ON SOME SEQUENCES OF INTEGERS

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Consider a sequence of integers \( a_1 < a_2 < \ldots \leq N \) containing no three terms for which \( a_i - a_{i-1} = a_{i-1} - a_{i-2} \), i.e. a sequence containing no three consecutive members of an arithmetic progression. Such sequences we call \( A \) sequences belonging to \( N \), or simply \( A \) sequences. We consider those with the maximum number of elements, and denote by \( r = r(N) \)

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the number of elements of such maximum sequences. In this paper we estimate \( r(N) \).

**Theorem 1.** \( r(2N) \leq N \) if \( N \geq 8 \).

**Remark.** It is interesting to observe that, as we shall see, the theorem is true for \( N = 4, 5, 6 \), but not for \( N = 7 \).

**Proof.** First we observe that, if \( a_1 < a_2 < \ldots < a_r \) represents an \( A \) sequence belonging to \( N \), then

\[
N + 1 - a_r < N + 1 - a_{r-1} < \ldots < N + 1 - a_1
\]

is also an \( A \) sequence.

The same holds for

\[
a_1 - k < a_2 - k < \ldots < a_r - k,
\]

for any integer \( k < a_1 \).

Hence, evidently,

\[
r(m+n) \leq r(m) + r(n).
\]

We prove Theorem 1 by induction. Consider first the case \( N = 4 \). If we have \( r(8) = 5 \), then, in consequence of (1) and (2), we may suppose that 1 and two other integers less than or equal to 4 occur in the maximum sequence. Hence the sequence contains either 1, 2, 4 or 1, 3, 4. But it is evident that neither of these sequences leads to \( r(8) = 5 \). Hence \( r(8) \leq 4 \), and, since 1, 2, 4, 5 is an \( A \) sequence, \( r(8) = 4 \).

Consider now \( r(10) \). If \( r(10) = 6 \), then, in consequence of \( r(8) = 4 \) and (2), 1, 2, 9, 10 occurs in the sequence. But then 3, 5, 6, and 8 cannot occur. Thus the only possibility is 1, 2, 4, 7, 9, 10; this is impossible because it contains 1, 4, 7. Hence \( r(10) \leq 5 \), and, since 1, 2, 4, 9, 10 is an \( A \) sequence, \( r(10) = 5 \).

Now we consider \( r(12) \). If \( r(12) = 7 \), by the above argument 1, 2, 11, 12 occurs in our sequence. In consequence of \( r(8) = 4 \) and (2), 4 and 9 must occur, too. Hence the sequence contains 1, 2, 4, 9, 11, 12; but it cannot contain any other integers. Thus \( r(12) = 6 \). Since 1, 2, 4, 5, 10, 11, 13, 14 is an \( A \) sequence, \( r(14) = 8 \) and \( r(13) = 7 \). In consequence of (3), we have \( r(16) \leq 8, r(18) \leq 9, r(20) \leq 10, r(22) \leq 11 \).

From these results we now easily deduce the general theorem.

\* \( r(9) = 5 \) and \( r(11) = 6 \), since 1, 2, 4, 8, 9 and 1, 2, 4, 8, 9, 11 are \( A \) sequences.
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Suppose that the theorem holds for $2N-8$. Then, by (3),

$$r(2N) \leq r(2N-8) + r(8) < N-3 + 4 = N+1,$$

i.e. the theorem is proved, for we have established it for the special cases 16, 18, 20, 22.

For sufficiently large $N$, we have a better estimate by

THEOREM II. For $\epsilon > 0$ and $N > N_0(\epsilon)$,

$$r(N) < \left(\frac{2}{3} + \epsilon\right)N.$$

First we prove that $r(17) = 8$. Since $r(14) = 8$, it is evident that $r(17) \geq 8$. In the case $r(17) = 9$, the numbers 1 and 17 must occur, since $r(14) = 8$. But then 9 cannot occur, and so, by (2),

$$r(17) = r(8) + r(8) = 8. \quad \text{Thus } r(34) \leq 16.$$

Further, $r(35) \leq 16$. For, if $r(35) \geq 17$, then, by $r(34) \leq 16$, the integers 1 and 35 must occur; but then 18 cannot occur, since the sequence would contain 1, 18, 35. Hence, as previously, $r(35) \leq 16$.

Similarly $r(71) \leq 32$, ..., $r(2^k + 2^{k-3} - 1) \leq 2^{k-1}$. Hence the result.

By a similar but very much longer argument we find that

$$r(18) = r(19) = r(20) = 8.$$

On the other hand, $r(21) = 9$, since 1, 3, 4, 8, 9, 16, 18, 19, 21 is an $A$ sequence; further,

$$r(22) = r(23) = 9.$$

Hence, as previously, we find that, for sufficiently large $N > N(\epsilon)$,

$$r(N) < \left(\frac{2}{3} + \epsilon\right)N.$$

At present this is the best result for $r(N)$. It is probable that

$$r(N) = o(N).$$

It may be noted that, from $r(20) = 8$, $r(41) \leq 16$. On the other hand, $r(41) = 16$, since 1, 2, 4, 5, 10, 11, 13, 14, 28, 29, 31, 32, 37, 38, 40, 41 is an $A$ sequence. G. Szekeres has conjectured that $r\left\{\frac{3}{2}(3^k+1)\right\} = 2^k$. This is proved* for $k = 1, 2, 3, 4$.

More generally, he has conjectured that, if we denote by $r_i(N)$ the maximum number of integers less than or equal to $N$ such that no $l$ of

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* It is easily seen that $r\left\{\frac{3}{2}(3^k+1)\right\} \geq 2^k$; for, if $u < \frac{3}{2}(3^k-1)$ is any integer not containing the digit 2 in the ternary scale, then the integers $u+1$ form an $A$ sequence.
them form an arithmetic progression, then, for any $k$, and any prime $p$,

$$r_p \left( \frac{(p-2)p^k+1}{p-1} \right) = (p-1)^k.$$ 

An immediate and very interesting consequence of this conjecture would be that for every $k$ there is an infinity of $k$ combinations of primes forming an arithmetic progression.

Another consequence of it would be a new proof of a theorem of van der Waerden which would give much better limits than any of the previous proofs. Namely, it would follow from the conjecture that, if we denote by $N = f(k, l)$ the least integer such that, if we split the integers up to $N$ into $l$ classes, at least one of them contains an arithmetic progression of $k$ terms, then

$$f(k, l) < k^c \log l.$$ 

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