Hardy and Ramanujan* proved that $\nu(m)$ is almost always $\log\log n$, *i.e.* that for any positive ϵ there are only $o(n)$ integers $m \leqslant n$ for which either $\nu(m) > (1+\epsilon)\log\log n$ or $\nu(m) < (1-\epsilon)\log\log n$.

We use the following notation:

1. T denotes the closed interval $[(\log n)^6,\ n^{(\log\log n)^{-3}}]$,

2. $\nu'(m)$ the number of different prime factors of m in T,

3. $q_1, q_2, \ldots, q_v$ symbols for the v primes q of T,

4. $a_1, a_2, \ldots$ the integers composed of q_i,

5. $a_1^{(k)}, a_2^{(k)}, \ldots$ the integers whose factors are powers of k different q_i $(k < 2\log\log n)$,

6. $A(m)$ the greatest a_i contained in m,

7. U_k the number of integers $m \leqslant n$ for which $A(m)$ is an $a^{(k)}$,

8. $c_1, c_2, \ldots$ absolute constants,

9. $x = \sum_q \frac{1}{q}$; from the formula $\sum_{p<y} \frac{1}{p} = \log\log y + c_1 + o(1)$, it immedately follows that $x = \log\log n - 4\log\log\log n - \log 6 + o(1)$.

We require four lemmas.

Lemma 1. *The number of integers $m \leqslant n$ for which*

$$\nu(m) - \nu'(m) > (\log\log\log n)^2$$

is $o(n)$.

We evidently have

$$\sum_{m=1}^{n} \left(\nu(m) - \nu'(m)\right) = \sum_{p \leqslant n} \left[\frac{n}{p}\right] - \sum_q \left[\frac{n}{q}\right]$$

$$= \sum_{p<(\log n)^6} \left[\frac{n}{p}\right] + \sum_{n^{(\log\log n)^{-3}} < p \leqslant n} \left[\frac{n}{p}\right]$$

$$= O(n \log\log\log n),$$

which implies Lemma 1.

Lemma 2.

$$\frac{x^k}{k!} - o\left(\frac{1}{(\log n)^2}\right) < \sum_i{}' \frac{1}{a_i^{(k)}} < \frac{x^k}{k!},$$

† Srinivasa Ramanujan, *Collected papers* (1927), 262–275.

where the dash in the summation means that the summation is extended over the square-free $a^{(k)}$'s only.

We have

$$\sum_i{}' \frac{1}{a_i^{(k)}} < \frac{\left(\sum_q \frac{1}{q}\right)^k}{k!} = \frac{x^k}{k!}.$$

By expanding $\left(\sum_q \frac{1}{q}\right)^k \Big/ k!$ by the multinomial theorem we see that the coefficient of the terms whose denominator is a square-free $a^{(k)}$ is 1, but the other terms contain in their denominator the square of a q, *i.e.* a square greater than $(\log n)^{12}$ and have coefficients less than 1. Finally, the denominators are all less than $n^{2/(\log\log n)^2}$, since $k < 2\log\log n$. Thus

$$\frac{\left(\sum_q \frac{1}{q}\right)^k}{k!} < \sum_i{}' \frac{1}{a_i^{(k)}} + \sum_{r > (\log n)^6} \frac{1}{r^2} \sum_{i<n} \frac{1}{i} = \sum_i{}' \frac{1}{a_i^{(k)}} + O\left(\frac{1}{(\log n)^5}\right),$$

$$\sum_i{}' \frac{1}{a_i^{(k)}} > \frac{\left(\sum_q \frac{1}{q}\right)^k}{k!} - O\left(\frac{1}{(\log n)^5}\right),$$

and hence

$$\sum_i{}' \frac{1}{a_i^{(k)}} > \frac{x^k}{k!} - o\left(\frac{1}{(\log n)^2}\right),$$

which establishes Lemma 2.

LEMMA 3. $$U_k = ne^{-x}\frac{x^k}{k!} + o\left(\frac{n}{(\log n)^2}\right).$$

First we evaluate the number of integers $m \leqslant n$ for which $A(m) = a_i^{(k)}$. The number of the $m \leqslant n$ divisible by the square of a q is less than $\sum_i \frac{n}{q^2} = O\left(\frac{n}{(\log n)^6}\right)$. If m is not divisible by the square of a q, $A(m)$ is square-free, and the number of the m for which $A(m) = a_i^{(k)}$ is equal to the number z of integers

$$m \leqslant \frac{n}{a_i^{(k)}},$$

no one of which is divisible by a q. We calculate z by Brun's method. We have

$$z = \left[\frac{n}{a_i^{(k)}}\right] - \sum_q \left[\frac{n}{q a_i^{(k)}}\right] + \sum_{q_1 < q_2} \left[\frac{n}{q_1 q_2 a_i^{(k)}}\right] - \cdots + (-1)^r \sum_{q_1 < q_2 < \ldots < q_r} \left[\frac{n}{q_1 q_2 \cdots q_r a_i^{(k)}}\right] + \ldots \quad (1)$$

