, Grünbaum, Hadwiger, Turán, and others in the context of classical topology, graph theory, and combinatorics (related, e.g., to Helly's theorem, Ramsey theory, perfect graphs). The rapid development of computational geometry and graph drawing algorithms in the last couple of decades gave further impetus to research in this field. A *topological graph* is a graph drawn in the plane so that its vertices are represented by points and its edges by possibly intersecting simple continuous curves connecting the corresponding point pairs. We give applications of the results concerning intersection patterns in the theory of topological graphs.

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1. From topological graphs to intersection graphs

A *topological graph* is a graph G drawn in the *plane* with possibly intersecting curvilinear edges. More precisely, the *vertices* of G are points in the plane and the *edges* are simple continuous curves connecting the corresponding point pairs and not passing through any other point representing a vertex. These curves are allowed to cross, but we assume for simplicity that any two intersect only in a finite number of points, no two are tangent to each other, and no three share an interior

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point. In the special case when the edges are straight-line segments, G is called a *geometric graph*. In notation and terminology, we do not distinguish between the vertices (edges) of a topological graph and the vertices (edges) of the underlying abstract graph.

In the past few decades, the theory of topological and geometric graphs has become a fast growing separate field of combinatorial geometry with interesting applications in graph drawing, in combinatorial and computational geometry, in additive number theory, and elsewhere. See, e.g., [5], [74], [108], [113], [29]. Many related contributions can be found in the proceedings of the annual Graph Drawing symposia, published in Springer's Lecture Notes series in Computer Science and in two collections of papers [86], [87]. For surveys, see Chapter 14 in [88], Chapter 10 in [60], and Chapters 1 and 3 in [37].

In this section, we would like to illustrate by an example how questions about topological graphs lead to the study of intersection graphs of geometric objects.

Definition 1.1. Two edges, e and f, of a topological graph are said to *cross* if they share an interior point at which e passes from one side of f to the other side. A topological graph is *simple* if any pair of its edges have at most one point in common, which is either an endpoint or an interior point at which they cross.

A topological graph is called k*-quasiplanar* for some integer $k \geq 2$ if no k of its edges are pairwise crossing.

Using this terminology, a planar graph is 2-quasiplanar.

Conjecture 1.2. *For any fixed $k \geq 2$, the number of edges of every k-quasiplanar topological graph with n vertices is $O(n)$.*

For $k = 2$, this follows from Euler's polyhedral formula. For $k = 3$, for *simple* topological graphs, Conjecture 1.2 was proved in [4]. Without the simplicity condition, the statement was first proved in [91]. The best known upper bound of roughly $8n$ was established by Ackerman and Tardos [3]. For $k = 4$, the conjecture has been verified by Ackerman [1].

For larger values of k, Conjecture 1.2 is still open. The upper bound $n(\log n)^{O(k)}$ for the number of edges of a simple k-quasiplanar topological graph was first proved in [92], and then for all k-quasiplanar topological graphs in [91]. This was further improved to $n(\log n)^{O(\log k)}$ by Fox and Pach [44]. For k-quasiplanar *geometric* graphs and, more generally, for simple topological graphs whose edges are represented by x-monotone arcs (that is, curves in the plane such that every vertical line intersects them in at most one point), Valtr [119], [120] showed that the number of edges cannot exceed $c_k n \log n$. Extending Valtr's ideas, Fox, Pach, and Suk [47] (see also [103]) proved the following.

Theorem 1.3. [47] *The number of edges of every k-quasiplanar topological graph of n vertices with all edges represented by x-monotone arcs is at most $2^{ck^6} n \log n$, for a suitable absolute constant c.*

Using similar ideas, Suk and Walczak [112] established another generalization of Valtr's result: the number of edges of any *simple* k-quasiplanar topological graph with n vertices is also $O_k(n \log n)$.

For *convex geometric graphs*, that is, for geometric graphs whose vertices form a convex n-gon, the Conjecture 1.2 was proved by Capoyleas and Pach [23].

Theorem 1.4. [23] *The maximum number of edges that a k-quasiplanar convex geometric graph with n vertices can have is $2(k-1)n - \binom{2k-1}{2}$, provided that $n \geq 2k-1$.*

The *intersection graph* of a set system $\mathcal{S}$ is a graph on the vertex set $\mathcal{S}$, in which two vertices are connected by an edge if and only if the corresponding sets have nonempty intersection.

A natural attempt to prove Conjecture 1.2, at least for geometric graphs, is the following. Let K_k denote a clique (complete graph) on k vertices.

