Intersecting balanced families of sets * †

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Abstract

Suppose that any t members ($t \geq 2$) of a regular family on an n element set have at least k common elements. It is proved that the largest member of the family has at least $k^{1/t}n^{1-1/t}$ elements. The same holds for balanced families, which is a generalization of the regularity. The estimate is asymptotically sharp.

0 Introduction

The notion of a balanced family of sets has played an important role in the analysis of the core of a cooperative game, a concept which is central to cooperative game theory and economics. Bondareva [2] , [3] and Shapley [11] showed that a TU game possesses a non-empty core if and only if it is a balanced game. A game is said to be balanced if a utility profile feasible for every coalition belonging to a balanced family of coalitions is also feasible for the grand coalition. Scarf [10] proved that every balanced NTU game has a non-empty core. For further work in this area, see, for example, Ichiishi [4], Ichiishi and Idzik [5] and Shapley [11]. Balanced families of sets have also been useful in the study of other game theoretic solution concepts, such as the bargaining set (see, for example, Maschler [8] and Vohra [13]) and the kernel (Maschler, Peleg and Shapley [9]).

In studying the non-emptiness of the bargaining set, Vohra [12], [13] emphasized the importance of balanced families with the additional property that all pairs of sets in the family have at least two common elements. There, this property was used to establish that every NTU game had a non-empty bargaining set if objecting coalitions were restricted in size.

The present paper is concerned with a more systematic analysis of balanced families which have intersecting sets. We show that if any t sets ($t \geq 2$) of a balanced family have at least k common elements ($k \geq 1$), then the largest set in this family has cardinality at least $k^{\frac{1}{t}} n^{1-\frac{1}{t}}$, where n is the cardinality of the sum of all sets in this family. We also show that this estimate is asymptotically sharp.

The first result along these lines was a theorem of Lovász [7] proving that if a regular intersecting family ($k = 1$) consists of sets of the same size l, then $l \geq \sqrt{n}$. Babai [1] proved the following generalization: if any two sets ($t = 2$) in a regular family have at least k elements in common, and all sets in the family have the same size l, then $l \geq \sqrt{kn}$. Finally Vohra [12] observed that if any two sets in a balanced family have at least two elements in common ($t = k = 2$), then one of the members have size at least $\sqrt{2n}$.

1 Preliminaries

Let $2^{[n]}$ denote the family of all subsets of $[n] = \{1, ..., n\}$ and R^n the n-dimensional Euclidean space (R is the set of reals and R_+ is the set of non-negative reals). By N we denote the set of natural numbers and by $|S|$ we denote the cardinality of a subset $S \subset N$. For any $S \in 2^{[n]}$, let e_S denote the vector in R^n whose i-th coordinate is 1 if $i \in S$ and 0 otherwise. We also use the notation e_i for $e_{\{i\}}$ and e for $e_{[n]}$.

Definition 1.1 *A non-empty family $\mathcal{B} \subset 2^{[n]}$ is balanced if there exist $\{\lambda_S\}_{S\in\mathcal{B}} \subset R_+$, called balancing coefficients, such that $\sum_{S\in\mathcal{B}} \lambda_S e_S = e$ (or equivalently, $\sum_{S\in\mathcal{B}:i\in S} \lambda_S = 1$ for every $i \in [n]$).*

Definition 1.2 *A family $\mathcal{D} = \{S_1, \ldots, S_m\} \subset 2^{[n]}$ is d-regular $(0 \leq d \leq m)$ if for every $l \in [n]$, $|\{j|l \in S_j\}| = d$.*

Proposition 1.3 *Every d-regular $(1 \leq d)$ family is balanced.*

Proof. Observe that $\sum_{i=1}^{m} \frac{1}{d} e_{S_i} = e$. □

Definition 1.4 *A family $\mathcal{F} = \{S_1, \ldots, S_m\}$ is called t-wise k-intersecting $(m, t \geq 2, k \geq 1$ are integers) if it satisfies the condition*

$$|S_{i_1} \cap \ldots \cap S_{i_t}| \geq k \quad \text{for any } t \text{ members of } \mathcal{F}. \tag{1.1}$$

