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POINTS OF MULTIPLICITY $\mathfrak{c}$ OF PLANE BROWNIAN PATHS*

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In a previous paper [2] we proved that almost all Brownian paths in the plane have points of arbitrary high finite multiplicity. In the present paper this result is strengthened by establishing the following:

THEOREM. *Almost all Brownian paths in the plane have points of multiplicity* $\mathfrak{c}$.

Here $\mathfrak{c}$ is the power of the continuum and a path $z(t;\omega)$, $0 \leqslant t < \infty$, is said to have a point of multiplicity $\mathfrak{c}$ if there exist a point ζ and a set T of positive numbers having the power of the continuum such that $z(t;\omega) = \zeta$ for all $t \in T$.

This result, combined with those of the previous papers [1] and [3], completely settles the question of points of highest multiplicity of Brownian paths in m-dimensional space. Thus the following holds with probability 1: For $m \geqslant 4$ the path contains only simple points, for $m = 3$ it contains double but no triple points, while for $m = 2$ (and also, of course, for $m = 1$) it contains points of multiplicity $\mathfrak{c}$.

We shall not repeat here the definition of Brownian motion in the plane. Suffice it to say that we are considering a probability space (Ω, E, P) such that with every $\omega \in \Omega$ there is associated a function $z(t;\omega) = [x(t;\omega), y(t;\omega)]$ from $0 \leqslant t < \infty$ into the plane, where the components x and y represent ordinary independent one-dimensional Brownian motions. A fuller description may be found in [2]. We recall, however, that the process is assumed separable, i.e. $z(t;\omega)$ is, with probability 1, continuous. Also the normalizations $z(0;\omega) = 0$ and $\sigma = 1$ in $E\,x^2(t;\omega) = E\,y^2(t;\omega) = \sigma^2 t$ are assumed; these have, of course, no bearing on the validity of the Theorem, but are used in the estimates leading to its proof.

We shall use vector notation in the plane, in particular $|\zeta - \zeta'|$ denotes the distance of the two points ζ and ζ'. For $0 \leqslant a < b \leqslant \infty$ we shall write $L(a,b;\omega)$ for the set of points $z(t;\omega)$, $a \leqslant t < b$. $P\{\}$ stands for the probability of the event in the braces, while c_1, c_2, c_3 are absolute positive constants.

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The proof will be achieved in stages. We start with:

(A). *Let* $\varrho > 0$ *and* α, β_1, β_2 *be points in the plane satisfying*

$$|\alpha| < 1/100, \ e\varrho < |\beta_1 - \beta_2| < 1, e\varrho < |\beta_i - \alpha| < 1 \ \ (i = 1,2).$$

Denote by A_i $(i = 1,2)$ *the event: there exist* t, t' *with*

$$0 \leqslant t \leqslant 1, \qquad 1/2 \leqslant t' - t \leqslant 1,$$

satisfying

$$|\alpha + z(t; \omega) - \beta_i| < \varrho, \ \ |\alpha + z(t'; \omega) - \beta_i| < \varrho.$$

Then we have

$$P\{A_i\} > c_1/\log^2(1/\varrho), \tag{1}$$

$$P\{A_1 \cap A_2\} < c_2\left[1 + \log(1/|\beta_1 - \beta_2|)\right]^4/\log^4(1/\varrho). \tag{2}$$

These estimates result immediately from the inequalities preceding (27) and (28) in [2] with $k = 2$ (the c's in the present paper are not, of course, the same as in [2]).

(B). *Let* k *be a positive integer and let* $\omega_1, \omega_2, \ldots, \omega_k$ *be chosen independently in* Ω. *Let* n *be a sufficiently large positive integer.*

Put

$$\nu = (\nu_1 - 1)\, n + \nu_2, \quad (\nu_i = 1,2, \ldots, n;\ i = 1,2) \tag{3}$$

and let ζ_ν *denote the point.*

$$\zeta_\nu = \left[(1/5) + (\nu_1/2n), (1/5) + (\nu_2/2n)\right], \ (\nu = 1, \ldots, n^2). \tag{4}$$

Let a_j, $(j = 1, \ldots, k)$ *be points satisfying* $|a_j| < 1/100$ *and denote by* B *the event: there exist, for all* $j = 1,2, \ldots, k$, *numbers* t_j, t'_j *with*

$$0 \leqslant t_j \leqslant 1, \qquad 1/2 \leqslant t'_j - t_j \leqslant 1, \tag{5}$$

for which

$$|a_j + z(t_j; \omega_j) - \zeta_\nu| < \varrho, \qquad |a_j + z(t'_j; \omega_j) - \zeta_\nu| < \varrho. \tag{6}$$