Problem 1.5. *Given two integers $k, m > 2$, determine the smallest number $\alpha = \alpha_k(m)$ with the property that the intersection graph of any system of m segments in the plane which contains no K_k as a subgraph has at least α independent vertices. The same problem can be raised for intersection graphs of continuous curves.*

Assume for a moment that for some k there exists $\varepsilon_k > 0$ such that $\alpha_k(m) > \varepsilon_k m$. This would immediately imply Conjecture 1.2 for geometric graphs. To see this, let G be a k-quasiplanar geometric graph with vertex set $V(G)$ and edge set $E(G)$. By definition, the intersection graph of the open segments representing the edges of G contains no K_k as a subgraph. By our assumption, G has an independent set of size at least $\varepsilon_k|E(G)|$. The corresponding segments induce a planar subgraph of G. Therefore, we obtain $\varepsilon_k|E(G)| \leq 3n - 6$, implying that $|E(G)| < (3/\varepsilon_k)|V(G)|$, as required.

However, generalizing a construction of Pawlik *et al.* [99], Walczak [121] proved that $\alpha_3(m) = O(1/\log\log m)$. Hence, the above attempt to verify Conjecture 1.2 fails. The best known lower bound on $\alpha_3(m)$ is 1 over a polynomial in m (see [44]). On the other hand, every K_k-free intersection graph of m unit segments in the plane has an independent set of size at least $\varepsilon_k m$, for a suitable constant $\varepsilon_k > 0$. Moreover, Suk [110] proved that for a fixed k, the chromatic numbers of these graphs are bounded by an absolute constant. In [112], a similar statement was proved for intersection graphs of continuous curves, each intersecting the x-axis in precisely one point.

Due to space limitations and personal preferences, many classical topics concerning geometric intersection patterns and graph representations will be suppressed or not mentioned at all in this survey. These include Helly-type results [122], geometric transversal theory [27], approximate embeddings of graphs into normed spaces [82], orthogonal and other geometric graph representations [77], [79], epsilon-nets and VC-dimension [106].

2. Forbidden subgraphs of intersection graphs

In combinatorics and computer science, several natural classes of geometric intersection graphs have been considered. On the line, the most frequently studied objects are *interval graphs*: intersection graphs of intervals. They serve as the simplest examples of *perfect graphs*, that is, graphs in which the chromatic number of every induced subgraph is the same as the *clique number* (the size of the largest clique). We know good characterizations of interval graphs [56],[57] in terms of forbidden subgraphs, and simple linear time algorithms for their recognition [19].

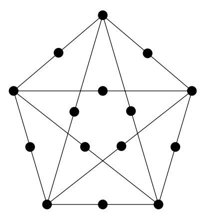
There are various natural generalizations of interval graphs in the plane: intersection graphs of (1) segments, (2) convex sets, (3) arcwise connected sets, etc. The first class is contained in the second, the second class in the third. It is easy to verify that class (3) coincides with the class of string graphs.

Definition 2.1. An intersection graph of simple continuous curves ("strings") in the plane is called a *string graph*.

The *rank* of a string graph G is the smallest integer r such that the vertices of G can be represented by continuous curves in the plane, any two of which intersect in at most r points, so that two vertices of G are connected by an edge if and only if the corresponding arcs intersect.

The investigation of string graphs was initiated by Benzer [16] and Sinden [107], in connection with genetic structures and printed electrical circuits.

Sinden [107] showed that the graph of *fifteen* vertices depicted below is not a string graph, therefore, it does not belong to any of the classes (1)-(3). Ten years later Ehrlich, Even, and Tarjan [28] constructed a string graph which is not a segment intersection graph, that is, it belongs to class (3), but not to class (1).



Thus, a string graph cannot contain this 15-vertex graph as an induced subgraph. However, we cannot hope that string graphs have a good characterization in terms of forbidden subgraphs, unless P=NP: it was shown by Kratochvíl [68] that recognizing string graphs is NP-hard, and by Schaefer, Sedgwick, and Štefankovič [105] that it belongs to NP. The problem of recognizing whether a graph is an intersection graph of segments is also known to be NP-hard (in fact, equivalent to the existential theory of reals [69], [104]). Even for relatively simple graphs, it may be a formidable task to decide whether they allow such a representation by segments. For example, it was a longstanding conjecture that every finite planar graph is an intersection graph of segments. It was finally verified by Chalopin and Gonçalves in 2009 [24].

Definition 2.2. A graph property P is called *hereditary* if every induced subgraph of a graph with property P also has property P. The property that G is a string graph obviously satisfies this condition.

The total number of graphs on n labeled vertices is $2^{\binom{n}{2}}$. Since most of them contain an induced subgraph isomorphic to the *fifteen*-vertex graph depicted above, most graphs are not string graphs. Using the extremal theory of graphs with some hereditary property, developed in [32], [102], [18], Pach and Tóth [96] established the following more precise asymptotic results.

