THE GAUSSIAN LAW OF ERRORS IN THE THEORY OF ADDITIVE NUMBER THEORETIC FUNCTIONS.*[1]

By P. ERDÖS and M. KAC.

The present paper concerns itself with the applications of statistical methods to some number-theoretic problems. Recent investigations of Erdös and Wintner[2] have shown the importance of the notion of statistical independence in number theory; the purpose of this paper is to emphasize this fact once again.

It may be mentioned here that we get as a particular case of our main theorem the following result:

If $\nu(m)$ denotes the number of prime divisors of m, and K_n the number of those integers from 1 up to n for which $\nu(m) < \lg\lg n + \omega\sqrt{2\lg\lg n}$ (ω an arbitrary real number), then

$$\lim_{n\to\infty}\frac{K_n}{n} = \pi^{-\frac{1}{2}}\int_{-\infty}^{\omega}\exp(-u^2)\,du.$$

This theorem refines some known results of Hardy, Ramanujan[3] and Erdös.[4]

1. In what follows p will denote a prime and ω will denote a real number.

Let $f(m)$ be an additive number-theoretic function, so that $f(mn) = f(m) + f(n)$ if $(m,n) = 1$. Suppose that $f(p^\alpha) = f(p)$ and $|f(p)| \leqq 1$. Obviously

$$f(m) = \sum_{p\mid m} f(p).$$

Furthermore put $\sum_{p<n} p^{-1}f(p) = A_n$ and $(\sum_{p<n} p^{-1}f^2(p))^{\frac{1}{2}} = B_n$. Then our main theorem may be stated as follows:

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[1] A preliminary account appeared in the *Proceedings of the National Academy*, vol. 25 (1939), pp. 206-207.

[2] P. Erdös and A. Wintner, "Additive arithmetic functions and statistical independence," *American Journal of Mathematics*, vol. 61 (1939), pp. 713-722.

[3] Srinivasa Ramanujan, *Collected Papers* (1927), pp. 262-275.

[4] P. Erdös, "Note on the number of prime divisors of integers," *Journal of the London Mathematical Society*, vol. 12 (1937), pp. 308-314.

THEOREM. *If $B_n \to \infty$ as $n \to \infty$, and K_n denotes the number of integers m from 1 up to n for which*

$$f(m) < A_n + \omega\sqrt{2}\, B_n$$

then

$$\lim_{n\to\infty} \frac{K_n}{n} = \pi^{-\frac{1}{2}} \int_{-\infty}^{\omega} \exp(-u^2)\,du = D(\omega).$$

2. We first prove the following

LEMMA 1. *Let*

$$f_l(m) = \sum_{\substack{p|m \\ p<l}} f(p).$$

Then denoting by δ_l the density of the set of integers m for which $f_l(m) < A_l + \omega\sqrt{2}\, B_l$ one has

$$\lim_{l\to\infty} \delta_l = D(\omega).$$

Let $\rho_p(n)$ be 0 or $f(p)$ according as p does not or does divide n. Then

$$f_l(m) = \sum_{p<l} \rho_p(m).$$

Since the $\rho_p(n)$ are statistically independent, $f_l(m)$ behaves like a sum of independent random variables and consequently the distribution function of $f_l(m) - A_l/\sqrt{2}\, B_l$ is a convolution (Faltung) of the distribution functions of $\rho_p(m) - p^{-1}f(p)/\sqrt{2}\, B_l$ [5] $(p < l)$. It is easy to see that the "central limit theorem of the calculus of probability" can be applied to the present case,[6] and this proves our lemma.

3. Lemma 1 is the only "statistical" lemma in the proof. Using this lemma, the main result will be established by purely number-theoretical methods.

LEMMA 2. *If m_n tends to ∞* (as $n \to \infty$) *more rapidly than any fixed*

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[5] *Loc. cit.* 2, where statistical independence of arithmetical functions is defined and discussed. See also P. Hartman, E. R. van Kampen and A. Wintner, *American Journal of Mathematics*, vol. 61 (1939), pp. 477-486.

[6] Cf. for instance the first chapter of S. Bernstein's paper, "Sur l'extension du théorème limite du calcul des probabilités aux sommes de quantités dépendantes," *Mathematische Annalen*, vol. 97, pp. 1-59. See also M. Kac and H. Steinhaus, "Sur les fonctions indépendantes II," *Studia Math.*, vol. 6 (1936), pp. 59-66.

power of s_n, then the number of integers from 1 up to m_n which are not divisible by any prime less than s_n is equal to

$$\frac{m_n e^{-C}}{\lg s_n} + o\left(\frac{m_n}{\lg s_n}\right),$$

where C denotes Euler's constant.

