ON SEQUENCES OF POSITIVE INTEGERS

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Let $a_1, a_2, \ldots, a_m$ be any finite set of distinct natural numbers, and let $b_1, b_2, \ldots$ be the sequence formed by all those numbers which are divisible by any of $a_1, a_2, \ldots, a_m$. This sequence has a density in the obvious sense and we denote this density by $A(a_1, a_2, \ldots, a_m)$. In fact

$$A(a_1, a_2, \ldots, a_m) = \frac{1}{a_1} + \left(\frac{1}{a_2} - \frac{1}{[a_1, a_2]}\right)$$
$$+ \left(\frac{1}{a_3} - \frac{1}{[a_1, a_3]} - \frac{1}{[a_2, a_3]} + \frac{1}{[a_1, a_2, a_3]}\right) + \ldots, \quad (1)$$

where $[a, b, \ldots]$ denotes the least common multiple of a, $b, \ldots$. For the first term above represents the density of the multiples of a_1, the second represents the density of those multiples of a_2 that are not multiples of a_1, and so on.

Now suppose we start from an infinite sequence a_1, $a_2, \ldots$ (arranged in increasing order) instead of from a finite set. It is plain that $A(a_1, a_2, \ldots, a_m)$ increases with m, and is always less than 1. We define

$$A = \lim_{m\to\infty} A(a_1, a_2, \ldots, a_m). \quad (2)$$

It is natural to expect that A should again be the density, in some sense, of the sequence $b_1, b_2, \ldots$ formed by all numbers which are divisible by any of $a_1, a_2, \ldots$. This cannot be true for the ordinary density, since it was proved by Besicovitch [1] that the b sequence may have different upper and lower densities.

There is one specially simple case in which the conclusion does hold, namely when the series $\Sigma\, 1/a_n$ converges. For, in this case, the number of b's up to x which are not divisible by any of $a_1, a_2, \ldots, a_m$ is at most

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$$\left[\frac{x}{a_{m+1}}\right]+\left[\frac{x}{a_{m+2}}\right]+\cdots;$$

and this after division by x is less than $\sum_{n=m+1}^{\infty} 1/a_n$. Hence the proportion of numbers that are b's differs from $A(a_1, a_2, \ldots, a_m)$ by an amount which tends to zero as $m \rightarrow \infty$, and the conclusion follows. Under these conditions, we can safely use the notation $A(a_1, a_2, \ldots)$ for A.

We proved some years ago [2] that, in the general case, the number A represents the *lower* density of the b sequence. We also proved that the b sequence has a *logarithmic density*, and that this also equals A. The logarithmic density is defined as

$$\lim_{x\to\infty} \frac{\beta(x)}{\log x},$$

where

$$\beta(x) = \sum_{b_i \leqslant x} \frac{1}{b_i}. \tag{3}$$

The proof of these two results was somewhat indirect, since it used Dirichlet series and appealed to a Tauberian theorem of Hardy and Littlewood. Our object in the present note is to give a direct and elementary proof.

Let us denote the upper and lower densities of the b sequence by d and D and the upper and lower logarithmic densities by δ and Δ. It is well known that

$$d \leqslant \delta \leqslant \Delta \leqslant D$$

for any sequence. In the present case, it is immediate that $d \geqslant A$. For the b sequence includes the multiples of $a_1, a_2, \ldots, a_m$, and so $d \geqslant A(a_1, a_2, \ldots, a_m)$ for each m, whence $D \geqslant A$. To complete the proof of the two results just enunciated, it suffices to prove that $\Delta \leqslant A$, in other words that

$$\overline{\lim_{x\to\infty}} \frac{\beta(x)}{\log x} \leqslant A. \tag{4}$$

The new proof of (4) is based on the consideration of what may be called the *multiplicative density* of a sequence. Let $p_1, p_2, \ldots, p_k$ be the first k primes. Denote by n' the general number composed entirely of these primes; then $\Sigma\, 1/n'$ converges, and

$$\Sigma \frac{1}{n'} = \prod_{i=1}^{k} \left(1 - 1/p_i\right)^{-1} = \Pi_k, \text{ say.} \tag{5}$$

Now denote by $b_1', b_2', \ldots$ those numbers of the b sequence that are composed entirely of $p_1, p_2, \ldots, p_k$. Let

$$B_k = \frac{\Sigma\, 1/b_i'}{\Sigma\, 1/n'} = (\Pi_k)^{-1} \Sigma\, 1/b_i'. \tag{6}$$

This fraction may be said to measure the density of the numbers b' among the numbers n'. If B_k tends to a limit as $k \to \infty$, we may call this limit the multiplicative density of the b sequence.

