1 THE CYCLE METHOD AND ITS LIMITS

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Abstract: A powerful tool of extremal set theory, the cycle method is surveyed in the paper. It works, however only when the non-emptyness of the pairwise intersections of the members of the family is assumed. If these intersections have to be at least 2, the method fails: the celebrated Complete Intersection Theorem by Ahlswede and Khachatrian cannot be proved by this method. We show the reasons and some attempts to overcome the difficulties.

1.1 THE BEGINNING

Let $X = \{1, 2, \ldots, n\}$ be a finite set of n elements, we will consider families $\mathcal{F}$ of its subsets: $\mathcal{F} \subset 2^X$. The family of all k-element subsets of X will be denoted by $\binom{X}{k}$. A family $\mathcal{F}$ of distinct subsets is called *intersecting* if $F, G \in \mathcal{F}$ implies $F \cap G \neq \emptyset$. One of the fundamental theorems of the theory of *extremal families* is the Erdős-Ko-Rado theorem ([8]). It answers the question, what is the maximum size of an intersecting family of subsets of an n-element set. If $k > \frac{n}{2}$ then the question is uninteresting, one can choose all k-element subsets, this family will be intersecting. This is not true when $k \leq \frac{n}{2}$. In this case one can choose all k-element subsets containing the element $1 \in X$. The theorem states that this is the best we can do.

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Theorem 1.1.1 (Erdős-Ko-Rado) *Let* $|X| = n, k \leq \frac{n}{2}$, *and suppose that* $\mathcal{F} \subset \binom{X}{k}$ *is an intersecting family. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}. \tag{1.1}$$

The cycle method will be illustrated by the proof of this theorem.

Proof ([20]) Place the elements of X listed along a cycle and consider the *intervals* along this cycle, that is, the sets of form $\{i, i+1, \ldots, i+l\}$ where these numbers are taken mod n. Solve the question of Erdős-Ko-Rado for intervals of length k, first. The number of intervals of length k containing the element 1 is obviously k and this family of intervals is intersecting. We will see that this is the best.

Lemma 1.1.2 *If* $A_1, A_2, \ldots, A_s$ *is a family of intersecting* k*-element intervals in* X *then*

$$s \leq k. \tag{1.2}$$

Proof of the lemma Suppose that one of the A's, say, $A_1 = \{1, 2, \ldots, k\}$. The intersection property implies that every other A has either its first or last element in A_1. However, i cannot be the last element of an A when $i+1$ is the first element of another A, since $2k < n$, the two intervals cannot meet at the "other end". Therefore there is at most one further A for each pair $i, i+1$ $(1 \leq i < k)$. The total number of As is at most $1 + k - 1 = k$, proving the lemma.

The rest of the proof is based on *double counting.* Let $\mathcal{F}$ be the family in the theorem. Count the number of pairs (C, F) where C is a cyclic permutation of X, $F \in \mathcal{F}$ is an interval in the permuted X. First fix F. The number of permutations of X where F is an interval is $k!(n-k)!$ since the elements of F and the other elements can be permuted independently. Therefore the number of pairs is $|\mathcal{F}|k!(n-k)!$.

Now fix the permutation C. The lemma can be applied for any permuted version of X therefore, by (1.2), there are at most k members $F \in \mathcal{F}$ which are intervals in this permutation. Since the number of cyclic permutations is $(n-1)!$, the number of pairs is at most $(n-1)!k$. Comparing the two countings:

$$|\mathcal{F}|k!(n-k)! \leq (n-1)!k.$$

This is equivalent to (1.1).

