THE SET ON WHICH AN ENTIRE FUNCTION IS SMALL.*

By R. P. Boas, Jr., R. C. Buck, and P. Erdös.

Let $f(z)$ be an entire function and $M(r)$ the maximum of $|f(z)|$ on $|z| = r$. We give some results on the density of the set of points at which $|f(z)|$ is small in comparison with $M(r)$; although simple, these results seem not to have been noticed before.

If E is a measurable set in the z-plane, we denote by $D_R(E)$ the ratio $m(z \in E, |z| \leq R)/\pi R^2$ and by $\bar{D}(E)$ and $\underline{D}(E)$ the upper and lower densities of E, that is the superior and inferior limits of $D_R(E)$ as $R \to \infty$. For a fixed function $f(z)$, let E_λ be the set of points z for which $\log|f(z)| \leq (1 - \lambda) \log M(|z|)$. Our results may be stated as follows.

Theorem 1. *For any $\lambda > 1$, there is a number K, the same for all functions $f(z)$, such that $\bar{D}(E_\lambda) \leq K$. Moreover, $0 < K \leq \lambda^{-1}$.*

In particular, for $\lambda = 2$, the upper density of the set where $|f(z)| \leq 1/M(|z|)$ is at most $1/2$. Much stronger results are known for entire functions of small finite order. The interest of Theorem 1 is that it holds for all entire functions and that, contrary to what might be expected, K is strictly positive. We shall show that a lower bound on K is given by $\delta^2/(1+\delta)^2$ where δ is the positive root of $\delta(2+\delta)^{\lambda-1} = 1$. For $\lambda = 2$, this can be improved to $.1925$; the same method will yield better values for other choices of λ. For lower density, the following is true.

Theorem 2. *As $\lambda \to \infty$, $\underline{D}(E_\lambda) = o(\lambda^{-1})$.*

It might be conjectured that this also holds for the upper density, and for the numbers $K = K(\lambda)$.

We first prove that λ^{-1} is an upper bound for $\bar{D}(E_\lambda)$. Consider the integral

$$I = (1/2\pi) \int_0^{2\pi} \{\log M(r) - \log|f(re^{i\theta})|\} d\theta.$$

Let $f(z) = z^p g(z)$, $g(0) \neq 0$. Then, by Jensen's theorem

* Received July 28, 1947.

$$I = \log M(r) - p \log r - (1/2\pi) \int_0^{2\pi} \log | g(re^{i\theta}) | \, d\theta$$
$$\leq \log M(r) - p \log r - \log | g(0) | .$$

Let $H_{r,\lambda}$ be the set of values of θ for which $\log | f(re^{i\theta}) |$ is less than $(1-\lambda) \log M(r)$. By applying to the integral I the identity $\int \phi(x) dx = \int_0^\infty \psi(r) dr$ where $\phi(x) \geq 0$ and $\psi(r)$ is the measure of the set on which $\phi(x) \geq r$, the integral I may also be expressed as

$$I = (2\pi)^{-1} \log M(r) \int_0^\infty m(H_{r,\lambda}) \, d\lambda.$$

Hence, writing $C = \log | g(0) |$, we have

$$(1) \qquad (1/2\pi) \int_0^\infty m(H_{r,\lambda}) \, d\lambda \leq 1 - \frac{p \log r + C}{\log M(r)} .$$

Choose R_0 so that $M(r) > 1$ for $r \geq R_0$; then

$$m(z \,\varepsilon\, E, R_0 \leq | z | \leq R) = \int_{R_0}^R m(H_{r,\lambda}) r \, dr.$$

Integrating this with respect to λ and using (1), we have

$$(2) \qquad \int_0^\infty \bar{D}_R(E_\lambda^*) \, d\lambda \leqq 1 - R_0^2/R^2 - (2/R^2) \int_{R_0}^R \frac{(p \log r + C) r \, dr}{\log M(r)}$$

where E_λ^* is E_λ with the circle $| z | \leq R_0$ deleted.