Theorem 2.3. [96] *The number of string graphs on n labeled vertices is $2^{(\frac{3}{4}+o(1))\binom{n}{2}}$.*

Theorem 2.4. [96] *For any fixed positive k, the number of string graphs of rank k on n labeled vertices is $2^{o(n^2)}$.*

Every graph which is an intersection graph of segments is a string graph of rank 1. For intersection graphs of segments, we have a much better result, which can be deduced using a theorem of Oleĭnik and Petrovsky [100], Milnor [85], and Thom [116] from real algebraic geometry (see also [15]). The number of n-vertex intersection graphs of segments is $2^{O(n \log n)}$, and the order of magnitude of the exponent is correct. The best known upper bound, $2^{O(n^{3/2} \log n)}$, for the number of n-vertex string graphs of rank 1 is due to Kynčl [72]. The structure of a "typical" string graphs was studied in [61].

The above results can be applied to estimate the number of combinatorially different ways a complete graph on n vertices can be drawn in the plane so that any pair of its edges cross at most r times, where r is a fixed positive integer [96], [72].

3. Ramsey-type properties of intersection graphs

By the quantitative form of Ramsey's theorem, established by Erdős and Szekeres [35], every graph of n vertices has a clique or an independent set of size at least constant times $\frac{1}{2}\log n$. In a seminal paper written in 1989, Erdős and Hajnal [33] showed that the family of all graphs that do not contain a fixed forbidden graph G as an induced subgraph, have much stronger Ramsey-type properties than the family of all graphs. More precisely, they proved the following.

Theorem 3.1. [33] *For any graph G, there exists a constant $c = c(G) > 0$ such that every graph of n vertices that does not contain G as an induced subgraph has a clique or an independent set of size at least $e^{c\sqrt{\log n}}$.*

They raised the question whether one can always find a complete or empty induced subgraph of size n^c. This remains one of the most challenging open problems in Ramsey theory.

A complete bipartite graph with $\lceil n/2 \rceil$ vertices in one class and $\lfloor n/2 \rfloor$ vertices in the other is called a *bi-clique of size n*. Erdős, Hajnal, and Pach [34] proved a bipartite variant.

Theorem 3.2. [34] *For any graph G, there is a constant $c = c(G) > 0$ such that every graph on n vertices that does not contain G as an induced subgraph has a bi-clique of size n^c or the complement of such a bi-clique.*

See [50] for a strengthening of this result.

Obviously, the last two theorems remain true for all hereditary families of graphs, that is, for any family other than the family of all finite graphs that is closed under taking induced subgraphs. The family of string graphs (intersection graphs of continuous curves or arcwise connected sets in the plane) and, hence, the families of all graphs that can be obtained as intersection graphs of segments, convex sets, etc. belong to this category.

In [40], we introduced the following terminology.

Definition 3.3. A family $\mathcal{F}$ of graphs has the

1. *(Weak) Erdős-Hajnal property* if there is a constant $c(\mathcal{F}) > 0$ such that every graph in $\mathcal{F}$ on n vertices contains a clique or an independent set of size $n^{c(\mathcal{F})}$;

2. *Strong Erdős-Hajnal property* if there is a constant $b(\mathcal{F}) > 0$ such that for every graph G in $\mathcal{F}$ on n vertices contains a bi-clique of size $b(\mathcal{F})n$ or the complement of such a bi-clique.

It was shown in [6] that if a hereditary family of graphs has the strong Erdős-Hajnal property, then it also has the Erdős-Hajnal property. The converse is false, as is shown, e.g., by the family of triangle-free graphs. The first nontrivial result showing that a geometric intersection graph has the Erdős-Hajnal property was found by Larman *et al.* [73].

Theorem 3.4. [73] *The intersection graph of n convex sets in the plane has a clique or an independent set of size at least $n^{1/5}$.*

It is enough to assume here that every set of the family is *vertically convex*, that is, a connected set with the property that every vertical line meeting it intersects it in an interval or in a point. It is an interesting open problem to improve the exponent $1/5$ in the theorem. The best known upper bound, due to Kynčl [71], is $\log 8/\log 169 \approx .405$ (cf. [62]), so there is plenty of room for improvement.

The family of intersection graphs of convex sets in the plane also has the *strong* Erdős-Hajnal property [49]. However, the family of intersection graphs of vertically convex sets does not [95]. By definition, any *x-monotone* curve, that is, any continuous curve in the plane such that every vertical line intersects it in at most one point, is vertically convex.

Theorem 3.5. [95] *For every n, there is an n-member family of x-monotone curves in the plane such that neither their intersection graph, nor its complement contains a bi-clique of size at least $cn/\log n$. Here c is an absolute constant.*

If we put an upper bound r on the number of times two curves are allowed to meet, then the corresponding intersection graphs, string graphs of rank r (see Definition 2.1) behave much nicer.

