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ON THE MAXIMUM OF THE FUNDAMENTAL FUNCTIONS OF THE ULTRASPHERICAL POLYNOMIALS

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In the present note we are going to prove the following theorem: Let $-1 \leqq x_1 < x_2 \cdots < x_n \leqq 1$ be the roots of the ultraspherical polynomial $P_n^{(\alpha)}(x)$ with $0 \leqq \alpha \leqq 3/2$. (The normalisation is of no importance.) $\alpha = \frac{1}{2}$ gives the Legendre polynomial $\alpha = 3/2$ gives $U_n(x) = T'_{n+1}(x)$, where $T_n(x)$ is the n^{th} Tchebicheff polynomial. Let

$$l_k^{(n)}(x) = \frac{P_n^{(\alpha)}(x)}{P_n'^{(\alpha)}(x_k)(x - x_k)}$$

be the fundamental polynomial of the Lagrange interpolation. Then

$$\max_{k=1,2,\cdots n, -1 \leq x \leq 1} | l_k^{(n)}(x) | = l_1^{(n)}(-1) = l_n^{(n)}(1).$$

Special cases of this theorem have been proved by Erdös-Grünwald[1] and Webster[2] (the cases $\alpha = 1/2$ and $\alpha = 3/2$). If there is no danger of confusion we shall omit the upper index n in $l_k^{(n)}(x)$.

PROOF OF THE THEOREM. It clearly suffices to consider the $l_k(x)$ with $-1 \leqq x_k \leqq 0$. From the differential equation of the ultraspherical polynomials[3] we obtain

$$l_k'(x_k) = \frac{P_n''^{(\alpha)}(x_k)}{\frac{1}{2}P_n'^{(\alpha)}(x_k)} = \frac{\alpha x_k}{1 - x_k^2}. \tag{1}$$

Thus for $x_k \leqq x \leqq x_{k+1}$ $0 \leqq l_k(x) \leqq 1$. Suppose now that $k \neq 1$, then we prove that in (x_{k-1}, x_k) $l_k(x)$ lies below its tangent at x_k. Denote by $y_1, y_2, \cdots y_{n-1}$ the roots of $l_k'(x)$ and by $z_1, z_2, \cdots z_{n-2}$ the roots of $l_k''(x)$. From (1) it follows that $x_{k-1} < y_{k-1} < x_k$. To prove our assertion it suffices to show that $z_{k-1} > x_k$. First we prove that $y_{k-1} > \dfrac{x_{k-1} + x_k}{2} = u$. From (1)

$$\frac{1}{2}\frac{\alpha}{1 + x_k} + \sum_{i<k} \frac{1}{x_k - x_i} = \frac{\alpha}{2(1 - x_k)} + \sum_{j>k} \frac{1}{x_j - x_k},$$

thus

$$\frac{1}{1 + x_k} + \sum_{i<k} \frac{1}{x_k - x_i} > \sum_{j>k} \frac{1}{x_j - x_k}. \tag{2}$$

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[1] ERDÖS-GRUNWALD, Bull. Amer. Math. Soc. 44 (1938), p. 515–518.

[2] WEBSTER, ibid. 45 (1939), p. 870–873.

[3] See e.g. G. SZEGÖ, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publications vol. XXIII p. 59. Our notation differs from that of Szegö. This α has to be replaced by $\alpha + 1$.

Now from (2)

$$\sum_{i<k}\frac{1}{u-x_i}=\sum_{i<k-1}\frac{1}{u-x_i}+\frac{1}{u-x_{k-1}}>\sum_{i<k-1}\frac{1}{x_k-x_i}+\frac{1}{u-x_{k-1}}$$

$$=\sum_{i<k}\frac{1}{x_k-x_i}-\frac{1}{x_k-x_{k-1}}+\frac{1}{u-x_{k-1}}$$

$$=\sum_{i<k}\frac{1}{x_k-x_i}+\frac{1}{x_k-x_{k-1}}>\sum_{j>k}\frac{1}{x_j-x_k}-\frac{1}{x_k+1}$$

$$+\frac{1}{x_k-x_{k-1}}>\sum_{j>k}\frac{1}{x_j-x_k}>\sum_{j>k}\frac{1}{x_j-u}$$

which proves $y_{k-1}>u$. Now evidently from $y_{k-1}>u$

$$\sum_{i<k}\frac{1}{x_k-y_i}=\sum_{i<k-1}\frac{1}{x_k-y_i}+\frac{1}{x_k-y_{k-1}}>\sum_{i<k-1}\frac{1}{x_k-x_i}+\frac{1}{x_k-u}$$