Observe that the "miracle" works because we found a subfamily (intervals) of $\binom{X}{k}$ in which the intersecting property ensures proportionally the same bound as in the original "big" case. Namely, as the lemma states we can have at most k sets out of the n intervals. The proportion is $\frac{k}{n}$. This proportional bound is sufficient for the original problem, since

$$\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}.$$

1.2 UNICITY IN THE SPERNER THEOREM

The very first theorem of the theory of extremal families was the theorem of Sperner ([28]). A family $\mathcal{F}$ of distinct subsets is called *inclusion-free* if $F, G \in \mathcal{F}$ implies $F \not\subset G$. It is obvious that the family of all k-element subsets is inclusion-free. The largest one of the numbers $\binom{n}{k}$ is $\binom{n}{\lfloor n/2 \rfloor}$, therefore we have an inclusion-free family of this size. Sperner's theorem states that this is the best.

Theorem 1.2.1 (Sperner) *Let $\mathcal{F} \subset 2^X$ be an inclusion-free family, then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \tag{2.1}$$

with equality only when

$$\mathcal{F} = \binom{X}{\lfloor \frac{n}{2} \rfloor} \quad or \quad \binom{X}{\lceil \frac{n}{2} \rceil}. \tag{2.2}$$

The simplest proof of (2.1) is due to Lubell [24]. His proof is somewhat simpler than the cycle method. The application of this latter method, however, gives an easy proof for the second part of the theorem, too.

Proof (Füredi [14]) The following lemma solves the analogous question for the cycle.

Lemma 1.2.2 *If $A_1, A_2, \ldots, A_s$ is a family of inclusion-free intervals in X then*

$$s \leq n \tag{2.3}$$

with equality only when the family consists of all possible intervals of a fixed length.

Proof of the lemma Since the family of intervals is inclusion-free, at most one of them can start with i ($1 \leq i \leq n$). This proves (2.3). In the case of equality $s = n$, suppose that the interval starting with i is denoted by A_i. It is easy to see that $|A_i| \leq |A_{i+1}|$ holds, otherwise $A_{i+1} \subset A_i$ contradicts our assumption. Finally $|A_1| \leq |A_2| \leq \cdots \leq |A_n| \leq |A_1|$ proves the statement.

Count the number of pairs (C, F) where C is a cyclic permutation of X, $F \in \mathcal{F}$ is an interval in C. For any fixed F the number of cyclic permutations in which F is an interval is $|F|!(n - |F|)!$, therefore the number of pairs is $|\mathcal{F}||F|!(n - |F|)!$. On the other hand, for any fixed C there are at most n intervals from $\mathcal{F}$. The number of pairs is at most $(n-1)!n$. Hence we obtained the inequality

$$|\mathcal{F}||F|!(n - |F|)! \leq n! \tag{2.4}$$

which is equivalent to (2.1).

Suppose that there is an equality in (2.4). Then there are exactly n intervals from $\mathcal{F}$ along each cycle. Using the second part of the lemma all intervals along a given cycle must have the same length. Let $F, G \in \mathcal{F}$. It is easy to see that there is a cycle in which both F and G are intervals. (It can be formed from the intervals $F-G, F\cap G, G-F, X-F-G$.) This proves $|F|=|G|$ for any two members. Hence

$$\mathcal{F}=\binom{X}{k}$$

for some k. The latter expression is maximum only when $k=\lfloor\frac{n}{2}\rfloor$ or $k=\lceil\frac{n}{2}\rceil$.

1.3 DOUBLE COUNTING WITH WEIGHT

Combine the above conditions and find the largest intersecting, inclusion-free family. It is easy to see that

$$\binom{X}{\lceil\frac{n+1}{2}\rceil}$$

satisfies these conditions. The following theorem states that this is the best one.

Theorem 1.3.1 (Milner [25]) *Let $\mathcal{F}\subset 2^X$ be an intersecting, inclusion-free family, then*

$$|\mathcal{F}|\le\binom{n}{\lceil\frac{n+1}{2}\rceil}. \tag{3.1}$$

Proof ([22]) We will use double counting with a weight function. This is why the lemma does not simply upperbound the number of intervals in question.