2 Lower estimate on the largest member

Theorem 2.1 *Let $m, t \geq 2, k \geq 1$ be integers. Assume that the balanced family $\mathcal{B} = \{S_1, \ldots, S_m\} \subset 2^{[n]}$ is t-wise k-intersecting. Then $\mathcal{B}$ contains a member S_j such that*

$$|S_j| \geq k^{\frac{1}{t}} n^{1-\frac{1}{t}}. \tag{2.1}$$

Proof. Let λ_j be the balancing coefficient of the the set S_j. Introduce the notation $s = \max\{|S_j| : S_j \in \mathcal{B}\}$.

For $i \in [n]$, let

$$S(i) = \{S_j \in \mathcal{B} | i \in S_j\},$$

$$d_i = |S(i)|,$$

$$\lambda(i) = \sum_{\{j|S_j \in S(i)\}} \lambda_j.$$

Consider the matrix M whose j-th column is $\lambda_j e_{S_j}$. Adding all the entries in the ith row yields $\lambda(i) = 1$. Therefore the sum of all entries of M is n. First adding the columns we obtain

$$\sum_{j=1}^{m} \lambda_j |S_j| = n.$$

Clearly, $s \geq |S_j|$ for all j, hence we have

$$s \sum_{j=1}^{m} \lambda_j \geq n$$

or

$$s \geq \frac{n}{\sum_{j=1}^{m} \lambda_j}. \tag{2.2}$$

Now, consider all possible products of t entries in the same row of M, where repetition is allowed and the order of the entries does matter. Of course, only those products are interesting which contain only non-zero factors. Denote the sum of all these products by A.

Determine the sum of the products in one row. We may suppose without loss of generality that the non-zero entries in this row are $\lambda_1, \ldots, \lambda_r (r \leq m)$, where $\sum_{i=1}^{r} \lambda_i = 1$. Our sum in question is

$$\sum_{(j_1,\ldots,j_t), 1 \leq j_i \leq r} \lambda_{j_1} \cdots \lambda_{j_t} = (\lambda_1 + \cdots + \lambda_r)^t = 1.$$

Therefore $A = n$.

On the other hand, classify the terms occuring in A according to the sequences $(u_1, \ldots, u_t), 1 \leq u_i \leq m$. These classes are obviously disjoint. They contain identical terms: $\lambda_{u_1} \cdots \lambda_{u_t}$. Let us see that the number of these terms is at least k. Since $|S_{u_1} \cap \ldots \cap S_{u_t}| \geq k$ holds, the matrix M contains at least k rows where all t columns $u_1, \ldots, u_t$ are non-zero. Therefore

$$A \geq k \sum_{(u_1,\ldots,u_t), 1 \leq u_i \leq m} \lambda_{u_1} \cdots \lambda_{u_t} = k(\lambda_1 + \cdots + \lambda_m)^t.$$

This proves the inequality

$$n \geq k(\lambda_1 + \cdots + \lambda_m)^t \tag{2.3}$$

Substituting (2.3) in (2.2) we have

$$s \geq k^{\frac{1}{t}} n^{1-\frac{1}{t}}$$

□

By Proposition 1.3 we obtain the following special case.

Corollary 2.2 *Let $m, t \geq 2, k \geq 1$ be integers. Assume that the d-regular $(1 \leq d)$ family $\mathcal{D} = \{S_1, \ldots, S_m\} \subset 2^{[n]}$ is t-wise k-intersecting. Then $\mathcal{D}$ contains a member S_j such that*

$$|S_j| \geq k^{\frac{1}{t}} n^{1-\frac{1}{t}}.$$

□

If $t = 2$ and $|S_i| = |S_j|$ for all $i, j \in [n]$, Corollary 2.2 reduces to Lemma 3 of Babai [1].

Now we give an alternative proof for the regular case. This actually gives an upper bound on the number of sets, too.