Theorem 3.6. [48] *The family of string graphs of rank r has the strong Erdős-Hajnal property.*

One of the most challenging unsolved problems in this area is to decide whether the family of *all* string graphs has the (weak) Erdős-Hajnal property.

4. Intersection graphs of semialgebraic sets

According to Tietze's theorem [117] cited in the abstract, every finite graph can be obtained as the intersection graph of 3-dimensional convex bodies. This may suggest that there is no hope to generalize the results in the previous section to higher dimensions. Actually, this is not the case. The proof method of Pach-Solymosi [93], where it was first shown that the family of intersection graphs of segments in the plane has the strong Erdős-Hajnal property, can be extended as follows.

Definition 4.1. [15] A *semialgebraic set S* in $\mathbb{R}^d$ is the locus of all points that satisfy a given finite Boolean combination of at most d polynomial equations and inequalities of degree at most d in the d coordinates. (Without loss of generality, these three parameters are bounded by the same integer d.) The *description complexity* of S is the smallest integer d for which S has such a representation.

Every element S of a family $\mathcal{F}$ of semialgebraic sets of constant description complexity d can be represented by a point S^* of a d^*-dimensional Euclidean space (in which the coordinates are, say, the coefficients of the monomials in the polynomials that define S). A graph (binary relation) $R \subset \mathcal{F} \times \mathcal{F}$ is *semialgebraic* if the corresponding set $\{(S^*, T^*) \in \mathbb{R}^{2d^*} \mid S, T \in \mathcal{F},\ (S,T) \in R\}$ is semialgebraic. *Semialgebraic hypergraphs* (*relations* of h variables, h-ary relations) can be defined analogously.

Theorem 4.2. [6] *For any d, the family of all graphs that are associated with a semialgebraic binary relation of description complexity at most d has the strong Erdős-Hajnal property.*

The relation that two semialgebraic sets, $S, T \in \mathcal{F}$, with description complexity d have nonempty intersection is semialgebraic. Thus, we have the following.

Corollary 4.3. [6] *Any family of intersection graphs of (real) semialgebraic sets of constant description complexity has the strong (and, therefore, the weak) Erdős-Hajnal property.*

Basu [14] extended this result for a broader class of algebraically defined sets (o-minimal sets).

An n-vertex graph is called *t-Ramsey* if it contains no clique and no independent set of size at least t. A probabilistic construction of Erdős [31] shows that there are n-vertex graphs that are $2\log n$-Ramsey, but it appears to be a formidable task to find comparably good efficient constructions. The best known polynomial time deterministic algorithm, due to Barak *et al.* [11], produces only $2^{(\log n)^{o(1)}}$-Ramsey graphs. The previous record was held by Frankl and Wilson [51]. Theorem 4.2 above shows that no $n^{o(1)}$-Ramsey graphs can be defined using semialgebraic relations of constant description complexity. This settles a conjecture of Babai [10].

Fox, Gromov, Lafforgue, Naor, and Pach [39] proved the following far-reaching generalization of Theorem 4.2.

Theorem 4.4. [39] *Let $\alpha > 0$, let $\mathcal{F}_1, \ldots, \mathcal{F}_h$ be finite families of semialgebraic sets of constant description complexity, and let R be a fixed semialgebraic h-ary relation on $\mathcal{F}_1 \times \cdots \times \mathcal{F}_h$ such that the number of h-tuples that are related (resp. unrelated) with respect to R is at least $\alpha \prod_{i=1}^{h} |\mathcal{F}_i|$. Then there exists a constant $c' > 0$, which depends on α, h and on the maximum description complexity d of the sets in $\mathcal{F}_i$ $(1 \le i \le h)$ and R, and there exist subfamilies $\mathcal{F}'_i \subseteq \mathcal{F}_i$ with $|\mathcal{F}'_i| \ge c'|\mathcal{F}_i|$ $(1 \le i \le h)$ such that $\mathcal{F}'_1 \times \cdots \times \mathcal{F}'_h \subseteq R$ (resp. $(\mathcal{F}'_1 \times \cdots \times \mathcal{F}'_h) \cap R = \emptyset$). Moreover, each subset $\mathcal{F}'_i$ consists of exactly those elements of $\mathcal{F}_i$ that satisfy a certain semialgebraic relation of constant description complexity.*

Apart from the fact that the last statement also handles semialgebraic hypergraphs (h-ary relations), it also strengthens Theorem 4.2 in another direction. It is not just a Ramsey-type theorem, which guarantees that at least one of two or several possibilities will occur. It is a so-called "density theorem," which tells us that if sufficiently many h-tuples are related by the relation R (that is, the h-uniform semialgebraic hypergraph R has sufficiently many hyperedges), then there are h large subsets $\mathcal{F}'_i \subseteq \mathcal{F}_i$ $(1 \le i \le h)$ such that no matter how we pick an element from each, the resulting h-tuple is related (is a hyperedge of R).