Lemma 1.3.2 *If $A_1, A_2, \dots, A_s$ is a family of intersecting, inclusion-free intervals in X then*

$$\sum_{i=1}^{s}\binom{n}{|A_i|}\le n\binom{n}{\lceil\frac{n+1}{2}\rceil}. \tag{3.2}$$

Without **proof,** see [22].

The number of pairs $(\mathcal{C},F)$ where $\mathcal{C}$ is a cyclic permutation of X, $F\in\mathcal{F}$ is an interval in $\mathcal{C}$ will be counted with the weight $\binom{n}{|F|}$, that is, the sum

$$\sum_{(\mathcal{C},F)}\binom{n}{|F|} \tag{3.3}$$

will be considered. On one hand it is equal to

$$\sum_{F\in\mathcal{F}}\ \sum_{\{\mathcal{C}:F \text{ is an interval in } \mathcal{C}\}}\binom{n}{|F|}=\sum_{F\in\mathcal{F}}|F|!(n-|F|)!\binom{n}{|F|}=|\mathcal{F}|n!. \tag{3.4}$$

On the other hand, (3.3) can also be written in the form

$$\sum_{\mathcal{C}}\ \sum_{\{F\in\mathcal{F}:\text{ is an interval in } \mathcal{C}\}}\binom{n}{|F|}$$

that is, by the lemma

$$\sum_C n\binom{n}{\lceil \frac{n+1}{2}\rceil} = n!\binom{n}{\lceil \frac{n+1}{2}\rceil} \tag{3.5}$$

is an upper bound on (3.3). Comparing (3.4) and (3.5) the statement of the theorem is obtained.

1.4 INEQUALITIES FOR INTERSECTING, INCLUSION-FREE FAMILIES

One can prove more complicated inequalities rather than just an upper bound on the number of members of $\mathcal{F}$.

Theorem 1.4.1 (Bollobás [2]) *If $\mathcal{F}$ is an intersecting, inclusion-free family of subsets of X then*

$$\sum_{\substack{F\in\mathcal{F}\\ |F|\le n/2}} \frac{1}{\binom{n-1}{|F|-1}} \le 1. \tag{4.1}$$

Proof Again, the analogous inequality for intervals is needed for the proof of the theorem.

Lemma 1.4.2 *If $\mathcal{A}$ is a family of intersecting, inclusion-free intervals in X then*

$$\sum_{\substack{A\in\mathcal{A}\\ |A|\le n/2}} \frac{1}{|A|} \tag{4.2}$$

holds.

Without **proof**, see [2] .

The obvious weight function will be used in the double counting, the sum

$$\sum_{(C,F)} \frac{1}{|F|} \tag{4.3}$$

will be considered. On one hand, it is equal to

$$\sum_{\substack{F\in\mathcal{F}\\ |F|\le n/2}} \quad \sum_{\{C:F \text{ is an interval in } C\}} \frac{1}{|F|} = \sum_{\substack{F\in\mathcal{F}\\ |F|\le n/2}} |F|!(n-|F|)!\frac{1}{|F|}. \tag{4.4}$$

On the other hand, by the lemma we have

$$\sum_C 1 = (n-1)! \tag{4.5}$$

as an upper bound for (4.3). The comparison of (4.4) and (4.5) proves (4.3).

The above theorem does not say anything about the large members of the family. The following theorem tries to improve this situation.

Theorem 1.4.3 ([18]) *If $\mathcal{F}$ is an intersecting, inclusion-free family of subsets of X then*

$$\sum_{\substack{F\in\mathcal{F}\\ |F|\le n/2}} \frac{1}{\binom{n}{|F|-1}} + \sum_{\substack{F\in\mathcal{F}\\ |F|> n/2}} \frac{1}{\binom{n}{|F|}} \le 1. \qquad (4.6)$$

Proof Here the small and large members need different kinds of weights.

Lemma 1.4.4 *If $\mathcal{A}$ is a family of intersecting, inclusion-free intervals in X then*

$$\sum_{A\in\mathcal{A},|A|\le n/2} \frac{n-|A|+1}{|A|} + \sum_{A\in\mathcal{A},|A|\le n/2} \frac{1}{n} \qquad (4.7)$$

holds.