$$=\sum_{i<k}\frac{1}{x_k-x_i}-\frac{1}{x_k-x_{k-1}}+\frac{1}{x_k-u}$$

$$=\sum_{i<k}\frac{1}{x_k-x_i}+\frac{1}{x_k-x_{k-1}}>\sum_{i<k}\frac{1}{x_k-x_i}+\frac{1}{x_k+1}$$

and

$$\sum_{j\geqq k}\frac{1}{y_j-x_k}<\sum_{j>k}\frac{1}{x_j-x_k}.$$

Thus by (2)

$$\sum_{i\leqq k-1}\frac{1}{x_k-y_i}>\sum_{j\geqq k}\frac{1}{y_j-x_k},$$

which proves $z_{k-1}>x_k$.

Thus we obtain for $k\neq 1$

$$\max_{x_{k-1}\leqq x\leqq x_{k+1}}|l_k(x)|<1+\frac{\alpha\,|x_k|}{1+|x_k|}\tag{3}$$

and of course from (1)

$$l_1(-1)>1+\frac{\alpha\,|x_k|}{1+|x_1|}\geqq 1+\frac{\alpha\,|x_k|}{1+|x_k|}.\tag{4}$$

Suppose now $1/2\leqq\alpha\leqq 3/2$. A well known theorem of M. Riesz[4] states: Let $f(x)$ be a polynomial of degree n which assumes its absolute maximum in $(-1,1)$ at x_0; then for every root x_k of $f(x)$ in $(-1,+1)$ we have $\vartheta_k-\vartheta_0\geqq\frac{\pi}{2n}$. Here $x_k=\cos\vartheta_k$, $x_0=\cos\vartheta_0$, $0<\vartheta_k\leqq\pi$, $0<\vartheta_0\leqq\pi$.

[4] M. Riesz, Jahresbericht der Deutschen Math Vereinigung, (1916) p. 354–368.

Let $-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$ be the roots of $P_n^{(\alpha)}(x)$; put $\cos\vartheta_k = x_k$ $0 < \vartheta_k < \pi$, then it is well known that[5]

$$\vartheta_\nu - \vartheta_{\nu+1} \leqq \frac{\pi}{n + (2\alpha + 1)/2} \leqq \frac{\pi}{n}.$$

Thus $| l_k(x) |$ can take its absolute maximum in $(-1, 1)$ only in (x_{k-1}, x_{k+1}), or at the points -1 and 1. We shall prove that for $k \neq 1$,

$$| l_k(-1) | < l_1(-1). \tag{5}$$

It clearly suffices to show that

$$| P_n^{(\alpha)\prime}(x_k)(1 + x_k) | > | P_n^{(\alpha)\prime}(x_1)(1 + x_1) |.$$

Or that

$$| P_n^{(\alpha)\prime}(x_k)(1 - x_k^2) | \geqq | P_n^{(\alpha)\prime}(x_1)(1 - x_1^2) |. \tag{6}$$

By the differential equation we have

$$(1 - x^2)P_n^{(\alpha)\prime\prime}(x) - (2\alpha + 4)xP_n^{(\alpha)\prime}(x) + n(n + 2\alpha + 3)P_n^{(\alpha)}(x) = 0.$$

Now apart from a constant factor $P_n^{(\alpha)\prime}(x) = P_{n-1}^{(\alpha+1)}(x)$. Thus we can write

$$(1 - x^2)P_n^{(\alpha)\prime}(x) + c_1 x P_n^{(\alpha)}(x) + c_2 P_{n-1}^{(\alpha-1)}(x) = 0.$$

Hence for the roots of $P_n^{(\alpha)}(x)$

$$| (1 - x_k^2)P_n^{(\alpha)\prime}(x_k) | = | c_2 P_{n+1}^{(\alpha-1)}(x_k) |.$$

The points x_k are the relative maxima of $P_{n+1}^{\alpha-1}(x)$. It is well known[6] that for $\alpha \leqq 1/2$ the successive maxima of $P_n^{(\alpha)}(x)$ increase toward the origin i.e. for $\alpha \leqq 3/2$

$$| P_{n+1}^{(\alpha-1)}(x_1) \leqq | P_{n+1}^{(\alpha-1)}(x_k) |.$$

This proves (6) and therefore (5). By the symmetry of the x it follows that for $k \neq n$

$$l_1(-1) = l_n(1) > | l_k(1) |. \tag{7}$$

Thus, finally, from (3), (4), (6) and (7) we obtain our theorem for $1/2 \leqq \alpha \leqq 3/2$.