In the case under consideration here, where the b sequence consists of all multiples of $a_1, a_2, \ldots$, we can easily prove that the multiplicative density exists and has the value A. Let us denote by $a_1', a_2', \ldots$ those a's which are composed entirely of the primes $p_1, p_2, \ldots, p_k$. Then the b' consists of all numbers of the form $a'\, n'$, but without repetition. Hence we have

$$\Sigma \frac{1}{b'} = \frac{1}{a_1'} \Sigma \frac{1}{n'} + \left(\frac{1}{a_2'} - \frac{1}{[a_1', a_2']}\right) \Sigma \frac{1}{n'} + \ldots = \Pi_k A(a_1'\, a_2', \ldots).$$

It follows that

$$B_k = A(a_1', a_2', \ldots). \tag{7}$$

By an earlier remark, since $\Sigma\, 1/a'$ is convergent, we know that this is the density, in the ordinary sense, of the sequence formed by all multiples of $a_1', a_2', \ldots$. It is plain from (7) that B_k increases with k, and is always less than 1. Hence

$$B = \lim_{k\to\infty} B_k$$

exists, and our next step is to prove that

$$B = A. \tag{8}$$

In the first place, if k is sufficiently large in relation to m, the numbers $a_1', a_2', \ldots$ include $a_1, a_2, \ldots, a_m$. Hence $B \geqslant A(a_1', a_2' \ldots) \geqslant A(a_1, a_2, \ldots, a_m)$, whence $B \geqslant A$. Next, since $\Sigma\, 1/a'$ converges, we have for fixed k (by an argument used earlier)

$$A(a_1', a_2', \ldots) \leqslant A(a_1', a_2', \ldots, a_m') + \sum_{n=m+1}^{\infty} \frac{1}{a_n'}.$$

Now, if we choose r so large that $a_1', a_2', \ldots, a_m'$ are all included in $a_1, a_2, \ldots, a_r$, we have

$$A(a_1', a_2', \ldots, a_m') \leqslant A(a_1, a_2, \ldots, a_r) \leqslant A.$$

Making $m \to \infty$, we obtain

$$A(a_1', a_2', \ldots) \leqslant A,$$

that is, $B_k \leqslant A$. Hence $B \leqslant A$, which proves (8).

After this preparation, we proceed to prove (4). We divide the numbers $b_i \leqslant x$ into two classes, placing in the first class those divisible by any of $a_1', a_2', \ldots$. Here $a_1', a_2', \ldots$ are again those a's that are composed entirely of $p_1, p_2, \ldots, p_k$. For fixed k, the b's in the first class have density B_k, by (7). Hence the sum $\beta_1(x)$ corresponding to the b's in the first class satisfies

$$\lim_{x\to\infty} \frac{\beta_1(x)}{\log x} = B_k. \tag{9}$$

To estimate the sum $\beta_2(x)$ corresponding to the b's in the second class, we introduce a prime p_h defined by $p_h \leqslant x < p_{h+1}$. The b's in the second class are composed entirely of $p_1, p_2, \ldots, p_h$, but are not divisible by any a composed entirely of $p_1, p_2, \ldots, p_k$. If we denote by b^* the b's of this kind, whether less than x or not, we have

$$\beta_2(x) \leqslant \Sigma 1/b^*. \tag{10}$$

The numbers b^* can be obtained by taking all numbers b composed entirely of $p_1, p_2, \ldots, p_h$, say all numbers b'', and removing from them all numbers $b' t$, where b' is composed entirely of $p_1, p_2, \ldots, p_k$ and t is any number composed entirely of $p_{k+1}, p_{k+2}, \ldots, p_h$. Hence

$$\Sigma 1/b^* = 1/b'' - (\Sigma 1/b')(\Sigma 1/t) = \Pi_h B_h - \Pi_k B_k \Sigma 1/t,$$

by two appeals to (6). Since

$$\sum \frac{1}{t} = \prod_{i=k+1}^{h} \left(1 - \frac{1}{p_i}\right)^{-1} = \Pi_h (\Pi_k)^{-1},$$

we have

$$\Sigma 1/b^* = \Pi_h (B_h - B_k). \tag{11}$$

Now it is well known that Π_h, defined by (5) with h in place of k, satisfies [3, p. 22]

$$\Pi_h < C \log p_h \leqslant C \log x,$$

where C is an absolute constant. Hence, by (10) and (11), we have

$$\beta_2(x) < C(B_h - B_k) \log x. \tag{12}$$

It follows from (9) and (12) that

$$\overline{\lim_{x\to\infty}} \frac{\beta(x)}{\log x} \leqslant B_k + C(B - B_k).$$

Since $B_k \to B$ as $k \to \infty$, this proves (4).

It may be observed, in conclusion, that the density taken in any sense which is essentially stronger than the logarithmic sense, need not exist. For example, if $a < 1$, the density in the sense of

$$\lim_{x\to\infty} (1-a)\, x^{a-1} \sum_{b_i \leqslant x} 1/b_i^a$$

need not exist. This follows from the example constructed by Besicovitch. If a function of b_i is used which increases very little less rapidly than b_i, for instance $b_i/(\log b_i)$, the density will exist in the sense that

$$2 (\log x)^{-2} \sum_{b_i \leqslant x} (\log b_i)/b_i \to A.$$

But this is at once seen to be equivalent to $\beta(x)/(\log x) \to A$, on applying partial summation; so that nothing essentially new is obtained.

Note added May 1951. It may be of interest to observe that results similar to those proved above about that b sequence can sometimes be proved for the sequence formed by those b's which satisfy a supplementary condition. Consider, for example, those b_i for which $b_{i+1}-b_i=k$, where k is a given positive integer. It can be proved that these b_i have a logarithmic density; and that they have a density in the ordinary sense, provided that the whole b sequence has a density. The method of proof is to start from the case of a finite set $a_1, a_2, \ldots, a_m$, in which case the b's form a periodic sequence.

REFERENCES

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