Extremal problems for finite sets and convex hulls — A survey[1]

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Abstract

Let $\mathscr{F}$ be a family of distinct subsets of an n-element set. Define $p_i(\mathscr{F})$ $(0 \leqslant i \leqslant n)$ as the number of i-element members of $\mathscr{F}$. Consider the profile vectors $(p_0(\mathscr{F}), \ldots, p_n(\mathscr{F}))$ for all families $\mathscr{F}$ belonging to a certain class $\boldsymbol{A}$ (e.g. $\boldsymbol{A}$ can be the class of all families where any two members have a non-empty intersection). Let $\varepsilon(\boldsymbol{A})$ denote the set of extreme points of the convex hull of the set of these profile vectors. Results determining $\varepsilon(\boldsymbol{A})$ for some classes $\boldsymbol{A}$ are surveyed. Facets and edges of these convex hulls are also described for some $\boldsymbol{A}$. Connections to the classical extremal problems are shown.

1. Introduction

Let X be a finite set of n elements. A family of its distinct subsets, $\mathscr{F} \subset 2^X$ is said to be *inclusion-free* if $F_1, F_2 \in \mathscr{F}$ implies $F_1 \not\subset F_2$. It is easy to see that all k-element subsets of X form an inclusion-free family. The largest one of these families is the one with $k = \lfloor n/2 \rfloor$. The classical theorem of *Sperner* states that this is the largest one.

Theorem 1 (Sperner [39]). *The maximum number of members of an inclusion-free family is*

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

In some applications (see [28]), however,

$$\max \sum_{F \in \mathscr{F}} |F|$$

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is needed rather than

$$\max \sum_{F \in \mathscr{F}} 1$$

for inclusion-free families. This question has been solved in [28]. Let us consider the following obvious generalization. Let $f(x)$ be a real function and try to find

$$\max \sum_{F \in \mathscr{F}} f(|F|) \tag{1}$$

for all inclusion-free families.

Introduce the notation $p_i(\mathscr{F}) = p_i = |\{F\colon F \in \mathscr{F},\ |F| = i\}|$ $(0 \leqslant i \leqslant n)$. The $(n+1)$-dimensional vector $p(\mathscr{F}) = (p_0, p_1, \ldots, p_n)$ is called the *profile vector* of $\mathscr{F}$. Then (1) can be written in the equivalent form

$$\max \sum_{i=0}^{n} p_i f(i), \tag{2}$$

where the maximum is taken for all profile vectors of inclusion-free families on X. $f(i)$ are constants, $c = \sum_{i=0}^{n} x_i f(i)$ is a hyperplane. We have to find the maximum c such that this hyperplane contains a profile vector. It is obvious that it is sufficient to consider the vertices or *extreme points* of the convex hull of the set of profile vectors. One of these extreme points will achieve the maximum in (2).

Theorem 2. *The extreme points of the convex hull of the set of profile vectors of inclusion-free families are the zero-vector and*

$$\left(0, \ldots, 0, \binom{n}{i}, 0, \ldots, 0\right) \quad (0 \leqslant i \leqslant n),$$

where the non-zero component is the ith one.

It is easy to see that these vectors are profile vectors. The empty family and the family containing all i-element subsets serve as constructions. The theorem claims, on the other hand, that all other profile vectors are convex linear combinations of these ones. In other words, all other profile vectors are in the convex polytope spanned by these points. The polytope can also be described by the bordering hyperplanes. These bordering hyperplanes are ones connecting $n+1$ extreme points (since the $n+2$ points form a simplex). We write them in the 'inequality form' showing which side is in the convex hull:

$$0 \leqslant x_i \quad (0 \leqslant i \leqslant n),$$

$$\sum_{i=0}^{n} \frac{x_i}{\binom{n}{i}} \leqslant 1.$$

The above inequalities give an equivalent description of the convex hull of the profile vectors of inclusion-free families. However the ones in the first row are trivial, the only interesting one is in the second row. This inequality, however, has been known since many years as the LYM-inequality [34,41,35]. (But I prefer the name YBLM-inequality since Bollobás [1] proved a more general statement very early.) The most elegant proof is due to Lubell [34].