Without **proof**, see [18] .

The rest of the proof is the same as in the case of Bollobás's theorem.

1.5 CONVEX HULLS

Introduce the notation $p_i(\mathcal{F}) = |\{F\in\mathcal{F} : |F| = i\}|$ $(1\le i\le n)$. Furthermore, the vector $p(\mathcal{F}) = (p_0, p_1, \dots, p_n) \in R^{n+1}$ is called the *profile vector* of $\mathcal{F}$. Then, e.g. the Bollobás inequality (4.1) can be written in the form

$$\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor} \frac{f_i}{\binom{n-1}{i-1}} \le 1.$$

Observe that this is a linear inequality which has to be satisfied for the profile vector of an intersecting inclusion-free family. The coefficients are

$$c_3(n,i) = \begin{cases} \frac{1}{\binom{n-1}{i-1}} & \text{if} \quad 1\le i\le \frac{n}{2}, \\ 0 & \text{if} \quad \frac{n}{2} < i. \end{cases}$$

Our other statements can also be written in a form of a linear inequality for the profile vector:

$$\sum_{i=1}^{n} f_i c(n,i) \le 1. \qquad (5.1)$$

Supposing $k\le\lfloor\frac{n}{2}\rfloor$ and choosing

$$c_1(n,i) = \begin{cases} \frac{1}{\binom{n-1}{k-1}} & \text{if} \quad i=k, \\ 0 & \text{if} \quad i\ne k. \end{cases}$$

(5.1) becomes the Erdős-Ko-Rado theorem.

$$c_2(n,i) = \frac{1}{\binom{n}{\lceil \frac{n+1}{2} \rceil}}$$

makes the Milner theorem from (5.1). Finally, if

$$c_4(n,i) = \begin{cases} \frac{1}{\binom{n}{i-1}} & \text{if } \quad 1 \le i \le \frac{n}{2}, \\ \frac{1}{\binom{n}{i}} & \text{if } \quad \frac{n}{2} < i. \end{cases}$$

then Theorem 4.3 is obtained from (5.1). One can determine all linear inequalities of type (5.1) which are satisfied for the profile vectors of intersecting, inclusion-free families (see [23]). These inequalities (hyperplanes) determine the *convex hull* of the profile vectors of intersecting, inclusion-free families. This convex hull can be easier described by its *extreme points* (= vertices).

Theorem 1.5.1 **(([6]))** *The extreme points of the convex hull of the profile vectors of intersecting, inclusion-free families on an n-element set are*

$$(0,\ldots,0),$$

$$\left(0,\ldots,\binom{n-1}{i-1},\ldots,0\right) \quad \left(0 \le i \le \binom{n}{2}\right),$$

$$\left(0,\ldots,\binom{n}{j},\ldots,0\right) \quad \left(\binom{n}{2} < j\right),$$

$$\left(0,\ldots,\binom{n-1}{i-1},\ldots,\binom{n-1}{j},\ldots,0\right) \quad \left(0 \le i \le \binom{n}{2}, n < i+j\right)$$

where the non-zero components are the ith and jth ones, resp.

Proof It is easy to see that there are intersecting, iclusion-free families with the above profiles. We only have to prove that any profile can be expressed as a convex linear combination of the given extreme points. This can be proved with the cycle method again. First we have to see the analogous problem for the intervals.

Lemma 1.5.2 *The extreme points of the convex hull of the profile vectors of intersecting, inclusion-free families of intervals on a cyclically ordered n-element set are*

$$(0,\ldots,0),$$

$$(0,\ldots,i,\ldots,0) \quad \left(0 \le i \le \binom{n}{2}\right),$$

$$(0,\ldots,n,\ldots,0) \quad \left(\binom{n}{2} < j\right),$$

$$(0,\ldots,i,\ldots,n-j,\ldots,0)\quad\left(0\le i\le\binom{n}{2},n<i+j\right),$$

where the non-zero components are the ith and jth ones, resp., and the non-zero 0th and nth components are replaced by 1.

