ON A DIOPHANTINE EQUATION

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Throughout this paper the letters n, k, l, x, y denote positive integers satisfying $l > 1$, $x > 1$, $y > 1$, $n \geqslant 2k$, and p denotes a prime. In a previous paper† I proved that the equation $\binom{n}{k} = x^l$ has no solutions‡ if $k \geqslant 2^l$; I also proved that $\binom{n}{k} = x^3$ has no solutions. Obláth§ proved that

* Received 26 April, 1950; read 18 May, 1950.

† *Journal London Math. Soc.*, 14 (1939), 245–249.

‡ The assumption $n \geqslant 2k$ is not a loss of generality since we have $\binom{n}{k} = \binom{n}{n-k}$.

§ *Ibid.*, 23 (1948), 252–253.

$\binom{n}{k} = x^4$ and $\binom{n}{k} = x^5$ have no solutions. On the other hand it is well known that $\binom{n}{2} = x^2$ has infinitely many solutions and that the only solution of $\binom{n}{3} = x^2$ is $n = 50$, $x = 140$.*

In the present paper we prove the following

Theorem. *Let $k > 3$; then $\binom{n}{k} = x^l$ has no solutions.*

Remark. The cases $k = 2$ and $k = 3$ are left open, and it will be clear that our method cannot deal with these cases.

For the sake of completeness we repeat some of the proofs from my previous paper.

A theorem of Sylvester and Schur† states that $\binom{n}{k}$ always has a prime factor greater than k. Denote one of these primes by p. If $\binom{n}{k} = x^l$, we must have for some i with $0 \leqslant i < k$,

$$n - i \equiv 0 \pmod{p^l},$$

since only one of the numbers $n-i$ can be a multiple of p. Hence

$$n \geqslant p^l > k^l. \tag{1}$$

Write now $n - i = a_i x_i^l$, where all the a's are integers which are not divisible by any l-th power and whose prime factors are all less than or equal to k. First we prove that all the a's are different. Assume $a_i = a_j$, $i < j$. Then

$$k > a_i x_i^l - a_i x_j^l \geqslant a_i[(x_j+1)^l - x_j^l] > l a_i x_j^{l-1} \geqslant l(a_i x_j^l)^{\frac{1}{2}} \geqslant l(n-k+1)^{\frac{1}{2}} > n^{\frac{1}{2}},$$

which clearly contradicts (1).

Next we prove that the a's are the integers $1, 2, \ldots, k$ in some order. To prove this it will clearly suffice to show (since the a's are all different) that

$$a_1 a_2 \ldots a_k \mid k!. \tag{2}$$

From $\binom{n}{k} = x^l$ we have

$$\frac{a_1 a_2 \ldots a_k}{k!} = \frac{u}{v^l}, \quad (u, v) = 1.$$

* I cannot find a reference to this fact.

† *Ibid.*, 9 (1934), 232–288.

Let $q \leqslant k$ be any prime. The number of multiples of q^{α} among the a's is clearly not greater than $\left[\frac{k}{q^{\alpha}}\right]+1$ (since the number of multiples of q^{α} among the integers $n-i$, $0 \leqslant i < k$, is at most $\left[\frac{k}{q^{\alpha}}\right]+1$). Also since no a is a multiple of q^{l}, $a_1 a_2 \ldots a_k/k\,!$ is divisible by q to a power which is not greater than

$$\sum_{\alpha=1}^{l-1}\left(\left[\frac{k}{q^{\alpha}}\right]+1\right)-\sum_{\alpha=1}^{\infty}\left[\frac{k}{q^{\alpha}}\right] \leqslant l-1.$$

Thus $u=1$, and (2) is proved.

Hence if $l=2$ and $k>3$, $\binom{n}{k}=x^2$ is impossible, since 4 being a square cannot be an a, and thus $a_1 a_2 \ldots a_k > k\,!$, which contradicts (2).

So far our proof is identical with the one contained in my previous paper*. Now we can assume $l>2$. Since $k \geqslant 4$, we can choose i_1, i_2, i_3 $(0 \leqslant i_\nu < k)$ so that

$$(3) \qquad n-i_1 = x_1^{\,l}, \quad n-i_2 = 2x_2^{\,l}, \quad n-i_3 = 4x_3^{\,l}.$$

Clearly $(n-i_2)^2 \neq (n-i_1)(n-i_3)$. For otherwise put $n-i_2=m$; then

$$m^2=(m-x)(m+y), \text{ or } (y-x)\,m=xy.$$

$x=y$ is clearly impossible. On the other hand, if $x \neq y$ we have, by (1),

$$xy=m(y-x) \geqslant m > n-k > (k-1)^2 \geqslant xy \text{ (since } x<k,\ y<k),$$

an evident contradiction. Hence $x_2^{2l} \neq x_1^{\,l} x_3^{\,l}$. We can assume without loss of generality that $x_2^{\,2} > x_1 x_3$; then

$$2(k-1)\,n > n^2-(n-k+1)^2 > (n-i_2)^2-(n-i_1)(n-i_3)$$
$$=4[x_2^{2l}-(x_1 x_3)^l] \geqslant 4[(x_1 x_3+1)^l - x_1^{\,l} x_3^{\,l}] > 4l\,x_1^{l-1} x_3^{l-1}.$$

Hence, since $n > k^3 > 6k$ and $l \geqslant 3$,

$$2(k-1)\,x_1 x_3\, n > 4l\,x_1^{\,l} x_3^{\,l} \geqslant l(n-k+1)^2 > l(n^2-2kn) > \frac{2ln^2}{3} \geqslant 2n^2.$$

Thus, since by (3) $x_i \leqslant n^{\frac{1}{3}}$,

$$kn^{\frac{2}{3}} \geqslant kx_1 x_3 > (k-1)\,x_1 x_3 > n, \text{ or } k^3 > n,$$

which contradicts (1). Thus our theorem is proved.

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* *Ibid.*, 14 (1939), 245–249.

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