Suppose now that $0 \leqq \alpha < 1/2$. Then it is well known that $\vartheta_1 \leqq 2n$.[7] Thus according to the theorem of M. Riesz it suffices to consider the interval (x_1, x_n). Suppose then that $l_k(x)$ assumes its absolute maximum at x_0, and that x_0 is not in (x_{k-1}, x_{k+1}). It is easy to see that[8]

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[5] G. Szegö, ibid. p. 121, theorem 6.3.1.

[6] Ibid. p. 163 164, proof of theorem 7.32.1.

[7] Ibid. p. 117, theorem 6.21.1. $\vartheta_1 \leqq \dfrac{\pi}{2n}$ follows from the remark that in case of $T_n(x)(\alpha = \frac{1}{2})\vartheta_1 = \dfrac{\pi}{2n}$.

[8] Erdös-Turan, Annals of Math. vol. 41 (1940) p. 429 lemma IV.

$$l_i(x_0) + l_{i+1}(x_0) > 1, \qquad x_i < x_0 < x_{i+1}.$$

According to a formula of Fejér[9]

$$\sum_{k=1}^{n} v_k(x_0) l_k^2(x_0) = 1, \quad \text{where} \quad v_k(x_k) = 1, \tag{8}$$
$$v_k\left(x_k + \frac{1 - x_k^2}{2\alpha x_k}\right) = 0, \qquad v_k(x) \text{ linear,}$$

hence

$$v_i(x_0) l_i^2(x_0) + v_{i+1}(x_0) l_{i+1}^2(x_0) + v_k(x_0) l_k^2(x_0) \leqq 1.$$

Thus from (8)

$$v_j(x_0) > 1 - \frac{2\alpha x_j}{1 + |x_j|} \geqq 1 - \frac{2\alpha |x_1|}{1 + |x_1|} = c, \qquad \tfrac{1}{2} < c \leqq 1$$

Clearly one of the numbers $v_i(x_0)$, $x_{i+1}(x_0)$ is greater than 1. Thus

$$v_i(x_0) l_i^2(x_0) + v_{i+1} l_{i+1}^2(x_0) > \min_{x+y=1, x,y>0} (x^2 + cy^2) = \frac{c}{1+c}.$$

Hence

$$|l_k(x_0)| < \sqrt{\frac{1}{c(1+c)}}.$$

From (4) we have

$$l_1(-1) > \frac{3-c}{2},$$

and it is easy to see that

$$\frac{3-c}{2} > \sqrt{\frac{1}{c(1+c)}} \qquad (1/2 < c \leqq 1)$$

which completes the proof.

If $\alpha > 3/2$ our theorem does not hold any more, since it is easy to see that $l_1(-1)$ remains bounded but $\max l_k(x)$ does not remain bounded.

Webster[10] proved that

$$l_1^{(n)}(-1) \to (1/2j_1)^{\alpha-2} \,|\, \Gamma(\alpha) y_\alpha(j_1) \,|^{-1},$$

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[9] L. FEJÉR, Math. Annalen, 106, (1932) p. 4 and p. 43.

[10] WEBSTER, Bull. Amer. Math. Soc. 47 (1941), p. 73.

where j_1 is the first zero of $J_{\alpha-1}$ ($J(x)$ denotes Bessel functions). I think it can be shown that

$$l_1^{(n)}(-1) < (1/2j_1)^{\alpha-2}\Gamma(\alpha)y_\alpha(j_1)\,|^{-1},$$

in fact $l_1^{(n)}(-1) < l_1^{(n+1)}(-1)$. If so, we could state the following theorem:

Let $0 \leqq \alpha \leqq 3/2$. Then

$$\max_{k=1,2,\cdots n,-1\leqq x\leqq 1} |\,l_k(x)\,| < (\tfrac{1}{2}j_1)^{\alpha-2}\,|\,\Gamma(\alpha)y_\alpha(j_1)\,|^{-1},$$

and this result is the best possible.

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