Without **proof**, see [6].

Proof of the theorem It is easy to see that there are intersecting, inclusion-free families with profile vectors listed in the theorem. It remained to prove that the profile vector of any such family is in the convex linear combination of these given vectors. The proof of this statement will use the cycle method with a vector-valued weight function:

$$w(F)=\left(0,\ldots,\frac{1}{(n-1)!},\ldots,0\right)$$

where the non-zero component is the $|F|$th one. As before, the double sum of this weight will be calculated for the pairs $(\mathcal{C},F)$ where $\mathcal{C}$ is a cyclic permutation of X, $F\in\mathcal{F}$ and F is an interval along $\mathcal{C}$. Let $\mathcal{F}(\mathcal{C})$ denote the family of those members $F\in\mathcal{F}$ which are intervals along $\mathcal{C}$.

For a fixed $\mathcal{C}$ we obtain

$$\sum_{F\in\mathcal{F}(\mathcal{C})}w(F)=\frac{1}{(n-1)!}p(\mathcal{F}(\mathcal{C})).$$

Denote the extreme points in the lemma by $e_1,\ldots,e_N$. The lemma implies that $p(\mathcal{F}(\mathcal{C}))$ is a convex linear combination of these vectors, that is,

$$p(\mathcal{F}(\mathcal{C}))=\sum_{i=1}^{N}\lambda_i(\mathcal{C})e_i$$

where the λ'ss are non-negative and their sum is 1.

Hence

$$\sum_{\mathcal{C},F}w(F)=\sum_{\mathcal{C}}\sum_{F}w(F)=\sum_{\mathcal{C}}\frac{1}{(n-1)!}\sum_{i=1}^{N}\lambda_i(\mathcal{C})e_i$$
$$=\sum_{i=1}^{N}\frac{1}{(n-1)!}\left(\sum_{\mathcal{C}}\lambda_i(\mathcal{C})\right)e_i$$

follows where $\sum_{i=1}^{N}\frac{1}{(n-1)!}\sum_{\mathcal{C}}\lambda_i(\mathcal{C})=1$. We have proved that $\sum_{\mathcal{C},F}w(F)$ is a convex linear combination of the e_i's.

Summing in the reverse order we obtain

$$\sum_{\mathcal{C},F}w(F)=\sum_{F}\sum_{\mathcal{C}}w(F)=\sum_{F}{}^{*}\left(0,\ldots,\frac{|F|!(n-|F|)!}{(n-1)!},\ldots,0\right)$$
$$=\left(p_0,p_1\frac{n}{\binom{n}{1}},\ldots,p_i\frac{n}{\binom{n}{i}},\ldots,p_{n-1}\frac{n}{\binom{n}{n-1}},p_n\right),$$

where $\sum^*$ denotes that $(1, 0, \ldots, 0)$ and $(0, \ldots, 0, 1)$ are taken for $F = \emptyset$ and $F = X$, resp., as the number of cyclic permutations along which F is an interval is $|F|!(n-|F|)!$ for $0 < |F| < n$ but it is $(n-1)!$ for $|F| = 0, n$. It follows that the last vector is a convex linear combination of $e_1, \ldots, e_N$, therefore $(p_0, \ldots, p_n)$ is convex linear combination of the vectors listed in the theorem, since they can be obtained from the e_i's by multiplication with $\binom{n}{i}/n$ $(0 < i < n)$.

1.6 OTHER RESULTS

There are many other applications of the method, see e.g. [4], [10], [12], [15], [16], [17] and [27]. Most of these are contained in the excellent book of Engel ([3]). In [7] the convex hull of several other classes of families are determined using the cycle method. [5] extends the method for more general structures. The most sophisticated application of the method is due to Pyber ([26]). He proved a special case of the following conjecture.