Theorem 3. *The profile vector of an inclusion-free family satisfies the inequality*

$$\sum_{i=0}^{n} \frac{p_i}{\binom{n}{i}} \leqslant 1.$$

That is, Theorem 2 is only another form of the old YBLM-inequality. This is not true, as we will see, in the case of other classes of families.

Knowing Theorem 3 it is very easy to determine

$$\max \sum_{F\in\mathscr{F}} |F| = \max \sum_{i=0}^{n} p_i i$$

for inclusion-free families. We have only to calculate $\sum_{i=0}^{n} p_i i$ for the extreme points and take the largest one: $\max_{0\leqslant i\leqslant n}\binom{n}{i} i$. ($i = \lceil n/2 \rceil$ gives the maximum.)

Theorem 3 gives a necessary condition for a vector to be a profile vector of an inclusion-free family. However this is not a sufficient condition. Not all vectors in the convex hull are profile vectors. There is a necessary and sufficient condition [3,5] based on the function $F(i,m)$ giving the minimum number of $(i-1)$-element subsets in an arbitrary m-member family of i-element sets ([33,26], for an easy proof see [19]). Unfortunately, this function is very complicated. After some obvious normalization it converges to a nowhere differentiable continuous function as it is proved in [21]. That is, it is hard to use this necessary and sufficient condition. This makes it desirable to find a good sufficient condition.

Open Problem 1. *Find a good sufficient condition for a vector to be a profile vector of an inclusion-free family by either*

(a) *giving a (non-linear) surface in the simplex determined in Theorem* 2 *such that all integral vectors below it are profile vectors, or*

(b) *determining the extreme points of the set of non-profile vectors.*

In the rest of the paper we consider other families.

2. Extreme points for some classes

Let $\boldsymbol{A}$ be a class of families of subsets of X, that is, $\boldsymbol{A} \subset 2^{2^X}$. $\mu(\boldsymbol{A})$ denotes the set of profile vectors of the families belonging to $\boldsymbol{A}$:

$$\mu(\boldsymbol{A}) = \{p(\mathscr{F})\colon \mathscr{F} \in \boldsymbol{A}\}.$$

The set of the extreme points of the convex hull of $\mu(\boldsymbol{A})$ is denoted by $\varepsilon(\boldsymbol{A})$.

The $\boldsymbol{A}$'s considered in the present survey are *hereditary*, that is, $\mathscr{G} \subseteq \mathscr{F} \in \boldsymbol{A}$ implies $\mathscr{G} \in \boldsymbol{A}$. For hereditary $\boldsymbol{A}$'s there is a simple way of reduction of the set of extreme points. Before showing it we have to introduce some more notations. $\mu^*(\boldsymbol{A})$ is the set of *maximal profile vectors*: it contains those elements $p \in \mu(\boldsymbol{A})$ for which $q \in \mu(\boldsymbol{A})$ and $p \leqslant q$ (componentwise) imply $p = q$. Furthermore let $\varepsilon^*(\boldsymbol{A}) = \varepsilon(\boldsymbol{A}) \cap \mu^*(\boldsymbol{A})$ be the set of *essential extreme points*.

Proposition 1. *Suppose that A is hereditary. Then any element of $\varepsilon(\boldsymbol{A})$ can be obtained by replacing some components of an element of $\varepsilon^*(\boldsymbol{A})$ by zero.*

The significance of the proposition is that it is sufficient to determine the set $\varepsilon^*(\boldsymbol{A})$. Replacing the components by zero a set of vectors is obtained, these should be individually checked if they are extreme points.

The set of essential extreme points $\varepsilon^*(\boldsymbol{A})$ is determined for many different $\boldsymbol{A}$'s. Before listing these results some definitions are needed.