Theorem 2.3 *Let $m, t \geq 2, k \geq 1$ be integers. Assume that the family $\mathcal{B} = \{S_1, \ldots, S_m\} \subset 2^{[n]}$ is t-wise k-intersecting. Let $d_i = |\{j | i \in S_j\}|$ for $i \in [n]$. Then*

$$m \leq \left(\frac{\sum_{i=1}^n d_i^t}{k} \right)^{\frac{1}{t}} \tag{2.4}$$

and $\mathcal{B}$ contains a member S_j such that

$$|S_j| \geq k^{\frac{1}{t}} \frac{\sum_{i=1}^n d_i}{(\sum_{i=1}^n d_i^t)^{\frac{1}{t}}}. \tag{2.5}$$

Proof. Consider the 0,1 matrix M' whose j-th column is e_{S_j}. The number of 1s in the i-th row is d_i. The total number of non-zero products of t entries in the same row of M', where repetition is allowed and the order of the entries does matter is

$$\sum_{i=1}^{n} d_i^t.$$

By the intersecting condition of the family, this is at least km^t. This inequality proves (2.4).

The total number of 1s in M' is $\sum_{i=1}^{n} d_i$, therefore there is a column containing at least $\sum_{i=1}^{n} d_i/m$ 1s. The substitution of (2.4) proves (2.5). $\square$

If $d_1 = \ldots = d_n = d$ that is the family is d-regular then (2.4) gives the upper bound

$$m \leq d \left(\frac{n}{k}\right)^{\frac{1}{t}}.$$

On the other hand, (2.5) leads to Corollary 2.2.

3 Constructions

Denote the minimum size of the largest member of a balanced t-wise k-intersecting family on n elements by $f(n,t,k)$. Theorem 2.1 states that

$$k^{\frac{1}{t}} n^{1-\frac{1}{t}} \leq f(n,t,k).$$

In the present section we give an upper estimate on $f(n,t,k)$.

Let q be a prime power, $t \geq 2$ and $\mathrm{PG}_t(q)$ a t-dimensional finite projective geometry over the q-element finite field. The number of points in $\mathrm{PG}_t(q)$ is $q^t + \cdots + q + 1$ (see e.g. [6]). The size of a $t-1$-dimensional hyperplane is $q^{t-1} + \cdots + q + 1$. The intersection of t such distinct hyperplanes is always one point. Therefore we have a family of $q^{t-1} + \cdots + q + 1$-element subsets of the $q^t + \cdots + q + 1$-element set so that the intersection of any t distinct ones has 1 element.

Replace each element in the above example by a k-element set. Then a family of $k(q^{t-1} + \cdots + q + 1)$-element subsets of a $k(q^t + \cdots + q + 1)$ element set is obtained so that the intersection of t distinct subsets has exactly k elements. This proves

$$f(k(q^t + \cdots + q + 1), t, k) \leq k(q^{t-1} + \cdots + q + 1). \tag{3.1}$$

Using this idea, one can easily prove that the estimate of Theorem 2.1 is asymptotically correct.

Theorem 3.1

$$f(n,t,k) \sim k^{\frac{1}{t}} n^{1-\frac{1}{t}}$$

holds for fixed t and k when n tends to infinity.

To prove Theorem 3.1 we need the following two lemmas.

Lemma 3.2 *For any $0 < \varepsilon < 1$, positive k and t and sufficiently large n there is a prime number q satisfying*

$$(1-\varepsilon)n < k(q^t + \cdots + q + 1) \leq n. \tag{3.2}$$

Proof. Let us start with the well-known fact that, for fixed $0 < \alpha < \beta$ and sufficiently large n there is a prime number q such that $\alpha n < q < \beta n$. Apply this statement for the necessary constants and $n^{\frac{1}{t}}$: if n is large enough, there is a prime number q such that

$$\frac{(1-\varepsilon)^{\frac{1}{t}} n^{\frac{1}{t}}}{k^{\frac{1}{t}}} < q < \frac{(1-\frac{\varepsilon}{2})^{\frac{1}{t}} n^{\frac{1}{t}}}{k^{\frac{1}{t}}}.$$