Conjecture 1.6.1 (Frankl-Füredi-Pyber) *Let $\mathcal{F}$ be an inclusion-free family of subsets of an n-element set, $2 \leq k \leq n$ be a fixed integer and suppose that any two members $F, G \in \mathcal{F}$ satisfy the conditions*

$$|F| \leq n - k,$$

$$1 \leq |F \cap G| \leq k - 1.$$

Then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}$$

holds.

This would be an extension of the Erdős-Ko-Rado theorem. One can easily modify the method of [11] to prove the conjecture for the case

$$\frac{100k^2}{\log k} \leq n.$$

Pyber proved it for the case

$$6k \leq n \leq \frac{r^2}{5}.$$

In all other applications of the method, an analogous problem is solved for the cycle and then double counting makes it valid for the original problem. Here Pyber considers mutual relationship between cycles. He uses statements, that if something happens in a cycle, then it strongly influences cycles which are not "far" from this cycle.

1.7 LARGER INTERSECTIONS

The most important recent theorem in extremal set theory is the following theorem what will be formulated here in a somewhat weaker form. We say that a family $\mathcal{F}$ is *t-intersecting* if $t \leq |F \cap G|$ holds for any pair of members $F, G \in \mathcal{F}$.

Theorem 1.7.1 (Ahlswede-Khachatrian [1]) *Let $1 \leq t \leq k \leq n, X = \{1, 2, \ldots, n\}$ and suppose that $\mathcal{F} \subset \binom{X}{k}$ is t-intersecting. Then $|\mathcal{F}|$ cannot exceed the size of the largest one of the following families*

$$\mathcal{A}_r = \left\{ A \in \binom{X}{k} \ : \ |A \cap \{1, 2, \ldots, t + 2r\}| \geq t + r \right\} \quad \left(0 \leq r \leq \frac{n-t}{2}\right) \tag{7.1}$$

The problem has a long history. It was posed in the original paper of Erdős, Ko and Rado ([8]). They proved that the family in (7.1) with $r = 0$ is the best when n is large enough, and posed the statement of Theorem 7.1 as a conjecture for the case when n is divisible by 4, $k = \frac{n}{2}$, $t = 2$ and $r = \frac{n-2}{2}$. Frankl has generalized this conjecture in the above form in [9]. He also determined in [9] the exact threshold in n when $15 \leq t$: the conjecture is true when $(k - t + 1)(t + 1) < n$ with $r = 0$, otherwise the construction with $r = 1$ gives a larger family. The cases $t = 2, \ldots, 14$ were solved by Wilson ([30]). Therefore the following theorem is a special case of Theorem 7.1, we formulate it separately because it will be used later.

Theorem 1.7.2 (Frankl-Wilson) *The largest t-intersecting family $\mathcal{F} \subset \binom{X}{k}$ has $\binom{n-t}{k-t}$ members if $(k - t + 1)(t + 1) \leq n$, otherwise it has more members.*

Frankl and Füredi ([13]) proved Frankl's conjecture (that is, the Ahlswede-Khachatrian theorem) for $c\sqrt{t/\log(t+1)}(k - t + 1) < n$.

Summarizing, a longstanding effort, for many decades was needed to solve the problem. Why does the cycle method which proved to be very effective in many cases fail when one of the conditions is the t-intersecting property with $2 \leq t$? Try the trivial generalization: determine the maximum number of t-intersecting intervals of length k. It is easy to see that the answer is $k - t + 1$ when $k \leq \frac{n+t-1}{2}$. The ratio selected/total number of intervals is much more than in the case of all sets: $\binom{n-t}{k-t}/\binom{n}{k}$.