We say that a family is *k-Sperner* $(1 \leqslant k)$ if it does not contain $k+1$ distinct members such that $F_1 \subset F_2 \subset \cdots \subset F_{k+1}$. A 1-Sperner family is simply an inclusion-free family. Furthermore, $\mathscr{F}$ is said to be *intersecting*, *cointersecting*, *complement-free* and *complementary* iff

$$F_1, F_2 \in \mathscr{F} \text{ implies } F_1 \cap F_2 \neq \emptyset,$$

$$F_1, F_2 \in \mathscr{F} \text{ implies } F_1 \cup F_2 \neq X,$$

$$F \in \mathscr{F} \text{ implies } X - F \notin \mathscr{F}$$

and

$$F \in \mathscr{F} \text{ implies } X - F \in \mathscr{F},$$

respectively.

$\varepsilon^*(\boldsymbol{A})$ is determined for the following $\boldsymbol{A}$'s.

(i) k-Sperner families [14],

(ii) intersecting families [14],

(iii) intersecting, inclusion-free families [13],

(iv) intersecting, cointersecting, inclusion-free families [8],

(v) complementary, inclusion-free families [8],

(vi) complement-free, inclusion-free families [8,9],

(vii) families $\mathscr{F}$ such that $F_1, F_2 \in \mathscr{F}$ implies either $F_1 \cap F_2 \neq \emptyset$ or $F_1 \cup F_2 = X$ [6],

(viii) families $\mathscr{F}$ such that $F_1, F_2 \in \mathscr{F}$ implies either that F_1 and F_2 divide X into 4 non-empty parts or they give a partition of X [7],

(ix) inclusion-free families $\mathscr{F}$ such that $F_1, F_2 \in \mathscr{F}$ implies either $F_1 \cap F_2 \neq \emptyset$ or $F_1 \cup F_2 \neq X$ [7],

(x) families $\mathscr{F}$ such that $F_1, F_2 \in \mathscr{F}$ implies either $F_1 \not\subset F_2$, $F_1 \cap F_2 \neq \emptyset$ or $F_1 \cup F_2 = X$ [7],

(xi) families $\mathscr{F}$ which are (not necessarily disjoint) union of $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_t$ where $F \in \mathscr{F}_i$, $G \in F_j$, $i \neq j$, $F \neq G$ imply $F \not\subset G$ [14].

We show (iii) as an illustration.

Theorem 4 (Erdős et al. [13]). *The essential extreme points of the convex hull of the set of profile vectors of the intersecting inclusion-free families are*

$$v_j = \left(0, \ldots, 0, \binom{n}{j}, 0, \ldots, 0\right) \quad \left(\frac{n}{2} < j \leqslant n\right)$$

and

$$w_{ij} = \left(0, \ldots, 0, \binom{n-1}{i-1}, 0, \ldots, 0, \binom{n-1}{j}, 0, \ldots, 0\right) \quad \left(1 \leqslant i \leqslant \frac{n}{2},\ n < i + j\right),$$

where the ith and jth components are non-zero.

It is easy to see that z, the zero vector, and w_i, the vector obtained from w_{ij} by replacing $\binom{n-1}{j}$ by zero, are also extreme points. Easy constructions show that all these points are really profile vectors of some intersecting, inclusion-free families. It is harder to prove that all other profile vectors are convex linear combinations of them. Here we can say only a few words about the methods proving such statements.

In some cases [20, 9, 6] one can prove directly that an arbitrary profile vector is a convex linear combination of the given extreme points. There is, however, another method which is proved to be efficient in most of the cases. It is the so-called *circle method* (see [27]). Fix a cyclic ordering of X and consider only those subsets of X which form an interval in X. Prove the analogous statement for families satisfying the same conditions and consisting of intervals. This statement is usually much easier to prove than the original one. Then a weighted double counting leads to the desired result.

Let us see the consequences of Theorem 4.

Suppose that $k \leqslant n/2$ and try to find the maximum number of members in an intersecting family consisting of k-element members. This family is obviously inclusion-free, therefore its size is $\leqslant \max p_k$ for profile vectors of intersecting, inclusion-free families. This is a linear function of the profile vector, therefore the maximum is achieved at one of the extreme points. As the coefficients are non-negative, it is achieved at one of the essential extreme points. The value of p_k in v_j $(n/2 < j \leqslant n)$ is zero, since $k < j$, while its value in w_{ij} is zero unless $i = k$ when $p_k = \binom{n-1}{k-1}$. The famous theorem of Erdős, Ko and Rado is obtained.

