$$-c^2 + 11.5866485c - 20.1285 = 0.$$

$$c_1 = 2.128067661.$$

$$c_2 = 11.5866485 - c_1 = 9.458580839,$$

or

$$c_2 = 20.1285 \div c_1 = 9.458580838.$$

The case of two angles and one side is handled by the Law of Sines, with the sines of the three angles found by series.

9. Solution of triangles with machine and tables. For the quickest possible method of solving triangles, use both tables and machine. Follow the methods of Section 8, and whenever it is necessary to find a trigonometric function of a given angle, or to find the value of the angle from one of the trigonometric functions, use the tables. The machine will be found helpful in the interpolation.

With the necessary multiplying, squaring, extracting the square root, and other computation done on the machine, every triangle can be solved with 3 applications to the tables. In some cases, this includes the check, in other cases, a fourth reference to the tables will be required for a check. This contrasts with 8 applications to the tables, when the computation is done by logarithms instead of machine.

MATHEMATICAL NOTES

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ON A CONJECTURE OF KLEE

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1. Introduction. Klee[1] denotes by $S_k(m)$ the number of solutions of $\phi(x)=m$, where x has exactly k prime factors which appear to the first power in the factorization of x. Lampek[1] observed that

$$\phi\left(\frac{(n!)^2}{\phi(n!)}\right) = n!.$$

Klee[1] remarks that except for the prime 2, all prime factors of $n!/\phi(n!)$ are multiple. Thus $S_0(n!)>0$. Klee[1] conjectures that for all n, $S_1(n!)>0$. Gupta[2] recently proved this conjecture, in fact he proved that $\lim_{n=\infty} S_1(n!)=\infty$. In the present note we prove that $\lim_{n=\infty} S_k(n!)=\infty$ for every k, and state without

[1] This MONTHLY, vol. 56 (1949), pp. 21–26.

[2] *Ibid.*, vol. 57 (1950), pp. 326–329.

proof a few other problems and results.

2. Lemmas. First we prove three lemmas.

Lemma 1. *Let $b|a$, and assume that a/b has the same prime factors as a (that is, all the prime factors of b occur in a with a higher exponent). Then*

$$\phi\left(\frac{a}{b}\right) = \frac{\phi(a)}{b}.$$

This follows immediately from the definition of the ϕ function.

Lemma 2. *The number of primes q, $n<q<2n$, $q\equiv 1 \pmod 6$, is greater than $c_1 n/\log n$ for a suitable constant c_1 and sufficiently large n.*

This follows immediately from the prime number theorem for arithmetic progressions (or also from a more elementary result).[3]

Lemma 3. *Let n be sufficiently large. Put*

$$A_{q_1,q_2,\cdots,q_k} = \frac{n!}{\prod (p-1)\cdot(q_1-1)(q_2-1)\cdots(q_k-1)},$$

where p runs through the primes $\leqq n$ and $n<q<2n$, $q\equiv 1 \pmod 6$. Then $A_{q_1,\cdots,q_k}$ is an integer, and $p|A_{q_1,\cdots,q_k}$, for $p\leqq n$.

First of all from Lemma 2, for sufficiently large n the number of q's is $>c_1 n/\log n>k$; thus $A_{q_1,\cdots,q_k}$ is defined. Let t be a prime. For $n/2<t\leqq n$, $t|A_{q_1,\cdots,q_k}$ since $t|n!$ while $t\neq p-1$, $q_i-1\not\equiv 0 \pmod t$ (since $q_i-1\equiv 0 \pmod 6$). Let next $3<t\leqq n/2$. The denominator of $A_{q_1,\cdots,q_k}$ can be written as

$$2^v \prod \frac{p-1}{2} \prod_{i=1}^{k} \frac{q_i-1}{2},$$

and here all the factors are distinct integers $\leqq n$. But t and $2t$ are never of the form $(q_i-1)/2$ (since $(q_i-1)/2\equiv 0 \pmod 3$). Further not both t and $2t$ can be of the form $(p-1)/2$, since either $2t+1$ or $4t+1$ is a multiple of 3. Thus any $3<t\leqq n/2$ occurs with a higher exponent in $n!$ than in the denominator of $A_{q_1,\cdots,q_k}$. If $t=3$ and $n\geqq 12$, then $3|A_{q_1,\cdots,q_k}$, since $12\neq(p-1)/2$, $12\neq(q_i-1)/2$. Let now $t=2$. The even numbers $6u+2$ are clearly not of the form $p-1$ ($6u+3 \not\equiv 0 \pmod 3$). Thus $n!/\prod(p-1)$ is a multiple of $2^{[(n-2)/6]}$. If $\prod_{i=1}^{k}(q_i-1)$ is a multiple of 2^u we clearly have $2^u<2^k n^k$ and, for sufficiently large n,

$$2^{[(n-2)/6]} > (2n)^k.$$

Thus $2|A_{q_1,\cdots,q_k}$, and Lemma 3 is proved.