One has to find a "more dense" substructure rather than the intervals along a cycle. A candidate is a *Steiner system* $\mathcal{S}(n, k, t)$, which is such a subfamily of $\binom{X}{k}$ that every t-element subset of X is contained in exactly one member. Observe that

$$|\mathcal{S}(n, k, t)| = \frac{\binom{n}{t}}{\binom{k}{t}}. \tag{7.2}$$

It is obvious that if $\mathcal{F}$ is a t-intersecting family of k-element subsets of X then $\mathcal{F}$ and $\mathcal{S}(n, k, t)$ have at most one common member. This is true for the family obtained from $\mathcal{S}(n, k, t)$ by permuting X. Consider the pairs $(\mathcal{P}, F)$ where $\mathcal{P}$ is a permutation of X, $F \in \mathcal{F}$ and $\mathcal{P}$ brings F to a member of $\mathcal{S}(n, k, t)$. There are $k!(n - k)!$ permutations bringing a given F to a given $S \in \mathcal{S}(n, k, t)$. Using

(7.2) we obtain that the number of pairs in question is

$$|\mathcal{F}|\frac{\binom{n}{t}}{\binom{k}{t}}k!(n-k)!. \tag{7.3}$$

On the other hand, if $\mathcal{P}$ is fixed, there is at most one F by the above remark. Therefore the number of pairs is at most $n!$, consequently (7.3) is $\leq n!$. This inequality implies $\mathcal{F} \leq \binom{n-t}{k-t}$.

Theorem 1.7.3 (Frankl-Katona) *If there is a Steiner system $S(n,k,t)$ for the given integers $2 \leq t < k < n$ and $\mathcal{F} \subset \binom{X}{k}$ is a t-intersecting family, then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}.$$

As the existence of Steiner systems is a difficult question, this result did not seem to be very effective. This is why it was not published before except for a short remark ($k = 3, t = 2$) in [21] (page 221). However, if it is combined with Theorem 7.2 then we obtain a new proof of an old theorem of Tits ([29]):

Theorem 1.7.4 (Tits) *In any non-trivial Steiner system $S(n,k,t)$*

$$(k-t+1)(t+1) \leq n$$

holds.

Another attempt to generalize the cyclic method for more-intersecting families can be found in [19]. For sake of simplicity we show the case $t = 2$, only. Consider the group S_n of all permutations of X. A subgroup Γ of S_n is called *2-transitive* if any ordered pair (x_1, y_1) of different elements can be mapped into any other pair (x_2, y_2) (of different elements) by one of $\phi \in \Gamma$. It is called *sharply 2-transitive* if there is exactly one such ϕ. If n is prime power then the function $ax + b$ ($a \neq 0$) is a permutation on $GF(n)$ for any $a, b \in GF(n)$. It is easy to see that the group of these functions (for composition) is a sharply 2-transitive subgroup of S_n. The number of elements of this subgroup is $n(n-1)$. Obviously, this must hold for any sharply 2-transitive subgroup Γ. (Note that the subgroup of cyclic shifts $\phi_j(i) = i + j \bmod n$ form a sharply 1-transitive subgroup.) Consider the sets obtained from a given k-element $A \subset X$ by applying the permutations $\phi \in \Gamma$ where Γ is a sharply 2-transitive subgroup: $\phi_1(A), \ldots, \phi_{n(n-1)}(A)$. If we can prove that a 2-intersecting subfamily of this family is of size at most $k(k-1)$ then the ratio of the selected subsets over the total number of subsets is the same as in the family of all sets. Let us formulate it as a theorem.

Theorem 1.7.5 (Howard-Károlyi-Székely) *A sharply 2-transitive group Γ acting on X is given. Let $A \subset X, |A| = k$. Suppose that any 2-intersecting subfamily of $\{\phi(A) : \phi \in \Gamma\}$ has at most $k(k-1)$ members. Then any 2-intersecting family $\mathcal{F} \in \binom{X}{k}$ satisfies $|\mathcal{F}| \leq \binom{n-2}{k-2}$.*

In [19] the authors find an infinite class of integers n and k for which they are able to use the above theorem to prove Theorem 7.1 in case of $t = 2$.

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