We write

$$s_r = \sum_{q_1<q_2<\ldots<q_r} \left[\frac{n}{q_1 q_2 \ldots q_r a_i^{(k)}}\right]$$

and

$$s_r' = \sum_{q_1<q_2<\ldots<q_r} \frac{n}{q_1 q_2 \ldots q_r a_i^{(k)}},$$

so that we have

$$z = \sum_{r=0}^{v} (-1)^r s_r. \tag{1'}$$

Now, evidently,

$$\sum_{r\leqslant 10\log\log n} (-1)^r s_r - \sum_{r>10\log\log n} s_r \leqslant z \leqslant \sum_{r\leqslant 10\log\log n} (-1)^r s_r + \sum_{r>10\log\log n} s_r, \tag{2}$$

but

$$\sum_{r>10\log\log n} s_r \leqslant \sum_{r>10\log\log n} s_r' < \frac{n}{a_i^{(k)}} \sum_{r>10\log\log n} \frac{\left(\sum_q \frac{1}{q}\right)^r}{r!}$$

$$< \frac{n}{a_i^{(k)}} \sum_{r>10\log\log n} \frac{(\log\log n)^r}{r!} < \frac{2n}{a_i^{(k)}} \frac{(\log\log n)^{[10\log\log n]}}{[10\log\log n]!}$$

$$< \frac{2ne^{10\log\log n}(10\log\log n+1)}{a_i^{(k)}\, 10^{10\log\log n}} < \frac{2n}{a_i^{(k)}\, 2^{10\log\log n}} \tag{3}$$

since

$$y! > \frac{y^y}{e^y}.$$

Hence, from (1′), on noting the right-hand inequalities in (2) and (3) and omitting the square brackets, we obtain

$$z = \sum_{r\leqslant 10\log\log n} (-1)^r s_r' + O\left((1+v)^{10\log\log n}\right) + O\left(\frac{n}{a_i^{(k)}\, 2^{10\log\log n}}\right), \tag{4}$$

the v term arising from a possible error $1+v+\binom{v}{2}+\ldots$ up to $10\log\log n$ terms.

From (3), (4), and $1+v<n^{(\log\log n)^{-3}}$, we obtain

$$z = \sum_r (-1)^r s_r' + O(n^{10/(\log\log n)^2} + O\left(\frac{n}{a_i^{(k)}\, 2^{10\log\log n}}\right). \tag{5}$$

Now we have

$$\sum_r (-1)^r s_r' = \frac{n}{a_i^{(k)}} \prod_q \left(1-\frac{1}{q}\right) = \frac{n}{a_i^{(k)}} e^{\sum_q(-(1/q)+O(1/q^2))} = \frac{n}{a_i^{(k)}} e^{-x} e^{\sum_q O(1/q^2)}$$

$$= \frac{n}{a_i^{(k)}} e^{-x} e^{O(1/(\log n)^6)} = \frac{n}{a_i^{(k)}} e^{-x}\left(1+O\left(\frac{1}{(\log n)^6}\right)\right).$$

Thus

$$z = \frac{n}{a_i^{(k)}} e^{-x}\left(1+O\left(\frac{1}{(\log n)^6}\right)\right)+O(n^{10/(\log\log n)^2})+O\left(\frac{n}{a_i^{(k)}\, 2^{10\log\log n}}\right). \quad (6)$$

From (6) we easily obtain

$$U_k = ne^{-x}\left(1+O\left(\frac{1}{(\log n)^6}\right)\right)\sum_i{}' \frac{1}{a_i^{(k)}}+O(n^{20/(\log\log n)^2})$$
$$+O\left(\frac{n}{2^{10\log\log n}}\sum_i \frac{1}{a_i^{(k)}}\right)+O\left(\frac{n}{(\log n)^6}\right), \quad (7)$$

since the number of the square-free $a_i^{(k)} \leqslant n$ is less than

$$(1+v)^k < n^{10/(\log\log n)^2},$$

and finally, from Lemma 2 and from

$$\sum_i \frac{1}{a_i^{(k)}} < \sum_{l<n} \frac{1}{l} = O(\log n),$$

we have
$$U_k = ne^{-x}\frac{x^k}{k!}+o\left(\frac{n}{(\log n)^2}\right). \quad (8)$$

Thus Lemma 3 is proved.

LEMMA 4. *The number of integers* $m \leqslant n$ *for which* $\nu'(m) > \log\log n$ *is* $\frac{1}{2}n+o(n)$.

Evidently $\nu'(m) = \nu[A(m)]$; thus we have only to consider the integers for which $\nu[A(m)] > \log\log n$.