Theorem 5 (Erdős et al. [11]). *If $k \leqslant n/2$ $(|X| = n)$ then the maximum number of members in an intersecting family of k-element subsets of X is $\binom{n-1}{k-1}$.*

Find the maximum number of members in an arbitrary intersecting, inclusion-free family. That is, we have to find $\max \sum_{i=0}^{n} p_i$. It achieves its maximum at an essential extreme point. Thus the above maximum is equal to

$$\max\left\{\max_{n/2<j\leqslant n}\binom{n}{j},\ \max_{i\leqslant n/2,\, n<i+j}\left(\binom{n-1}{i-1}+\binom{n-1}{j}\right)\right\}.$$

It is an easy task to determine this maximum and the following theorem is obtained.

Theorem 6 (Milner [36]). *The maximum number of members in an intersecting inclusion-free family in X is* $\binom{n}{\lceil (n+1)/2\rceil}$.

One can prove inequalities like the following one.

Theorem 7 (Bollobás [2]). *If p is the profile vector of an intersecting, inclusion-free family then*

$$\sum_{i\leqslant n/2}\frac{p_i}{\binom{n-1}{i-1}}\leqslant 1. \tag{3}$$

The left-hand side of (3) is a linear function of p, therefore it achieves its maximum at an essential extreme point. However the essential extreme points give either zero or 1 for this function, that is, its maximum is really 1.

The following inequality can be proved in the same way.

Theorem 8 (Greene et al. [24]). *If p is the profile vector of an intersecting, inclusion-free family then*

$$\sum_{i\leqslant n/2}\frac{p_i}{\binom{n}{i-1}}+\sum_{n/2<j\leqslant n}\frac{p_j}{\binom{n}{j}}\leqslant 1.$$

This sequence of theorems illustrates the main significance of the 'extreme point' theorems. They contain many other extremal theorems and inequalities, like Theorem 4 implies Theorems 5–8.

Let us show the limits. A family $\mathscr{F}$ is 2-intersecting if $F_1, F_2 \in \mathscr{F}$ implies $2\leqslant |F_1\cap F_2|$. If the set of extreme points of the profile vectors were known, we could determine the maximum of p_i, as well. However this is unsolved in general. A very old unsolved problem ([11]) is, for instance, the case when $n=4r$, $i=2r$ hold for an integer r. That is, the 'largest points' are unknown even along the axes. If k is small relative to n then this maximum is $\binom{n-2}{i-2}$ (see [11], the exact bound of validity was found in [18] and [40]). However [25] determines the extreme point 'in the middle', that is the maximum of $|\mathscr{F}|$.

Finally let us mention two other branches of this area.

Suppose that X is partitioned: $X=X_1\cup X_2$. A family $\mathscr{F}$ is given on X and p_{ij} is defined as the number of members F of $\mathscr{F}$ such that $|F\cap X_1|=i$ and $|F\cap X_2|=j$. The definitions of the profile matrix and the extreme points of the convex hull of all profile matrices of families satisfying a given property are obvious. Ref. [15] proves theorems for these extreme points or extreme matrices.

Consider a ranked poset $\boldsymbol{P}$ rather than subsets of a finite set. Let $\mathscr{F}$ be a subset of $\boldsymbol{P}$. One can define the profile vector with components p_i, it is the number of elements of $\mathscr{F}$ on the ith level. One can look for the extreme points of the convex hull of all profile vectors of families satisfying a certain condition. Results of this type can be found in [12].