By repeated application of this statement, one can obtain an even stronger Szemerédi-type partition theorem. An *equipartition* of a finite set P is a partition $P = P_1 \cup \ldots \cup P_k$ into almost equal parts. That is, $|P_i| = \lfloor |P|/k \rfloor$ or $\lceil |P|/k \rceil$ for every i.

Theorem 4.5. [39] *For any h, d and for any $\varepsilon > 0$, there exists $K = K(\varepsilon, h, d)$ satisfying the following condition. For any $k \ge K$, for any semialgebraic relation R on h-tuples of points in a Euclidean space $\mathbb{R}^d$ with description complexity at most d, every finite set $P \subseteq \mathbb{R}^d$ has an equipartition $P = P_1 \cup \ldots \cup P_k$ such that all but at most an ε-fraction of the h-tuples $(P_{i_1}, \ldots, P_{i_h})$ have the property that either all r-tuples of points with one element in each P_{i_j} are related with respect to R or none of them are.*

The investigation of semialgebraic versions of Ramsey's theorem for h-ary relations was initiated in [25]. Let $N_h^d(n)$ be the smallest integer N such that for any semialgebraic relation R on h-tuples of N points in $\mathbb{R}^d$ with description complexity at most d, there is a *homogeneous* subset of size n, that is, a subset with the property that either all of its h-tuples belong to R or none of them does. It was shown that the function $N_h^d(n)$ grows in n as a tower of height $h-1$, and that in some sense this result is optimal. This is one exponential better than the behavior of the general Ramsey function for arbitrary h-ary relations.

For some related results and geometric applications, see [39], [20], [13], [30], [12], [111].

5. Intersection graphs and partially ordered sets

Given a partially ordered set $(P, <)$, its *incomparability graph* is the graph with vertex set P, in which two elements are adjacent if and only if they are incomparable. Incomparability graphs are fairly well understood. In 1950, Dilworth [26] proved that every incomparability graph is a perfect graph, so the chromatic number of an incomparability graph is equal to its clique number. Gallai [56] gave a characterization of incomparability graphs in terms of minimal forbidden induced subgraphs, and there exist polynomial time algorithms to recognize them [58].

There is a curious relation between incomparability graphs and string graphs (Definition 2.1), which was first observed by Golumbic, Rotem, and Urrutia [59] and, independently, by Lovász [76].

Theorem 5.1. [59], [76] *Every incomparability graph is a string graph.*

The converse is obviously not true. For example, a cycle of length *five* is a string graph, but it is not perfect, therefore, it cannot be an incomparability graph. Kleitman and Rothschild [64] showed that the number of incomparability graphs on n vertices is only $2^{(1/2+o(1))\binom{n}{2}}$, which is much smaller then the number of string graphs, asymptotically given in Theorem 2.3.

Nevertheless, it was shown by Fox and Pach [45] that most string graphs contain huge subgraphs that are incomparability graphs. The geometric conditions somehow seem to enforce a partial order on the curves.

Theorem 5.2. *For every $\varepsilon > 0$ there exists $\delta > 0$ with the property that if $\mathcal{F}$ is a family of curves whose string graph has at least $\varepsilon|\mathcal{F}|^2$ edges, then one can select a subcurve γ' of each $\gamma \in \mathcal{F}$ such that the string graph of the family $\{\gamma' : \gamma \in \mathcal{F}\}$ has at least $\delta|\mathcal{F}|^2$ edges and is an incomparability graph.*

This implies that every dense string graph contains a dense *spanning* subgraph (i.e., a dense subgraph on the same vertex set) which is an incomparability graph. However, it is not true that every dense string graph contains a dense *induced* subgraph with a linear number of vertices that is an incomparability graph. Indeed, since every incomparability graph is perfect, this would imply that every string graph has a clique or an independent set of size at least constant times $\sqrt{n}$. This is certainly false, e.g., for the construction of Kynčl, mentioned after Theorem 3.4.

Fox [38] proved that incomparability graphs "almost" have the strong Erdős-Hajnal property.

Theorem 5.3. [38] *If n is large enough, the incomparability graph of every n-element partially ordered set, or its complement, the comparability graph, has a bi-clique of size at least $\frac{n}{4\log_2 n}$. This bound is tight up to a constant factor.*

The second part of this statement, combined with Theorem 5.1, immediately implies that the family of string graphs does not have the strong Erdős-Hajnal property (which was Theorem 3.5).

