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MATHEMATICS

A THEOREM ON THE RIEMANN INTEGRAL

BY

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Recently DE BRUIJN communicated to me the following conjecture: *Let $f(x)$, $-\infty < x < \infty$ be a real function. Assume that for all h*

(1) $$\int_{-\infty}^{\infty} |f(x+h) - f(x)|\, dx = 0,$$

where all integrals in this paper are understood to be Riemann integrals. Then for a certain constant c

(2) $$\int_{-\infty}^{\infty} |f(x) - c|\, dx = 0.$$ [1])

DE BRUIJN and I proved this conjecture almost simultaneously. In fact DE BRUIJN proved a good deal more. But perhaps my direct and simple proof is not entirely without interest.

Capital letters will denote sets of real numbers, small letters will denote real numbers. $a \in B$ means that a is in B. $\bar{A}$ will denote the complement of A. $A + x$ denotes the translation of the set A by x, $A \subset B$ means that A is contained in B and $A \cap B$ denotes the intersection of A and B. $\bigcup_{k=1}^{\infty} A_k$ denotes the union of the sets $A_1, A_2, \ldots$.

A set S is said to be of the first category if it is the union of countably many nowhere dense sets. A set not of the first category is called a set of second category. It is well known (theorem of BAIRE) that an interval is of second category.

First of all we make two remarks:

1) If $\int_{-\infty}^{\infty} |g(x)| dx = 0$ then $g(x)$ must be 0 except in a set of first category. For if not then for some k the set in x satisfying $|g(x)| > {}^1/_2 k$ must be dense in some interval, which implies that $\int_{-\infty}^{\infty} |g(x)| dx \neq 0$.

2) Let $\int_{-\infty}^{\infty} |r(x)| dx \neq 0$ [2]). Then there exists a countable set $\{y_n\}$ so that if we define

$$r^+(x) = \begin{cases} r(x) & \text{if } x = y_n \\ 0 & \text{otherwise} \end{cases}$$

[1]) The analogous statement for Lebesgue integrals is false, see N. G. DE BRUIJN, Nieuw Archief voor Wiskunde (2) **23**, 194–218 (1951).

[2]) This notation means: either the integral does not exist in the Riemann sense, or it does exist but is $\neq 0$.

we have $\int_{-\infty}^{\infty} |r^+(x)|\, dx \neq 0$. This remark is evident from the definition of the Riemann integral as the limit of sums.

Denote by $E(a, b)$ the set in x for which

$$a \leqslant f(x) \leqslant b.$$

Assume first that there are two disjoint intervals (a, b) and (c, d), $a < b < c < d$ so that $E(a, b)$ and $E(c, d)$ are both of second category. We then show that (1) can not be satisfied.

Let $\{ y_n \}$ be an arbitrary countable set dense in some interval. First we show that there exists a $z \in E(a, b)$ so that for all n $z + y_n \in E(a, b)$. If this were not so

$$E(a, b) \subset \bigcup_{n=1}^{\infty} (\overline{E(a, b)} - y_n). \tag{3}$$

But since $E(a, b)$ is of second category it follows from (3) that for some n

$$E(a, b) \cap (\overline{E(a, b)} - y_n)$$

is of second category, or $f(x - y_n) - f(x) \neq 0$ on a set of second category, which means by remark 1 that (1) is not satisfied for $h = - y_n$.

Similarly there exists $w \in E(c, d)$ so that for all n, $w + y_n \in E(c, d)$. But then

$$f(x + w - z) - f(x) \geqslant c - b \quad \text{for} \quad x = z + y_n, \ n = 1, 2, \ldots . \tag{4}$$

(i.e. if $x = z + y_n$, $f(x)$ is in (a, b) and $f(x + w - z) = f(w + y_n)$ is in (c, d).

Since $\{z + y_n\}$ is dense in some interval (4) clearly contradicts (1) for $h = w - z$.

Let us next assume that there are no two disjoint intervals (a, b) and (c, d) so that both $E(a, b)$ and $E(c, d)$ are of second category. Then there clearly exists a sequence of nested interval (a_k, b_k) with $b_k - a_k \to 0$, so that $\overline{E(a_k, b_k)}$ is of first category. Denote by t the intersection of these intervals, and by $E(t)$ the set of points by satisfying $f(y) = t$. Clearly

$$\overline{E}_t = \bigcup_{k=1}^{\infty} \overline{E(a_k, b_k)}$$

is of first category, or $f(x) = t$ except for a set of first category.

Now we use remark 2. If

$$\int_{-\infty}^{\infty} |f(x) - t|\, dx \neq 0$$

then there exists a countable set $\{y_n\}$ so that

$$\int_{-\infty}^{\infty} | F(x) |\, dx \neq 0 \tag{5}$$

where

$$F(x) = \begin{cases} f(y_n) - t & \text{for } x = y_n \\ 0 & \text{otherwise.} \end{cases}$$

Since $\overline{E}_t$ is of first category $\bigcup_{n=1}^{\infty} (\overline{E}_t - y_n)$ is of first category. Thus there exists an h not in it, or $h \in (E_t - y_n)$ for all n. Thus $|f(x+h) - f(x)| \geqslant F(x)$, or by (5)

$$\int_{-\infty}^{\infty} |f(x+h) - f(x)|\, dx \neq 0.$$

This contradiction completes the proof of the conjecture.

The method of our proof was similar to that of P. LAX [3]) who proved that if S and $\overline{S}$ have both power of the continuum and m is any cardinal less than that of the continuum, there exists an h so that $(\varphi + h) \cap \overline{\varphi}$ has power greater or equal m.

The same method would give the following result: Let S and $\overline{S}$ both be dense in some interval (not necessarily in the same interval). Then for some h $(S+h) \cap \overline{S}$ is dense in some interval.

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3) P. ERDÖS, Annals of Math. **44**, 145–146 (1943).