Then, if

$$\varrho = e^{-n^{1/k}H}. \qquad H = (c_2/c_1)^{1/2}\,(20k)^2, \tag{7}$$

we have

$$P\left\{\bigcup_{\nu=1}^{n^2} B_\nu\right\} > c_3(k), \text{ where } c_3(k) = (c_3/k)^{4k} \tag{8}$$

Indeed, due to the independence assumption, we have from (1):

$$P\,\{B_\nu\} > c_1^k\, \log^{-2k}\varrho \tag{9}$$

while, since by (4) $|\zeta_\nu - \zeta_{\nu'}| < 2^{-1/2}$, we have from (2):

$$P\{B_\nu \cap B_{\nu'}\} < c_2^k \,(5 \log |\zeta_\nu - \zeta_{\nu'}|)^{4k} \log^{-4k} \varrho, \quad (\nu \neq \nu'). \tag{10}$$

Substituting (9) and (10) in

$$P\{\bigcup_{\nu=1}^{n^2} B_\nu\} \geqslant \sum_{\nu=1}^{n^2} P\{B_\nu\} - \sum_{1 \leqslant \nu < \nu' \leqslant n^2} P\{B_\nu \cap B_{\nu'}\},$$

we obtain

$$P\{\bigcup_{\nu=1}^{n^2} B_\nu\} > n^2 c_1^k \log^{-2k} \varrho - (5^4 c_2)^k \log^{-4k} \varrho \sum_{1 \leqslant \nu < \nu' \leqslant n^2} \log^{4k} |\zeta_v - \zeta_{v'}|. \tag{11}$$

Now by (4)

$$\sum_{1 \leqslant \nu < \nu' \leqslant n^2} \log^{4k} |\zeta_\nu - \zeta_{\nu'}| < n^2 \sum_{\nu=2}^{n^2} \log^{4k} |\zeta_\nu - \zeta_1| < n^2 \sum_{\nu=2}^{n} (2\nu - 1) \log^{4k} |\zeta_\nu - \zeta_1|$$

$$= n^2 \sum_{\nu=2}^{n} (2\nu - 1) \log^{4k} [2n/(\nu - 1)]$$

$$< 3n^2 \sum_{\nu=2}^{n} (\nu - 1) \log^{4k} [2n/(\nu - 1)]$$

$$< 3n^3 \max_{1 \leqslant u \leqslant n} [u \log^{4k} (2n/u)]$$

$$= 6n^4 (4k/e)^{4k}.$$

From this, (7), and (11) we have:

$$P\{\bigcup_{\nu=1}^{n^2} B_\nu\} > c_1^k H^{-2k} - 6(5c_2)^k (4k/e)^{4k} H^{-4k},$$

which, in view of (7), yields (8) and completes the proof of (B).

Next we deduce:

(C). *Let $\omega_1, \ldots, \omega_k$ be chosen independently and let the points a_j, $(j = 1, \ldots, k)$ satisfy $|a_j| < 1/100$, then there exist, with probability not less than c_3^k, a point ζ and numbers t_j, t'_j $(j = 1, \ldots, k)$ satisfying* (5) *such that*

$$a_j + z(t_j; \omega_j) = a_j + z(t'_j; \omega_j) = \zeta, \quad (j = 1, \ldots, k).$$

To see this, let

$$g(\omega_1, \ldots, \omega_k) = \inf \sum_{j=1}^{k} [| a_j + z(t_j; \omega_j) - \zeta | + | a_j + z(t'_j; \omega_j) - \zeta |], \quad (12)$$

the inf being taken over all t_j, t'_j satisfying (5) and all points ζ in the plane. Define $g_n(\omega_1, \ldots, \omega_k)$ similarly, except that instead of letting the inf in (12) be taken over all points, let it be taken only over $\zeta = \zeta_\nu$ given by (4); clearly $g \leqslant g_n$, and it is easily seen that g and g_n are random variables. From (B) it follows that $P\{g_n < 2k\varrho\} > c_3(k)$ for sufficiently large n. Since, by (7), $\varrho \to 0$ as $n \to \infty$, it follows that $P\{g = 0\} \geqslant c_3(k)$. This is precisely the required result.

From (C) we conclude that independent Brownian paths have a common double point, or formulated more explicitly:

(D). *Let $\omega_1, \ldots, \omega_k$ be chosen independently. Then there exists, with probability* 1, *a common double point for all the paths $L(0,\infty; \omega_j)$, $(j = 1, \ldots, k)$.*

Indeed, due to the well-known ergodic character of Brownian paths in the plane (e.g. see [4]), there exist with probability 1 sequences $T_{j,m}$ $(j = 1, \ldots, k; m = 0,1,2\ldots)$. satisfying $T_{j,0} = 0$ and $T_{j,m+1} \geqslant T_{j,m} + 2$ such that $|z(T_{j,m}; \omega_j)| < 1/100$. Let $C_m (m = 0,1,\ldots)$ denote the event: there exist for all $j = 1, \ldots, k$ numbers t_j, t_j' satisfying (5) for which all $2k$ points $z(T_{j,m} + t_j; \omega_j)$, $z(T_{j,m} + t'_j; \omega_j)$ coincide. By the Markovian character of the Brownian process the events C_m are independent, while, by (C), $P\{C_m\} \geqslant c_3(k)$ for all m. An application of the Borel lemma gives $P\{\bigcup_{m=1}^{\infty} C_m\} = 1$, thus proving (D).