This implies (3.2) for large n. □

Lemma 3.3 *Given the positive integers $d \leq a$ and u, one can find subsets $B_1, \ldots, B_a$ of the u-element set U so that every element of U is contained in exactly d subsets, and $|B_i| \leq \lceil \frac{du}{a} \rceil (1 \leq i \leq a)$. The subsets B_i are not necessary different.*

Proof. Define b and r by du by a: $du = ab + r, 0 \leq r < a$. It is easy to see that one can find sets with sizes $|B_1| = \cdots = |B_{a-r}| = b, |B_{a-r+1}| = \cdots = |B_a| = b + 1$ in such a way that the elements of U are contained in the same number of sets. The sum of the degrees is equal to the sum of the sizes: $(a-r)b + r(b+1) = ab + r = du$. Hence it follows that the degrees are all d. □

Proof of Theorem 3.1 Choose q according to Lemma 3.2. Follow the above geometric construction with a slight modification. Replace each point of $\mathrm{PG}_t(q)$ by a k-element set. The new underlying set K has $k(q^t + \cdots + q + 1)$

elements. The $t-1$-dimensional hyperplanes of $\mathrm{PG}_t(q)$ are enlarged, too, each of these enlarged sets has $k(q^{t-1}+\cdots+q+1)$ elements. Denote them by H_i ($1 \le i \le q^t+\cdots+q+1$). This family on K is t-wise k-intersecting and regular of degree $q^{t-1}+\cdots+q+1$. Let $0 \le u = n - k(q^t+\cdots+q+1)$ and take a u-element set U disjointly to K. Apply Lemma 3.3 with $a = q^t+\cdots+q+1, d = q^{t-1}+\cdots+q+1$ and u. Using (3.2) we obtain that the sizes of the Bs are at most

$$\left\lceil \frac{du}{a} \right\rceil \le \left\lceil \frac{(q^{t-1}+\cdots+q+1)\varepsilon n}{q^t+\cdots q+1} \right\rceil \le \frac{k\varepsilon}{1-\varepsilon}(q^{t-1}+\cdots+q+1). \tag{3.3}$$

The family $H_i \cup B_i$ ($1 \le i \le q^t+\cdots+q+1$) on the n-element underlying set $K \cup U$ is obviously t-wise k-intersecting and regular (of degree $q^{t-1}+\cdots+q+1$). The sizes can be upperbounded using (3.2) and (3.3):

$$|H_i \cup B_i| \le (1+\frac{\varepsilon}{1-\varepsilon})k(q^{t-1}+\cdots+q+1)$$

$$\le (1+\frac{\varepsilon}{1-\varepsilon})k(q+1)^{t-1} \le (1+\frac{\varepsilon}{1-\varepsilon})k^{\frac{1}{t}}(k(q+1)^t)^{\frac{t-1}{t}}.$$

Here $k(q+1)^t < k(1+\varepsilon)(q^t+\cdots+q+1) \le (1+\varepsilon)n$ holds for large ns. Therefore

$$|H_i \cup B_i| < (1+\frac{\varepsilon}{1-\varepsilon})k^{\frac{1}{t}}(1+\varepsilon)^{\frac{t-1}{t}}n^{\frac{t-1}{t}}.$$

We have proved

$$f(n,t,k) \le (1+\frac{\varepsilon}{1-\varepsilon})(1+\varepsilon)^{\frac{t-1}{t}}k^{\frac{1}{t}}n^{\frac{t-1}{t}}$$

for sufficiently large ns. This inequality, combined with Theorem 2.1 finishes the proof. □

The case $t=2, k=1, n=q^2+q+1$ is much easier.

Theorem 3.4 $f(q^2+q+1,2,1) = q+1$.

