ON A PROBLEM IN ELEMENTARY NUMBER THEORY

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Denote by $v(n)$ the number of different prime factors of n, and by $\varphi(x, n)$ the number of integers not exceeding x which are relatively prime to n. In a previous paper[1] I proved that for every n there exists an x so that, if $\varphi(n, n) = \varphi(n)$ denotes Euler's φ function,

$$\left| \varphi(x, n) - x \frac{\varphi(n)}{n} \right| > c\, 2^{v(n)/2} / (\log (v)n)^{1/2}.$$

On the other hand it is easy to see that[2]

$$\left| \varphi(x, n) - x \frac{\varphi(n)}{n} \right| < 2^{v(n)-1}.$$

It can be conjectured that if $v(n) \longrightarrow \infty$

$$(1) \qquad \left| \varphi(x, n) - x \frac{\varphi(n)}{n} \right| = o\left(2^{v(n)}\right).$$

The proof of (1) seems difficult. In the present paper we prove the following related result:—

THEOREM. *We have ($\mu(d)$ is the Moebius symbol)*

$$(2) \qquad \left| \sum_{\substack{d \mid n \\ a \leqslant d \leqslant b}} \mu(d) \right| \leqslant \binom{v(n)}{[v(n)/2]}.$$

For every value k of $v(n)$, (2) is the best possible result.

First we show that (2) is best possible. Let p_1 be a sufficiently large prime, and let $p_1 < p_2 < \cdots < p_k$ be k consecutive primes $\geqslant p_1$. Put $n = p_1 \cdot p_2 \cdots p_k$, $a = p_1^{[k/2]}$, $b = p_k^{[k/2]}$. A simple argument shows that every $d \mid n$ in the interval (a, b) has $\left[\frac{k}{2}\right]$ prime factors and every $d \mid n$ with $v(d) = \left[\frac{k}{2}\right]$ is in (a, b). Thus (2) holds with the sign of equality. Q.E.D.

Now we prove (2). It will clearly suffice to prove (2) if n is squarefree. We evidently have $\left(\sum_{d \mid n} \mu(d) = 0\right)$

$$(3) \qquad \sum_{\substack{d \mid n \\ a \leqslant d \leqslant b}} \mu(d) = \sum_{\substack{d \mid n \\ d \leqslant b}} \mu(d) + \sum_{\substack{d \mid n \\ a \leqslant d}} \mu(d) = \sum\nolimits_1 + \sum\nolimits_2.$$

Define now for even k, $A_k = B_k = C_k = D_k = \binom{k-1}{\left[\frac{k-1}{2}\right]}$;

for $k=4t+1$, $A_k = D_k = \binom{4t}{2t}$, $B_k = C_k = \binom{4t}{2t-1}$;

for $k=4t+3$, $A_k = D_k = \binom{4t+2}{2t}$, $B_k = C_k = \binom{4t+2}{2t+1}$. We prove

(4) $$-B_k \leqslant \sum_1 \leqslant A_k ; \quad -D_k \leqslant \sum_2 \leqslant C_k.$$

Suppose (4) is already proved. A simple argument shows that $A_k + C_k = B_k + D_k = \binom{k}{\left[\frac{k}{2}\right]}$. Thus clearly (3) and (4) imply (2). Thus it will suffice to prove (4).

First we show that $\sum_1 \leqslant A_k$. Denote by $U(r, b)$ the number of integers $d|n$, $d \leqslant b$, $v(d)=r$. Clearly if d is in $U(r, b)$ and $p|d$ then d/p is in $U(r-1, b)$. Thus to every integer in $U(r, b)$ correspond r integers of $U(r-1, b)$. On the other hand it is easy to see that there are at most $k-r+1$ integers of $U(r, b)$ to which correspond the same integer in $U(r-1, b)$. Thus we obtain

(5) $$U(r-1, b) \geqslant \frac{r}{k-r+1} U(r, b).$$

We obtain from (5) that for $r > k/2$

(6) $$U(r-1, b) \geqslant U(r, b).$$

If $r < k/2$ we obtain from (5)

$$U(r, b) - U(r-1, b) \leqslant \left(1 - \frac{r}{k-r+1}\right) U(r, b) \leqslant \left(1 - \frac{r}{k-r+1}\right)\binom{k}{r}$$
$$= \binom{k}{r} - \binom{k}{r-1}.$$

Denote by $2s$ the greatest even number not exceeding $k/2$. We have from (6) and (7)

$$\sum_1 = \sum_{r=0}^{k} (-1)^r U(r, b) \leqslant \sum_{r=0}^{2s} (-1)^r U(r, b) \leqslant \sum_{r=0}^{2s} (-1)^r \binom{k}{r} = A_k, \quad \text{Q.E.D.}$$

The last equation follows from a simple argument on binomial coefficients. $-B_k \leqslant \sum_1$ can be proved in the same way. $-C_k \leqslant \sum_2 \leqslant D_k$ can also be proved in the same way. (Only instead of considering the numbers d/p with $p|d$, we consider the numbers pd with $p \mid \frac{n}{d}$.) This proves the theorem.

Footnotes

1. *Bull. Amer. Math. Soc* 52 (1946), p. 179-184.
2. See for example D. H. Lehmer, *Bull. Amer. Math. Soc.* 54 (1948); p. 1185-1190.
3. A similar argument is used by Sperner, *Math Zeitschrift*, 27 (1928) p. 544-548.

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