From the homogeneity property of the Brownian process (invariance in respect to simultaneous change of the space scale by a factor λ and of the time scale by a factor λ^2), it follows that the probability that the k paths $L(0,\lambda^2; \omega)$ have a common double point is independent of λ^2. This permits strengthening (D) to:

(E). *Let $\omega_1, \ldots, \omega_k$ be chosen independently and let $\varepsilon > 0$ be arbitrary. Then there exists, with probability* 1, *a common double point of the k paths $L(0, \varepsilon; \omega_j)$.*

A straightforward application of Fubini's theorem yields the following result about conditional probabilities:

(F). *Let Q denote the event described in* (E), *then*

$$P\{Q \mid z(\varepsilon; \omega_j) = z_j, (j = 1, \ldots, k)\} = 1$$

for almost all points $z_1, \ldots, z_k$.

Here the bar denotes conditional probability and the "almost all" may be understood in the usual Lebesque sense. Next we prove:

(G). *With probability* 1 *there exist numbers* $t_n(j)$, $(n = 0,1, \dots ; j = 1, \dots 2^n)$ *satisfying* $t_0(1) = 0$, *and*

$$t_n(j) < t_{n+1}(2j-1) < t_{n+1}(2j) < t_n(j+1)$$
$$t_{n+1}(2j) - t_n(j) \leqslant 2^{-n}$$

(with $t_n(j+1)$ *replaced by* 1 *for* $j = 2^n$*), for which*

$$z[t_n(1); \omega] = z[t_n(2); \omega] = \dots = z[t_n(2^n); \omega]. \tag{13}$$

Assume the existence of the sets $t_m(j)$, $(j = 1, \dots, 2^n)$ for $m \leqslant$ n has already been established. Let α_n denote the common point (13) and put

$$\varepsilon = 2^{-1} \min_{1 \leqslant j \leqslant 2^n} [t_n(j+1) - t_n(j)]$$

where again we understand $t_n(2^n+1) = 1$. Since the paths $L[t_n(j), t_n(j) + \varepsilon; \omega]$, $(j = 1, \dots, 2^n)$ are transformed, on substracting α_n from them, by a measure preserving transformation into independent paths $L(0, \varepsilon; \omega_j)$, subject to the conditions $z(\varepsilon; \omega_j) = z[t_n(j) + \varepsilon; \omega] - \alpha_n$, it follows from (F) that, with probability 1, there exists a common double point of the 2^n paths $L(t_n(j), t_n(j) + \varepsilon; \omega)$. Thus we have established the existence, with probability 1, of the set $t_{n+1}(j)$, $(j = 1,2, \dots, 2^{n+1})$, and (G) follows by induction.

We are now in a position to complete the proof of our Theorem. Let ω be such that $z(t; \omega)$ is continuous and that there exist for it $t_n(j)$ as described in (G). Let α_n again denote the 2^n-tuple point (13). Obviously $\alpha_n \in L(0,1; \omega)$, and since $L(0,1;\omega)$ is bounded, there exists a sub-sequence $\alpha_{n_p}(p = 1,2, \dots)$ converging to a limit α. Let the set of non-negative numbers T be defined by

$$T = \bigcap_{q=1}^{\infty} \overline{\bigcup_{p=q}^{\infty} \{t_{n_p}(j), j = 1, \dots, 2^{n_p}\}} ,$$

(the bar denoting closure). Then T is a perfect set and for every $t \in T$ there exists a sequence $n(m)$ increasing to infinity and integers j_m with $1 \leqslant j_m \leqslant 2^{n(m)}$, for which $t_{n(m)}(j_m) \to t$ as $m \to \infty$. From the continuity of the path it follows that $z(t; \omega) = \alpha$ for every $t \in T$. Since T is of the power of the continuum, the required result follows from (G). Q.E.D.

From the homogeneity of the Brownian process we immediately deduce:

COROLLARY 1. *For every* $0 \leqslant a < b \leqslant \infty$ *the points of multiplicity* $\mathfrak{c}$ *of* $L(a, b; \omega)$ *are, with probability* 1, *everywhere dense in* $L(a, b; \omega)$.

Taking account of the ergodic character of Brownian paths in the plane, we have, in particular:

COROLLARY 2. *For almost all Brownian paths the set of points of multiplicity* $\mathfrak{c}$ *is everywhere dense in the entire plane.*

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