N. G. DE BRUIJN and P. ERDÖS: *On a combinatioral problem.*

(Communicated at the meeting of November 27, 1948.)

Let there be given n elements $a_1, a_2, \ldots, a_n$. By $A_1, A_2, \ldots, A_m$ we shall denote combinations of the a's. We assume that we have given a system of $m > 1$ combinations $A_1, A_2, \ldots, A_m$ so that each pair (a_i, a_j) is contained in one and only one A. Then we prove

Theorem 1. We have $m \geq n$ [1]), with equality occurring only if either the system is of the type $A_1 = (a_1, a_2, \ldots, a_{n-1})$, $A_2 = (a_1, a_n)$, $A_3 = (a_2, a_n) \ldots A_n = (a_{n-1}, a_n)$, or if n is of the form $n = k(k-1)+1$ and all the A's have k elements, and each a occurs in exactly k of the A's.

Corollary: If the elements a_i are points in the real projective plane the theorem can be stated as follows: Let there be given n points in the plane, not all on a line. Connect any two of these points. Then the number of lines in this system is $\geq n$. In this case equality occurs only if $n-1$ of the points are on a line.

This corollary can be proved independently of Theorem 1 by aid of the following theorem of GALLAI (= GRÜNWALD) [2]):

Let there be given n points in the plane, not all on a line. Then there exists a line which goes through two and only two of the points.

Remark: The points of inflexion of the cubic show that it is essential that the points should all be real, thus GALLAI's theorem permits no projective and a fortiori no combinatorial formulation. Also the result clearly fails for infinitely many points.

We now give GALLAI's ingenious proof: Assume the theorem false. Then any line through two of the points also goes through a third. Project one of the points, say a_1 to infinity, and connect it with the other points. Thus we get a set of parallel lines each containing two or more points a_i (in the finite part of the plane). Consider the system of lines connecting any two of these points, and assume that the line $(a_i\, a_j\, a_k)$ forms the smallest angle with the parallel lines. (This line again contains at least three points). But the line connecting a_j with a_1 (at infinity) contains at least another (finite) point a_r, and clearly (see figure) either the line $(a_i\, a_r)$

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[1]) This was also proved by G. SZEKERES but his proof was more complicated.

[2]) This theorem was first conjectured by SYLVESTER, GALLAI's proof appeared in the Amer. Math. Montly as a solution to a problem by P. ERDÖS. The corollary to Theorem 1 also appeared as a problem in the Monthly.

See also H. S. M. COXETER, Amer. Math. Monthly **55**, 26—28 (1948), where very simple proofs due to KELLY and STEINBERG are given.

or the line (a_r, a_k) forms a smaller angle with the parallel lines then $(a_i\, a_j\, a_k)$. This contradiction establishes the result.

Remark: Denote by $f(n)$ the minimum number of lines which go through exactly two points. It is not known whether $\lim f(n) = \infty$. All that we can show is that $f(n) \geq 3$.

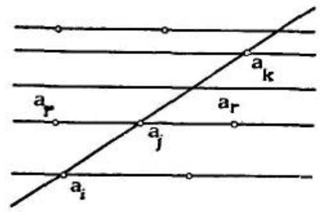

Fig. 1.

Now we prove the corollary as follows: We use induction. Assume that the number of lines determined by $n-1$ points, not all on a line, is $\geq n-1$. Then we shall prove that n points, not all on a line, determine at least n lines.

Let (a_1, a_2) be a line going through two points only. Consider the points $a_2, a_3, \ldots, a_n$. If they are all on a line, then (a_1, a_i), $i = 2, 3, \ldots, n$ and $(a_2, a_3, \ldots, a_n)$ clearly determine n lines. If they do not all lie on a line, then they determine at least $n-1$ lines, and (a_1, a_2) is clearly not one of these lines. Thus together with (a_1, a_2) we again get at least n lines. The same induction argument shows that we get exactly n lines only if $n-1$ of the points lie on a line, q.e.d.

Proof of theorem 1. For simplicity we shall call the elements $a_1, a_2, \ldots, a_n$ points and the sets $A_1, A_2, \ldots, A_m$ lines. Denote by k_i the number of lines passing through the point a_i, and by s_j the number of points on the line A_j. We evidently find (by counting the number of incidences in two ways)

$$\sum_{j=1}^{m} s_j = \sum_{i=1}^{n} k_i \quad \ldots\ldots\ldots\ldots \quad (1)$$

Further if A_j does not pass through a_i, then

$$s_j \leqslant k_i \quad \ldots\ldots\ldots\ldots \quad (2)$$

(2) follows from the fact that a_i can be connected by a line (i.e. an A) to all the s_j points of A_j, and any two of these lines are different, since otherwise they would have two points in common.

Assume now that k_n is the smallest k_i and that $A_1, A_2, \ldots, A_{k_n}$ are the lines through a_n. We may suppose that each line contains at least two points, since otherwise it could be omitted. Also $k_n > 1$, for otherwise all the points are on a line. Thus we can find points a_i on A_i, $a_i \neq a_n$, $i = 1, 2, \ldots, k_n$. Also if $i \neq j$, $i \leq k_n$, $j \leq k_n$ then a_i is not on A_j (for otherwise A_i and A_j would have two points in common). Hence by (2) (putting $k_n = \nu$)

$$s_2 \leqslant k_1, \quad s_3 \leqslant k_2, \ldots, \quad s_\nu \leqslant k_{\nu-1}, \quad s_1 \leqslant k_\nu; \quad s_j \leqslant k_n \text{ for } j > \nu. \quad (3)$$

From (1), (3) and the minimum property of k_n we obtain $m \geq n$, which proves the first part of Theorem 1.

We now determine the cases where $m = n$. If $m = n$, then all the inequalities of (3) have to be equalities. Consequently we can renumerate the points so that $s_1 = k_1$, $s_2 = k_2, \ldots, s_n = k_n$. We may suppose that $k_1 \geq k_2 \geq \ldots \geq k_n > 1$. There are two cases:

a) $k_1 > k_2$. Hence by $s_1 = k_1 > k_i$ $(2 \leq i \leq n)$, (2) shows that all the a_i $(i \geq 2)$ lie on A_1. Of course a_1 does not lie on A_1 and we have the first case of Theorem 1.

b) $k_1 = k_2$. If no k_i is less than k_1 then clearly $k_i = s_j$ $(1 \leq i, j \leq n)$. We shall show that this is the only possibility. If $k_j < k_1$, then we have by (2) that a_j lies on both A_1 and A_2. Hence k_n is the only k which can be less than k_1. Now $s_n = k_n$ different lines contain a_n. Any line through a_n contains one further point and all but one contain two further points, since $k_1 = k_2 = \ldots = k_{n-1} > k_n \geq 2$. Thus there are at least two lines which do not contain a_n; for both of these lines we have by (2) $s_j \leq k_n$. This contradicts $s_1 = s_2 = \ldots = s_{n-1} > k_n$.

Apart from case a) we only have the case where $s_i = k_j = k$, $(1 \leq i, j \leq n)$. It is easily seen that then $n = k(k-1) + 1$, and also that any pair of lines has exactly one intersection point. For if A_i does not intersect A_j; and if a_l lies on A_i then we infer from (2) that $k_l \geq s_j + 1$ which is not possible since $k_l = s_j = k$. The two dimensional projective finite geometries with $k - 1 = p^\alpha$, p prime, are known to be systems of this type, but F. W. LEVI [3]) constructed a non-projective example with $k = 9$.

[3]) F. W. LEVI, Finite geometrical systems, Calcutta 1942.