In [42], Theorem 5.3 was generalized to several partial orders. Note that the proof of Theorem 3.4 is based on the fact that on any family of (vertically) convex sets in the plane, we can define *four* partial orders so that two sets have nonempty intersection if and only if they are incomparable by all of them. Using the generalized version of Theorem 5.3, we obtain that the intersection graphs of (vertically) convex sets in the plane also "almost" have the strong Erdős-Hajnal property. As was mentioned in Section 3, for convex sets a stronger statement is true (which does not hold under the weaker assumption of vertical convexity).

Theorem 5.4. [49] *The family of intersection graphs of finitely many convex sets in the plane has the strong Erdős-Hajnal property.*

The *dimension* of a partially ordered set $(P, >)$ is the minimum number of linear extensions of the relation "$>$" such that their intersection is "$>$." For the proof of Theorem 5.4, one has to consider a new type of extremal problem for incomparability graphs: What is the maximum number of edges that an n-vertex incomparability graph of a partial order of dimension d can have if it does not contain, say, a complete bipartite subgraph $K_{r,r}$, for a fixed r? The same question can be asked about comparability graphs and also for the case where the condition on the dimension is dropped.

In the same paper, a stronger form of Theorem 5.3 was proved for dense graphs.

Theorem 5.5. [49] *For every $\varepsilon > 0$, there exists $\delta > 0$ such that every incomparability graph with n vertices and at least εn^2 edges contains a bi-clique of size $\delta n/\log n$.*

Combining this result with Theorem 5.1, we obtain

Corollary 5.6. *For every $\varepsilon > 0$, there exists $\delta > 0$ such that every string graph with n vertices and at least εn^2 edges contains a bi-clique of size $\delta n/\log n$.*

The formulation of Theorem 5.3 may suggest a certain kind of symmetry between incomparability and comparability graphs. However, Theorem 5.1 has no analogue for comparability graphs. The following strengthening of Theorem 5.3 is also slightly asymmetric.

Theorem 5.7. [49] *There is constant $c > 0$ such that the incomparability graph of every n-element partially ordered set has a bi-clique of size at least $cn/\log n$, or its complement, the comparability graph, has a bi-clique of size at least cn.*

6. Intersection graphs and planar separators

Given a family of continuous curves (strings) in the plane, introducing a vertex at each intersection point and each endpoint of the curves, we obtain a planar graph. Under some fairly natural conditions, there are few strings that connect far-away parts of this planar graph. In such cases, there is a good chance that we can use the *Lipton-Tarjan separator theorem* for planar graphs [75].

A *separator* for a graph $G = (V, E)$ is a subset $V_0 \subset V$ such that there is a partition $V = V_0 \cup V_1 \cup V_2$ with $|V_1|, |V_2| \le \frac{2}{3}|V|$ and no vertex in V_1 is adjacent to any vertex in V_2. The Lipton-Tarjan separator theorem states that every planar graph with n vertices has a separator of size $O(\sqrt{n})$. By a classical theorem of Koebe [65], every planar graph can be represented as the intersection graph of closed disks in the plane with disjoint interiors. Miller, Teng, Thurston, and Vavasis [84] found a generalization of the Lipton-Tarjan separator theorem to higher dimensions. They proved that the intersection graph of any family of n balls in $\mathbb{R}^d$ such that no k of them have a point in common has a separator of size $O(dk^{1/d}n^{1-1/d})$.

Fox and Pach [41] established the following common generalization of the separator theorems of Lipton and Tarjan and of Miller *et al.* in the plane.

Theorem 6.1. [41] *If $\mathcal{F}$ is a finite family of Jordan regions with a total of m boundary crossings, then the intersection graph of $\mathcal{F}$ has a separator of size $O(\sqrt{m})$.*

Corollary 6.2. [41] *If $\mathcal{F}$ is a finite family of curves in the plane with a total of m crossings, then the intersection graph (string graph) of $\mathcal{F}$ has a separator of size $O(\sqrt{m})$.*

Using Theorem 6.1 and Theorem 1.4, one can deduce the following.

Theorem 6.3. [41] *Every K_k-free intersection graph of convex bodies in the plane with m edges has a separator of size $O(\sqrt{km})$.*

Notice that in this statement, the size of the separator is bounded in terms of the number of edges of the intersection graph, rather than the number of vertices. Nevertheless, since planar graphs are K_5-free and (by Koebe's theorem) can be obtained as intersection graphs of convex bodies, Theorem 6.3 also implies the Lipton-Tarjan separator theorem.

Fox and Pach [43] made the following conjecture, much stronger than Corollary 6.2.

Conjecture 6.4. [43] *Every string graph with m edges has a separator of size $O(\sqrt{m})$.*

In [43], a weaker bound, $O(m^{3/4}\sqrt{\log m})$, was established. This bound was used to deduce the following interesting property of string graphs. Let $K_{k,k}$ denote the complete bipartite graph with k vertices in each of its classes (that is, a bi-clique of size $2k$).