First we prove that the number of integers for which $\nu[A(m)] > x$ is $\frac{1}{2}n+o(n)$, *i.e.*

$$\sum_{k>x} U_k = \tfrac{1}{2}n+o(n).$$

Since $\sum_{r=1}^{n} d(r) = O(n\log n)$, the number of integers $m \leqslant n$ for which $\nu(m) > 2\log\log n$ is $O(n\log n/2^{2\log\log n}) = o(n)$, so that we have to prove

$$\sum_{k>x}^{k\leqslant 2\log\log n} U_k = \tfrac{1}{2}n+o(n),$$

i.e., by Lemma 3,

$$ne^{-x}\sum_{k>x}^{k\leqslant 2\log\log n}\frac{x^k}{k!} = \tfrac{1}{2}n+o(n). \quad (9)$$

But it is known that*

$$\sum_{k>x}\frac{x^k}{k!} = \tfrac{1}{2}e^x+o(e^x) \quad (10)$$

* Srinivasa Ramanujan, *Collected papers*, 323, Question 294.

and

$$\sum_{k>2\log\log n} \frac{x^k}{k!} < 2\,\frac{x^{2\log\log n}}{[2\log\log n]!} < \frac{2x^{2\log\log n}\,e^{2\log\log n}(2\log\log n+1)}{2^{2\log\log n}(\log\log n)^{2\log\log n}}$$

$$< \frac{2e^{2\log\log n}(2\log\log n+1)}{2^{2\log\log n}} = o(e^x), \tag{11}$$

and (9) is an immediate consequence of (10) and (11). We now have to prove that there are only $o(n)$ integers $m \leqslant n$ for which

$$x \leqslant \nu'(m) \leqslant \log\log n.$$

From Lemma 3 we see that, since $x^k/k!$ assumes its maximum value for $k = [x]$, the number of integers $m \leqslant n$ for which $\nu'(m) = k$ is, by Stirling's formula, at the utmost

$$ne^{-x}\,\frac{x^{[x]}}{[x]!} + o\left(\frac{n}{\log^2 n}\right) < \frac{c_2 n}{\sqrt{x}}. \tag{12}$$

Hence the number of integers $m \leqslant n$ for which $x < \nu'(m) \leqslant \log\log n$ is

$$O\left(\frac{n}{\sqrt{x}}\,(\log\log n - x)\right) = O\left(\frac{n\log\log\log n)}{(\log\log n)^{\frac{1}{2}}}\right) = o(n),$$

which completes the proof of Lemma 4.

We now proceed to prove our main theorem.

By Lemma 4, we have only to prove that the number of integers $m \leqslant n$ for which $\nu'(m) \leqslant \log\log n$ but $\nu(m) > \log\log n$ is $o(n)$.

We divide these integers into two classes.

In the first class are the integers for which

$$\nu'(m) < \log\log n - (\log\log\log n)^2.$$

For these, $\nu(m) - \nu'(m) > (\log\log\log n)^2$, and so, from Lemma 1, the number of them is $o(n)$.

For the integers of the second class

$$\log\log n - (\log\log\log n)^2 \leqslant \nu'(m) \leqslant \log\log n.$$

From (12), it follows that the number of them is less than

$$\frac{c_2 n}{\sqrt{x}}\left((\log\log\log n)^2 + 1\right) = O\left(\frac{n(\log\log\log n)^2}{(\log\log n)^{\frac{1}{2}}}\right) = o(n).$$

Thus our theorem is established.

In consequence of the exceedingly slow increase of $\log\log n$ we can easily deduce from our theorem that the number of integers $m \leqslant n$ for which $\nu(m) > \log\log m$ is also $\frac{1}{2}n + o(n)$.

Let $f(m)$ be the number of prime factors of m, multiple factors being counted multiply. We easily deduce that for every ϵ there exists a c_3 such that the number of integers $m \leqslant n$ for which $f(m) - \nu(m) > c_3$ is less than ϵn, and from this it is clear that the number of integers $m \leqslant n$ for which

$$f(m) > \log \log n$$

is $\frac{1}{2}n + o(n)$.

By similar methods we can prove the following theorems:

THEOREM 1. *Let $\nu_1(m)$ and $\nu_2(m)$ denote the numbers of prime factors of m of the forms $4k+1$ and $4k+3$ respectively. The number of integers $m \leqslant n$ for which $\nu_1(m) > \nu_2(m)$ is $\frac{1}{2}n + o(n)$. The same holds for $\nu_1(m) < \nu_2(m)$ and hence the number of integers $m \leqslant n$ for which $\nu_1(m) = \nu_2(m)$ is $o(n)$.*

THEOREM 2. *Let $A_1(m)$ and $A_2(m)$ denote the product of all prime factors of m of the forms $4k+1$ and $4k+3$ respectively, multiple factors being counted multiply. The number of integers $m \leqslant n$, for which $A_1(m) > A_2(m)$ is $\frac{1}{2}n + o(n)$.*

THEOREM 3. *The number of integers $m \leqslant n$, the greatest prime factor of which is a prime of the form $4k+1$, is $\frac{1}{2}n + o(n)$.*

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