The proof of this statement is implicitly contained in the reasoning of V. Brun on page 21 of his famous memoir "Le crible d'Erasosthène et le théorème de Goldbach"[7] and may therefore be omitted.

Let $\phi(n)$ represent a function which tends, as $n \to \infty$, to 0 in such a way that $n^{\phi(n)} \to \infty$. The function $n^{\phi(n)}$ will be denoted by α_n and $n^{\sqrt{\phi(n)}}$ by β_n. Let $a_1(n), a_2(n), \cdots$ be the integers whose prime factors are all less than α_n, and let $\psi(m; n)$ be the greatest a_i which divides m. We then have the following

LEMMA 3. *The number of integers $m \leqq n$ for which $\psi(m; n) = a_i(n)$, where $a_i(n) \leq \beta_n$ is equal to*

$$\frac{e^{-C} n}{a_i(n)\phi(n) \lg n} + o\left(\frac{n}{a_i(n)\phi(n) \lg n}\right).$$

This is a direct consequence of Lemma 2. For consider all those integers $\leqq n$ which are of the form $r \cdot a_i(n)$ and such that r is not divisible by any prime $< \alpha_n$. Evidently, the integers thus defined are all the integers $\leqq n$ for which $\psi(m; n) = a_i(n)$. Their number is equal to the number of integers r which are $\leqq n/a_i(n)$ and not divisible by any prime $< \alpha_n$. The restriction $a_i(n) < \beta_n$ makes $n/a_i(n)$ tend to ∞ more rapidly than any power of α_n and therefore Lemma 2 can be applied (put $m_n = n/a_i(n)$ and $s_n = \alpha_n$). This completes the proof.

LEMMA 4. *The number y of integers $\leq M$ divisible by an $a_i(n) > \beta_n$ is less than $bM\sqrt{\phi(n)}$, where b is an absolute constant. (It follows from this that the density of the integers which are divisible by an $a_i(n) > \beta_n$ is less than $b\sqrt{\phi(n)}$.)*

We have

$$\prod_{m=1}^{M} \psi(m; n) = \prod_{p < a_n} p^{\sum_{r=1}^{\infty} [M/p^r]} < \prod_{p < a_n} p^{2M/p};$$

and since

$$\lg \prod_{p < a_n} p^{2M/p} = 2M \sum_{p < a_n} p^{-1} \lg p \sim 2M\phi(n) \lg n$$

[7] *Skrifter Videns*, Kristiania, 1920.

one has

$$\prod_{m=1}^{M} \psi(m; n) < n^{bM\phi(n)}.$$

Hence, finally

$$(\beta_n)^y = (n^{\sqrt{\phi(n)}})^y < n^{bM\phi(n)}, \quad \text{i. e.,} \quad y < bM\sqrt{\phi(n)}.$$

4. LEMMA 5. *Denote by l_n the number of integers from 1 up to n for which*

(i) $$f_{a_n}(m) < A_{a_n} + \omega\sqrt{2}\, B_{a_n}.$$

Then

$$\lim_{n\to\infty} \frac{l_n}{n} = D(\omega).$$

Divide the integers from 1 up to n which satisfy (i) into classes $E_1, E_2, \cdots$ so that m belongs to E_i if and only if $\psi(m; n) = a_i(n)$; and denote by $|E_i|$ the number of integers in E_i. One obviously has

$$l_n = \sum_i |E_i| = \sum_{a_i \leq \beta_n} |E_i| + \sum_{a_i > \beta_n} |E_i|.$$

By Lemma 4 $\sum_{a_i > \beta_n} |E_i| < bn\sqrt{\phi(n)}$ and therefore it is sufficient to prove that $n^{-1} \sum_{a_i \leq \beta_n} |E_i| \to D(\omega)$ as $n \to \infty$. On the other hand by Lemma 3