Theorem 6.5. [43] *For any positive integer k, there is a constant $c(k)$ such that every $K_{k,k}$-free string graph with n vertices has at most $c(k)n$ edges.*

This is in sharp contrast with the general behavior of graphs. According to the Kővári-Sós-Turán theorem [67], for a fixed k, every $K_{k,k}$-free graph with n vertices has at most $O(n^{2-1/k})$ edges. For $k > 2$, this bound is not known to be optimal, but the right exponent is definitely at least $2 - 2/k > 1$ (see, e.g., [17]). It is a rich and active subfield of extremal graph theory to estimate the maximum number of edges of a B-free graph of n vertices, for a given bipartite graph B. Theorem 6.5 shows that for string graphs there is no such theory: no matter what B is, the maximum is $O(n)$.

Matoušek [83] came close to proving Conjecture 6.4. He adapted some powerful techniques developed by Feige, Hajiaghayi, and Lee [36], who used the framework of multicommodity flows to design efficient approximation algorithms for finding small separators. See also [66].

Theorem 6.6. [83] *Every string graph with m edges has a separator of size at most $O(\sqrt{m}\log m)$.*

In [46], the last theorem was utilized to deduce that Theorem 6.5 is true with $c(k) = k(\log k)^{O(1)}$, which is not far from being optimal. It is conjectured that the best possible value of $c(k)$ for which the theorem still holds satisfies $c(k) = O(k \log k)$.

7. The theory of topological graphs

It was probably Erdős who first suggested in the 1960s that some of the basic questions in extremal graph theory have natural analogues for geometric or topological graphs. For instance, what is the maximum number of edges that a geometric graph of n vertices can have without containing a fixed "forbidden" configuration, that is, a set of edges such that their intersection pattern is specified. The first such result, in which the forbidden configuration consisted of 2 disjoint edges (that cannot have any endpoints or internal points in common) was published by Avital and by Erdős's close friend, Hanani [9]. The answer is n. Thirteen years later, in his master's thesis [70], Kupitz started to explore these questions systematically. Alon and Erdős [7] proved that every geometric graph with no 3 disjoint edges has $O(n)$ edges. The first general bound was established in [98] and uses partial orders.

Theorem 7.1. [98] *For any integer $k \geq 2$, the maximum number of edges of a geometric graph with n vertices that contains no k disjoint edges is $O_k(n)$.*

The best known value of the constant hidden in the O_k-notation is $O(k^2)$ (see [118]). It is perfectly possible that this bound can be improved to $O(k)$, which would be best possible.

It is conjectured that Theorem 7.1 remains true for simple topological graphs, i.e., for topological graphs in which every pair of edges intersect in at most one point (Definition 1.1). For the case $k = 2$, Conway made the following stronger conjecture, which has become known as the "thrackle conjecture".

Conjecture 7.2. [123] *Every simple topological graph with $n \geq 3$ vertices that contains no 2 disjoint edges has at most n edges.*

It is known that every such graph has a linear number of edges in n (see [78], [21], [52]). The thrackle conjecture has been verified for simple topological graphs with x-monotone edges ([94], cf. Theorem 3.5) and in the case where all vertices lie on a circle and all edges in its interior [22].

We do not know whether the analogue of Theorem 7.1 is true for simple topological graphs, when $k \geq 3$. All we know is that, according to [97], the maximum number of edges of a simple topological graph with n vertices that contains no k disjoint edges is $n(\log n)^{O(k)}$. In the most optimistic scenario, this bound could be improved to $O(kn)$. Suppose that this is the case. This would imply that a complete simple topological graph with n vertices (and $\binom{n}{2}$ edges) must have at least cn disjoint edges, for a suitable constant $c > 0$. Suk [109] proved a weaker bound.

Theorem 7.3. [109] *Every complete simple topological graph with n vertices must have at least $cn^{1/3}$ disjoint edges, for a suitable constant $c > 0$.*

An alternative proof of this theorem was found by Fulek and Ruiz-Vargas [53]. Ruiz-Vargas has recently announced the improved bound $cn^{1/2-\varepsilon}$, for every $\varepsilon > 0$. Both proofs break down if we want to extend Theorem 7.3 to all *dense* simple topological graphs, that is, to graphs with at least δn^2 edges for some $\delta > 0$. We cannot generalize this statement even for complete *bipartite* simple topological graphs.