3. Theorem. We shall establish the following result.

[3] R. Breusch, Math. Zeitschrift, vol. 34 (1932), pp. 505–526; see also P. Erdos, *ibid.*, vol. 39 (1935), ppl 473–491.

THEOREM. *For sufficiently large n, we have*

$$S_k(n!) > c_2 \frac{n^k}{(\log n)^k}.$$

Proof: We have, by Lemmas 1 and 3, with

$$B_{q_1,\cdots,q_k} = \prod_{i=1}^{k} q_i \frac{(n!)^2}{\phi(n!)\prod_{i=1}^{k}(q_i-1)},$$

$$\phi(B_{q_1,\cdots,q_k}) = \phi\left(\prod_{p\leq n} p \prod_{i=1}^{k} q_i \frac{n!}{\prod_{p\leq n}(p-1)\prod_{i=1}^{k}(q_i-1)}\right)$$

$$= \phi\left(\prod_{p\leq n} p \prod_{i=1}^{k} q_i A_{q_1,\cdots,q_k}\right) = n!.$$

It follows from Lemma 2 that there are more than $c_2 n^k/(\log n)^k$ choices for $q_1, \cdots, q_k$; also, by Lemma 3, $B_{q_1,\cdots,q_k}$ contains exactly k prime factors which appear to the first power in the factorization of $B_{q_1,\cdots,q_k}$. This completes the proof of the Theorem.

4. Further questions. One can ask the question how large has n to be in order that $S_k(n!)>0$. Our proof gives that n has to be greater than $c_3 k \log k$. By a more complicated argument we can show that for a suitable constant c_4, we have $\phi_k|[c_4 k]!|>0$. It is probable that for every $\epsilon>0$ and sufficiently large n we have $\phi_k([(1+\epsilon)k]!)>0$. It is easy to see that $S_n(n!)=0$ for $n>2$.

We can also show that $\lim_{n=\infty} S_k(n!)^{1/n}=1$. On the other hand there exists an absolute constant c_5 so that the number of solutions of $\phi(x)=n!$ is greater than $(n!)^{c_5}$. Previously it was known that there are infinitely many integers m, so that the number of solutions of $\phi(x)=m$ is greater than m^{c_5}. It is an open question whether c_5 can be chosen arbitrarily close to 1.

It seems a difficult question to decide whether $\phi(x)=n!$ is always solvable in squarefree integers x. Similarly it seems difficult to decide whether for sufficiently large n, the equation $\sigma(x)=n!$ is solvable ($\sigma(x)$ denotes the sum of the divisors of x).

If one wants to prove Gupta's[2] result, $S_1(n!)>0$ for all n, it suffices to remark that for $n\geq 4$ there always is a prime $q\equiv 1 \pmod 6$ in the interval $(n, 2n)$.[3] Also that for $n\geq 8$,

$$2^{[(n-2)/6]+2} \left| \frac{n!}{\prod(p-1)} \right.$$

(since 8 contains 2 with exponent 3). Further since $q\leq 2n$, $q\equiv 1 \pmod 6$, if $2^u|(q-1)$ we have $2^u<2n/3$. Thus if

$$2^{[(n-2)/6]} > n/6,$$

then $S_1(n!) > 0$, and this holds for $n \geqq 14$. For $n < 14$, the relation $S_1(n!) > 0$ can be shown by a short computation. By a slightly longer computation we can show that $S_2(n!) > 0$ for all $n \geqq 2$.

PERFECT SQUARES OF SPECIAL FORM*

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1. Introduction. This note carries further** the determination of systems of numeration in which there exist pairs of perfect squares having the form

$$aabb = (cc)^2, \qquad bbaa = (dd)^2.$$

2. Necessary and sufficient conditions. It is easy to show† that necessary and sufficient conditions for the above are:

(1) $a + b = B + 1,$ (2) $1 \leqq a, b, c < B,$

(3) $c^2 = a(B - 1) + 1,$ (4) $d^2 = b(B - 1) + 1,$

where B, a, b, c are positive integers.

3. Special cases. The form of (3) suggests an examination of the special cases where

(5) $$c = ma \pm 1,$$

with m an arbitrary positive integer.

From (1), (3) and (5), the following equations result immediately:

(6) $$B = m(ma \pm 2) + 1,$$

(7) $$b = (m \pm 1)[(m \mp 1)a \pm 2].$$

Now (4), (6) and (7) combine to give

(8) $$d^2 = m(m \pm 1)(ma \pm 2)[(m \pm 1)a \pm 2] + 1.$$

This equation is satisfied by $a = 0$, $d = 2m \pm 1$, and by $a = 4$, $d = 4m^2 \pm 2m - 1$. Hence, since the coefficient of a^2 is not a square, there will be infinitely many solutions for each positive integer m (and for many fractional values of m as well).

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* Translated (and abridged) from the French by E. P. Starke, Rutgers University.

** See V. Thébault, Mathesis, 1936 (Supplement); This MONTHLY, *Two Classes of Remarkable Perfect Square Pairs*, 1949, pp. 443–448.

† Or see V. Thébault, Mathesis, *loc. cit.*