(ii) $$\sum_{a_i \leq \beta_n} |E_i| = \left(\frac{e^{-C} \cdot n}{\phi(n) \lg n} + o\left(\frac{n}{\phi(n) \lg n}\right)\right) \sum_{a_i \leq \beta_n}' \frac{1}{a_i(n)},$$

where the dash in the summation indicates that it is extended over the a_i's satisfying $f_{a_n}(a_i) < A_{a_n} + \omega\sqrt{2}\, B_{a_n}$. In order to evaluate Σ', divide all the integers into classes $F_1, F_2, \cdots$ having the property that m belongs to F_i if and only if $\psi(m; n) = a_i(n)$ and let $\{F_i\}$ denote the density of F_i. Consider now the set $\Sigma' F_i$, where the dash in summation has the same meaning as above. By putting $l = \alpha_n$ and using Lemma 1 we have that $\{\Sigma' F_i\} \to D(\omega)$ as $n \to \infty$ or $\{\Sigma' F_i\} = D(\omega) + o(1)$. Now

(iii) $$\Sigma' F_i = \sum_{a_i \leq \beta_n}' F_i + \sum_{a_i > \beta_n}' F_i$$

and by Lemma 4

(iv) $$\{\sum_{a_i > \beta_n}' F_i\} < b\sqrt{\phi}(n).$$

Furthermore there is only a finite number of a_i's which are less than β_n and therefore $\{\sum_{a_i < \beta_n}' F_i\} = \sum_{a_i < \beta_n}' \{F_i\}$. But

$$\{F_i\} = \frac{1}{a_i(n)} \prod_{p < a_n} \left(1 - \frac{1}{p}\right) = \frac{1}{a_i(n)} \left(\frac{e^{-C}}{\phi(n) \lg n} + o\left(\frac{1}{\phi(n) \lg n}\right)\right)$$

and this implies that

$$\text{(v)} \qquad \{ \sum_{a_i < \beta_n}' F_i \} = \left(\frac{e^{-C}}{\phi(n) \lg n} + o \left(\frac{1}{\phi(n) \lg n} \right) \right) \sum_{a_i < \beta_n}' \frac{1}{a_i(n)}.$$

Finally (iii), (iv) and (v) give

$$D(\omega) - b\sqrt{\phi(n)} < \left(\frac{e^{-C}}{\phi(n) \lg n} \right) + o \left(\frac{1}{\phi(n) \lg n} \right) \Big) \sum_{a_i < \beta_n}' \frac{1}{a_i(n)} < D(\omega) + 0(1).$$

The combination of this formula with (ii) completes the proof of our Lemma.

5. We now come to the proof of the main theorem. Notice first that for $m \leq n$, $|f(m) - f_{a_n}(m)| < 1/\phi(n)$. In fact, $|f(p)| \leq 1$ implies that $|f(m) - f_{a_n}(m)|$ is less than the number of those prime divisors of m which are $\geq \alpha_n$. This number is obviously $< 1/\phi(n)$, since $(\alpha_n)^{1/\phi(n)} = n$. Notice furthermore that $|f(p)| \leq 1$ and the well known results concerning the sum $\sum_{p<n} p^{-1}$ imply that $|A_n - A_{a_n}| < - C_1 \lg \phi(n)$ and $|B_n - B_{a_n}| < - C_2 \lg \phi(n)$, where C_1 and C_2 are absolute constants.

Now choose $\phi(n)$ so that $1/\phi(n) = o(B_n)$. Evidently every $m \leq n$ satisfying the inequality $f(m) < A_n + \omega\sqrt{2}\, B_n$ also satisfies, for sufficiently large n, the inequality $f_{a_n}(m) < A_{a_n} + (\omega + \epsilon)\sqrt{2}\, B_{a_n}$. In addition every $m \leq n$ satisfying $f_{a_n}(m) < A_{a_n} + (\omega - \epsilon)\sqrt{2}\, B_{a_n}$ satisfies, for sufficiently large n, the inequality $f(m) < A_n + \omega\sqrt{2}\, B_n$. Hence, by Lemma 5,

$$D(\omega - \epsilon) \leq \liminf \frac{K_n}{n} \leq \limsup \frac{K_n}{n} \leq D(\omega + \epsilon);$$

and this proves the theorem, since $\epsilon > 0$ is arbitrary.

6. The theorem mentioned in the introduction is obviously a particular case of our main theorem. It corresponds to the case $f(p) = 1$. Because of the large number of applications of $\nu(m)$ it is of special interest. It should be mentioned that the assumption $f(p^a) = f(p)$ can be removed; also $|f(p)| \leq 1$ may be replaced by a much weaker condition. This however, would complicate the statement of the main theorem

We may perhaps point out that Lemma 2 (Brun) is the "deepest" part of the proof and that the "statistical" part is relatively superficial. However, the statistical considerations seemed to be suggestive and fruitful in leading to new and perhaps striking results.

INSTITUTE FOR ADVANCED STUDY AND CORNELL UNIVERSITY.