Proof. Theorem 2.1 gives $f(q^2+q+1,2,1) \ge \lceil\sqrt{q^2+q+1}\rceil$. The obvious $q^2 < q^2+q+1 < (q+1)^2$ proves $\lceil\sqrt{q^2+q+1}\rceil = q+1$. (3.1) completes the proof. □

Theorem 3.4 was proved by Lovász [7] in the special case when the sizes of the sets in the family are all equal, i.e. for uniform d-regular families.

4 Further questions

The degree of the regular family in our construction is relatively large, its order of magnitude is $d \sim k^{\frac{1}{t}-1} n^{1-\frac{1}{t}}$. The obvious question arises whether the size of the largest set in the family increases when the degree (of regularity) is fixed to be small. Let $f(n, t, k, d)$ be the minimum (for families) of the size of the largest member in a t-wise k-intersecting regular family of degree d on an n-element underlying set.

The above definition makes sense at all when $t \leq d$. This is why studying $f(n, t, k, t)$ is a good starting point.

Theorem 4.1

$$f\left(k\binom{l+t-1}{t}, t, k, t\right) = k\binom{l+t-2}{t-1}$$

holds for all integers $t \geq 2, k \geq 1, l \geq 1$.

Proof. Let us start with the construction for the case $k = 1$. The underlying set of the family will consist of all t-element subsets of $[l+t-1]$, that is, the t-element subsets will play the role of elements. Let the family $\mathcal{D}(t, l) = \{S_1, \dots, S_m\}$ be defined by $S_i = \{A | i \in A, |A| = l + t - 2\} (1 \leq i \leq m = l + t - 1)$. It is easy to see that $S_{i_1} \cap \dots \cap S_{i_t} = \{\{i_1, \dots i_t\}\}$, i.e. $|S_{i_1} \cap \dots \cap S_{i_t}| = 1$ holds for distinct $i_1, \dots, i_t$. $\{i_1, \dots i_t\}$ is an element of S_i iff $i = i_1, \dots, i_t$, that is, the degree of each element of the underlying set is t. Finally, $|S_i| = \binom{l+t-2}{t-1}$ proves

$$f\left(\binom{l+t-1}{t}, t, 1, t\right) \leq \binom{l+t-2}{t-1}.$$

If the elements of the underlying set are replaced by finite sets, the new family will be t-wise intersecting of degree t, again, only the sizes of the members will become larger. Denote the class of these modified families by $\Delta(t, l)$. Given a family $\mathcal{F}$ on $[n]$ we say that two elements $1 \leq i, j \leq n$ are equivalent if $i \in F$ and $j \in F$ hold for the same members $F \in \mathcal{F}$. The set of equivalent elements is called an *atom*. It is easy to see that each atom of $\mathcal{D}(t, l)$ has one element. On the other hand the elements obtained from one element when making another family $\mathcal{D} \in \Delta(t, l)$ from $\mathcal{D}(t, l)$ form an atom.

The family $\mathcal{D} \in \Delta(t,l)$ with atoms of size k serves as a construction proving

$$f\left(k\binom{l+t-1}{t}, t, k, t\right) \le k\binom{l+t-2}{t-1}.$$

Before proving the other direction we will show a lemma stating that these are the only constructions. Let $k \ge 1, t \ge 2, l \ge 1$ be integers. The set of families $\mathcal{D}$ consisting of $l+t-1$ sets, being t-wise k-intersecting and regular of degree t is denoted by $\Delta^*(t,l,k)$, moreover $\cup_{k=1}^{\infty}\Delta^*(t,l,k) = \Delta^*(t,l)$.

Lemma 4.2 *The elements of $\Delta^*(t,l)$ differ only in the sizes of the atoms and the permutation of the elements.*

Proof. Use induction on t and l. If $l = 1$, t is arbitrary then the degree condition ensures that the family $\mathcal{D} \in \Delta^*(t,1,k)$ consists of t identical sets. The statement is trivial for this case.