We may suppose that $f(z)$ is not a polynomial. (In this case, it is easily seen that $\bar{D}(E_\lambda) = 0$ for all $\lambda > 0$.) Since $\log M(r)$ is convex in $\log r$, it follows that $\log r = o(\log M(r))$ as r tends to infinity, and hence that the right side of (2) is $1 + o(1)$ as $R \to \infty$. As λ increases, the sets E_λ^* decrease and $D_R(E_\lambda^*)$ is monotone for fixed R. Thus, $\lambda D_R(E_\lambda^*) \leq \int_0^\infty D_R(E_\lambda^*) \, d\lambda$ and letting R increase, we have $\lambda \bar{D}(E_\lambda) = \lambda \bar{D}(E_\lambda^*) \leq 1$.

The proof of Theorem 2 also falls out of the inequality (2). Letting R tend to infinity, we have

$$\int_0^\infty D(E_\lambda) \, d\lambda \leq 1$$

and since the integrand is monotonic, $\lim_{\lambda \to \infty} \lambda \, \underline{D}(E_\lambda) = 0$.

To obtain a lower bound on K, the least upper bound of $\bar{D}(E_\lambda)$ for all functions $f(z)$, we investigate a special function. Consider the product

$$f(z) = \prod_{n=1}^{\infty} (1 - z/a^n)^{b^n}, \qquad a > b > 1,$$

which defines an entire function of order $\log b / \log a$. Put

$$\phi(z) = |f(z)| M(r)^{\lambda-1} = \prod_{k=1}^{\infty} \{|1 - z/a^k|(1 + r/a^k)^{\lambda-1}\}^{b^k}.$$

Suppose that z lies in the region S described by

$$(3) \qquad |1 - z/a^n| (1 + r/a^n)^{\lambda-1} \leq \beta < 1.$$

Let r/a^n be less than γ for all z in S. Then,

$$\phi(z) \leq \prod_{k<n} (1 + r/a^k)^{\lambda b^k} \beta^{b^n} \prod_{k>n} (1 + r/a^k)^{\lambda b^k}$$
$$\leq (\lambda\gamma a^{n-1})^{\lambda b} (\lambda\gamma a^{n-2})^{\lambda b^2} \cdots (\lambda\gamma a)^{\lambda b^{n-1}} \beta^{b^n} \exp \{\lambda\gamma a^n \sum_{k>n} (b/a)^k\}$$

and

$$\log \phi(z) \leq b^n \left\{ \frac{\lambda \log \lambda\gamma}{b-1} + \frac{\lambda b \log a}{(b-1)^2} + \frac{\lambda b \gamma}{a-b} + \log \beta \right\}.$$

As b and a tend to infinity in such a manner that $b^{-1} \log a$ and b/a approach zero (e. g., $a = b^2$), the bracket approaches $\log \beta$ which is negative. Thus, for any $\beta < 1$ and for suitable a and b, $\phi(z) < 1$ for all z in S, and for the special function that we have constructed, $S \subseteq E_\lambda$.

There is a set of type S enclosing each of the points $z = a^n$. We now estimate the upper density of the union of these sets, and hence the upper density of E_λ. We may take $\beta = 1$. Put $w = z/a^n = \rho e^{i\phi}$; the set S corresponds to the set S^* bounded by the curve $|1 - w|(1 + \rho)^{\lambda-1} = 1$. The circle $|w - 1| < \delta$ where $\delta(2 + \delta)^{\lambda-1} = 1$ lies in S^*. The ratio $D_{1+\delta}(S^*)$ is at least $\delta^2/(1 + \delta)^2$ and since this is independent of n, this number is a lower bound for $K(\lambda)$. A better bound can be obtained by computing the radius ρ_0 for which $D_{\rho_0}(S^*) = m(w \varepsilon S^*, |w| \leq \rho_0)/\pi\rho_0^2$ is greatest. This number is then the desired lower bound. In the special case $\lambda = 2$, numerical integration gives the value .1925 for this ratio.

With reference to generalizations, we observe that the relations (1) and (2) hold with $p = 0$ with any subharmonic function $v(z)$ replacing the function $\log |f(z)|$, and with $\mu(r) = \max_\theta v(re^{i\theta})$ replacing $\log M(r)$, provided that $C = v(0)$ is finite. In addition, there is equality instead of inequality in (1) and (2) if $v(z)$ is a harmonic function without singularities.

Brown University.
Society of Fellows, Harvard University.
Syracuse University.