3. Essential facets

Suppose that the set of extreme points $\varepsilon(\boldsymbol{A})$ is known for a certain class $\boldsymbol{A}$ of families. The duality theorem of linear programing determines all the hyperplanes (inequalities) not crossing the convex hull in an inner point. In this way all such inequalities are obtained which are satisfied for all profile vectors of families in $\boldsymbol{A}$. This is, however, a too large class of inequalities. What we really need is the set of inequalities describing the facets of the convex hull. This is the 'minimal' set of inequalities determining the convex hull. In the introduction we determined the extreme points for the inclusion-free families. This easily implied the inequalities of the facets. One of them is the YBLM-inequality, the others are trivial.

The situation is harder for intersecting, inclusion-free families. Theorem 4 determines the extreme points. The method of how to obtain the facets of the convex hull from these extreme points will be sketched here.

A polytope P in the $(n+1)$-dimensional Euclidean space is called *anti-blocking type* if $P \neq \emptyset$, $x \in P$ implies $0 \leqslant x$ and $0 \leqslant y \leqslant x \in P$ implies $y \in P$ (see [37]). P is *full* if it contains elements $(0, \ldots, 0, x_i, 0, \ldots, 0)$ $(0 < x_i)$ for all i $(0 \leqslant i \leqslant n)$. It is easy to see that the polytope determined by the points given in Theorem 4 (that is, the convex hull of the class of intersecting inclusion-free families) is full and anti-blocking type.

Define (see [37])

$$A(P) = \{z\colon 0 \leqslant z,\ zx \leqslant 1,\ \text{for all } x \in P\}.$$

By a theorem of *Fulkerson* [22,23] (see Theorem 9.4 in [37]) $A(P)$ is full and anti-blocking type, again.

An extreme point x of a polytope P is called *essential* if there is no other extreme point $y \in P$, $x \leqslant y$. It is easy to see that an anti-blocking type polytope is uniquely determined by its essential extreme points.

A *facet* of an $(n+1)$-dimensional polytope P is an n-dimensional hyperplane, given by an equation $a_0x_0 + \cdots + a_nx_n = 1$ or $a_0x_0 + \cdots + a_nx_n = 0$, containing at least $n+1$ extreme points of P and satisfying the inequality of the same direction for all points of P.

If P is full and anti-blocking type then $x_i = 0$ is a facet for each i and no other facet has 0 on the right-hand side. The latter ones are the *essential facets*.

Lemma 1. *$\sum a_ix_i = 1$ is an essential facet of the full anti-blocking type polytope P iff $a = (a_0, \ldots, a_n)$ is an essential extreme point of $A(P)$.*

Let P be the convex hull of the points given in Theorem 4. Then $A(P)$ can be easily determined. By Lemma 1 we have to determine only the essential extreme points of $A(P)$. This can be done by some step by step reduction (see [30]). Finally it leads to the following theorem.

Theorem 9 (Katona and Schild [30]). *The essential facets of the convex hull of the class of intersecting Sperner families are the following ones. Let* $1 = i_1 < \cdots < i_{r+1} = \lceil (n+1)/2 \rceil$ $(1 \leqslant r)$ *be some integers.*

$$\sum_{k=1}^{r} \left(\sum_{i=i_k}^{i_{k+1}-1} \frac{p_i}{\binom{n-1}{i-1} \frac{n}{n-i_k+1}} \right) + \frac{p_{\frac{n+1}{2}}}{\binom{n}{\frac{n+1}{2}}}$$

$$+ \sum_{k=1}^{r} \left(\sum_{n-i_{k+1}+1<j\leqslant n-i_k+1} \frac{p_j}{\binom{n-1}{j} \frac{n}{i_k-1}} \right) \leqslant 1,$$

where the middle term appears only for odd ns, the terms with i_1 $(=1)$ *should be taken to be* 0 *and the term with* $j = n$ *is simply* p_n.

Observe that $i_2 = \lceil (n+1)/2 \rceil$ leads to Bollobás's inequality (with a slight modification) while the case $i_k = k$ $(1 \leqslant k < \lceil (n+1)/2 \rceil)$ gives the inequality of Greene, Katona and Kleitman. The total number of inequalities in the theorem is exponential.

The essential facets are determined for case (ii) in [10] and cases (iv), (vii)–(x) by Engel [7].