Let $t = 2, l > 1$ and suppose that the lemma is true for $t = 2, l-1$. Take a family $\mathcal{D} \in \Delta^*(2,l,k)$. Its members are denoted by $D_1, \dots D_{l+1}$. The intersections $D_1 \cap D_i$ $(2 \le i \le l+1)$ must form a partition of D_1. Here $|D_1 \cap D_i| \ge k$. On the other hand $D_i - D_1$ $(2 \le i \le l+1)$ is a family of l sets, it is 2-wise k-intersecting and regular of degree 2, that is, it is in $\Delta^*(2,l-1)$. By the induction hypothesis it is uniquely determined in the given sense. The statement is proved for this case.

Suppose that $t > 2, l > 1$ and the statement is true for $t-1$ with all values of l and for t with $l-1$. Choose a family $\mathcal{D} \in \Delta^*(t,l,k)$. The $t-1$-wise intersections of $D_1 \cap D_i$ $(2 \le i \le l+t-1)$ form a partition of D_1 with classes of size at least k. Therefore the family $D_1 \cap D_i$ $(2 \le i \le l+t-1)$ is in $\Delta^*(t-1,l-1,k)$, by the induction, it is uniquely determined (in the give sense). On the other hand $D_i - D_1$ $(2 \le i \le l+t-1)$ is a family of $l+t-2$ sets, it is t-wise k-intersecting and regular of degree t, that is, it is in $\Delta^*(t,l-1,k)$. By the induction hypothesis it is uniquely determined in the given sense. $\square$

Corollary 4.3 $\Delta^*(t,l) = \Delta(t,l)$.

Proof. It is easy to check that $\mathcal{D}(t,l) \in \Delta^*(t,l)$. Lemma 4.2 completes the proof. $\square$

Now return to the proof of Theorem 4.1. By the corollary above, it is enough to consider the members of $\Delta(t,l)$. So, suppose that $\mathcal{D} \in \Delta(t,l)$ has atoms of size $k \leq a_i$ where

$$k\binom{l+t-1}{t} \leq n \tag{4.1}$$

and

$$\sum a_i = n \tag{4.2}$$

hold. Denote the members of $\mathcal{D}$ by $T_1, \dots T_{l+t-1}$. The regularity of $\mathcal{D}$ and (4.2) imply

$$\sum_{i=1}^{l+t-1} |T_i| = \sum a_j t = nt.$$

Hence there exist an i satisfying

$$\left\lceil \frac{nt}{l+t-1} \right\rceil \leq |T_i|. \tag{4.3}$$

On the other hand, if the sizes of the atoms are chosen nearly equal, then this inequality is sharp. The result is a monotone function of l, therefore we have to choose l the largest possible, satisfying (4.1). By this, the value of $f(n,t,k,t)$ is determined for all values of n. In the special case of the theorem when $n = k\binom{l+t-1}{t}$ the left hand side of (4.3) is really $k\binom{l+t-2}{t-1}$. $\square$

It is easy to deduce the following result from Theorem 4.1.

Corollary 4.4 *For fixed t and k and sufficiently large n*

$$f(n,t,k,t) \sim \frac{t}{(t!)^{\frac{1}{t}}} k^{\frac{1}{t}} n^{1-\frac{1}{t}}$$

holds.

It is surprising that the order of magnitude of the largest set is almost the same under this strong condition on the degree as in the case of unlimited degree (Theorem 3.1).

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1. Noga Alon remarked that a family is balanced iff (non-negative integral) multiplicities can be associated with the members in such a way that the so obtained "multi-family" is regular. (It is easy to see that the balancing coefficients in Definition 1.1 can be chosen rational. Multiply the equation with the lowest common multiple d of the denominators of these rational numbers. We obtain $\sum_{S\in\mathcal{B}} l_S e_S = (d,\ldots,d)$.) The proof of Theorem 2.3 is valid for multi-families, therefore it gives an alternative proof also for the general case of balanced families.

2. π-balanced families are generalizations of balanced families (see e.g. [5] or [11] for the definition. Unfortunatelly, our theorems are no longer true for this class of families as the "star" consisting of two-element sets shows.

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