In Section 1, we considered the "dual" problem, where the forbidden configuration consists of k pairwise *crossing* edges. Recall that topological graphs with no k pairwise crossing edges are called *k-quasiplanar* (see Definition 1.1). What is the maximum number of edges that a k-quasiplanar topological graph of n vertices can have? The conjectured answer is $O_k(n)$ (or perhaps even $O(kn)$; cf. Conjecture 1.2). As was mentioned in Section 1, this is known to be true only for $k \leq 4$. Presently, the best upper bound is $n(\log n)^{O(\log k)}$.

If the stronger conjecture was true, i.e., every k-quasiplanar graph of n vertices had at most $O(kn)$ edges, it would follow that every complete topological graph of n vertices has at least cn pairwise crossing edges, for a suitable constant $c > 0$. For geometric graphs, Aronov *et al.* [8] established a weaker statement, dual to Theorem 7.3: Every complete geometric graph with n vertices must have at least $cn^{1/2}$ pairwise crossing edges, for a suitable constant $c > 0$. A similar statement holds for all reasonably dense topological graphs, in which any pair of edges intersect at most a bounded number of times.

Theorem 7.4. [44] *For every $\varepsilon > 0$ and for every integer $t > 0$, there exists $\delta = \delta(\varepsilon, t) > 0$ with the following property. Every topological graph with n vertices, in which no two edges intersect in more than t points, has at least n^{δ} pairwise crossing edges.*

It follows from the results in [46] that if we drop the assumption in the last theorem that every pair of edges intersect in at most t points, then we can guarantee the existence of only $n^{\delta/\log\log n}$ pairwise crossing edges.

More complicated forbidden configurations have also been considered. For instance, let k be a positive integer and let G be a geometric graph with n vertices that contains no two sets of edges, $E_1, E_2 \subset E(G)$, each consisting of k pairwise crossing edges, such that every edge in E_1 is disjoint from every edge in E_2.

Fulek and Suk [54] proved that then G has at most $O_k(n \log n)$ edges, and they conjectured that the correct order of magnitude is linear for every fixed k.

Let k and l be fixed positive integers. A (k,l)-*grid* in a topological graph is a pair of subsets, $E_1, E_2 \subset E(G)$, with $|E_1| = k, |E_2| = l$ such that every edge in E_1 crosses every edge in E_2. If, in addition, each E_i consists of disjoint edges, the (k,l)-grid is called *natural*. It is known that every n-vertex topological graph with no (k,l)-grid has $O_{k,l}(n)$ edges [89], [115].

Conjecture 7.5. [2] *For any positive integers k and l, there exists a constant $c_{k,l}$ such that every simple topological graph on n vertices with no natural (k,l)-grid has at most $c_{k,l}n$ edges.*

This conjecture would immediately imply that Theorem 7.1 generalizes to simple topological graphs. It would also imply that every simple topological graph on n vertices which contains no (k,l)-grid such that all $2(k+l)$ endpoints of its edges are distinct, has at most $O_{k,l}(n)$ edges. We cannot even verify this weaker conjecture. We can prove only the following.

Theorem 7.6. [2] *For every positive integer k, there is a constant c_k such that every topological graph on n vertices that contains no (k,k)-grid with distinct vertices has at most $c_k n \log^* n$ edges, where $\log^*$ denotes the iterated logarithm function.*

It was already pointed out by Klazar and Marcus [63], in a slightly different formulation, that the proof of the Marcus-Tardos theorem [80] can be easily modified to prove that Conjecture 7.5 is true for *convex* geometric graphs, that is, for geometric graphs whose vertices form the vertex set of a convex n-gon.

The above mentioned results and conjectures might suggest that for every nontrivial forbidden configuration F of a fixed size, the maximum number of edges that an F-free geometric or topological graph with n vertices can have is linear in n. However, this is not the case. It was shown in [90] that the maximum number of edges of a geometric graph with n vertices, containing no self-intersecting path of length 3, is at most $cn \log n$ for a suitable constant c, and that the order of magnitude of this bound cannot be improved. This result was extended by Tardos [114]: for every $k \geq 3$, he constructed geometric graphs with a superlinear number of edges that contain no self-intersecting path of length k. As a corollary, one can obtain a simple characterization of all abstract graphs G, for which all geometric graphs with n vertices that contain no self-intersecting subgraph isomorphic to G have $O(n)$ edges: these graphs are forests with at least two components that are not isolated vertices.

Note that there exist arbitrarily large (abstract) graphs with a superlinear number of edges that contain no cycle of a fixed length k. For example, it is well known that for $k = 4$, there are C_4-free graphs with n vertices and $(\frac{1}{2} + o(1))n^{3/2}$ edges (see [17], [55]). On the other hand, improving an argument of Pinchasi and Radoičić [101], Marcus and Tardos [81] obtained the following almost tight result.

Theorem 7.7. [81] *Every topological graph on n vertices that contains no self-intersecting cycle of length 4 has at most $O(n^{3/2} \log n)$ edges.*

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