One can also determine the 1-dimensional facets, that is, the edges of the convex hull. It is done for cases (ii) and (iii) in [10]. (It is trivial for case (i).)

4. Applications

We think that the main aim of this 'theory' is to compress information, as we said it in Section 2. However, direct applications were also found. We list here some of them.

The results (inequalities) of [4] became easier and more clear after finding the extreme points in case (xi). Ref. [16] determined all extremal families for an old two-part Sperner theorem. The paper used the convex hull, determining the profile vectors which are inner points of a facet. Later, however, Shahriari [38] proved the same statement without using this method. Ref. [17] gives some results on 3-part Sperner families.

The weight function $f(i) = p^i(1-p)^{n-i}$ is very frequent in probability theory. This fact leads to the applications in [6], [31] and [29].

We show one of the applications more detailed. The following theorem will be proved by this method.

Theorem 10 (Kleitman and Milner [32]). *Let $\mathscr{F}$ be an inclusion-free family and $k \leqslant n/2$ an integer. Suppose that*

$$\binom{n}{k} \leqslant |\mathscr{F}|. \tag{4}$$

Then the average size of the members of $\mathscr{F}$ is at least k.

Let p be the profile of $\mathscr{F}$. The condition of the theorem can be written in the form of an inequality:

$$\binom{n}{k} \leqslant \sum_{i=0}^{n} p_i.$$

That is, p is 'above' the hyperplane

$$\binom{n}{k} = \sum_{i=0}^{n} x_i \tag{5}$$

in the convex hull of inclusion-free families. This means that the convex hull H should be cut by this plane. Denote the new polytope by I. The extreme points of I can be obtained by intersecting the edges of H. It has two types of edges. They either connect the origin with a non-zero vertex, or they connect two non-zero vertices. Their lines are described by the following equations:

$$x_0 = x_1 = \cdots = x_{i-1} = 0, \quad x_{i+1} = \cdots = x_n = 0, \tag{6}$$

$$x_l = 0 \text{ unless } l = i, j, \quad \frac{x_i}{\binom{n}{i}} + \frac{x_j}{\binom{n}{j}} = 1 \text{ for all } i \neq j. \tag{7}$$

The intersection of (5) and (6) satisfies $x_i = \binom{n}{k}$. It is outside of H if $i < k$ or $n - k < i$. In this way the intersection points are $(0, \ldots, 0, \binom{n}{k}, 0, \ldots, 0)$ where the non-zero component is the ith one ($k \leqslant i \leqslant n - k$).

The intersection of (5) and (7) is

$$\left(0, \ldots, 0, \binom{n}{i} \frac{\binom{n}{j} - \binom{n}{k}}{\binom{n}{j} - \binom{n}{i}}, 0, \ldots, 0, \binom{n}{i} \frac{\binom{n}{i} - \binom{n}{k}}{\binom{n}{i} - \binom{n}{j}}, 0, \ldots, 0\right).$$

(Suppose $i < j$, the ith and jth components are non-zero.) As an easy calculation shows, this is in H iff either $i < k$, $k < j < n - k$ or $k < i < n - k$, $n - k < j$.

Finally, the vertices of H which are in the 'good' side of (5) are also extreme points of I: $(0, \ldots, 0, \binom{n}{i}, 0, \ldots, 0)$ (the ith component is non-zero where $k \leqslant i \leqslant n - k$).

We have to prove that the average size of the members of $\mathscr{F}$ is at least k if $p(\mathscr{F})$ is in I. That is,

$$k \leqslant \frac{\sum i p_i}{\sum p_i}$$

or

$$0 \leqslant \sum p_i (i - k). \tag{8}$$

This is a linear function of p, therefore the validity of (8) in I can be proved by checking it for the above extreme points.

The inequalities obtained from the first and second types of extreme points are trivial. The ones obtained from the second type of extreme points need more effort, the properties of binomial coefficients should be used.

We will also show in a forthcoming paper how to obtain analogous theorems by using this method for cases (i) and (iii).

Added in proof. The open problem of [11] mentioned after Theorem 8 has been recently solved by R. Ahlswede and L.H. Khachatrian.

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