SHEDDING SOME LIGHT ON SHADOWS

*G. O. H. Katona**

Mathematical Institute
Hungarian Academy of Sciences
Budapest, P.O. Box 127, Hungary

Abstract

Let $\mathcal{A}$ be a family of k-element subsets of a finite set. The shadow of $\mathcal{A}$ is the family of such $(k-1)$-element subsets which are subsets of members of $\mathcal{A}$. An old theorem of Kruskal and the author finds the minimum size of the shakow if k and the size of $\mathcal{A}$ is given. This paper gives a short survey of some related results. In the last two sections applications in reliability theory are shown.

I. Introduction

Let X be a finite set of n elements and $\mathcal{A} = (A_1, A_2, \ldots, A_m\}$ a family of some (distinct) k-element subsets of it. If $\binom{X}{k}$ denotes the family of all k-element subsets of X, then $\mathcal{A} \subseteq \binom{X}{k}$ can be written. $\binom{X}{k}$ can be considered as the k-th level of the partially ordered set of all subsets of X ordered by inclusion. Then $\mathcal{A}$ is a set sitting on the k-th level of this partially ordered set.

Suppose that the elements of X are ordered: $X = \{1, 2, \ldots, n\}$. The *characteristic vector* of a set $A \subseteq X$ is a zero-one vector of dimension n, its i-th component is 1 iff $i \in A$. *E.g.* the characteristic vector of $A = \{2, 3, 5\}$ is $(0, 1, 1, 0, 1, 0, 0, \ldots)$. The *shadow* of $\mathcal{A}$ is defined by

$$\sigma(\mathcal{A}) = \{B : |B| = k-1 \text{ and } B \subset A \in \mathcal{A}\}. \tag{1}$$

E.g. if $A = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 5\}\}$ Then $\sigma(\mathcal{A}) = \{\{1,2\}, \{1,3\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,5\}\}$. The name of this concept comes from the partial ordered set. $\sigma(\mathcal{A})$ is the "shadow" of $\mathcal{A}$ on the $(k-1)$-th level.

Consider the (linear) ordering of the characteristic vectors of the elements of $\binom{X}{k}$. It defines an ordering in $\binom{X}{k}$ that is called the *lexicographic ordering* of $\binom{X}{k}$. *E.g.* If $|X| = 5$, $k = 3$, then the ordering is the following: (0, 0, 1, 1, 1), (0, 1, 0, 1, 1), (0, 1, 1, 0, 1), (0, 1, 1, 1, 0), (1, 0, 0, 1, 1), (1, 0, 1, 0, 1), (1, 0, 1, 1, 0), (1, 1, 0, 0, 1), (1, 1, 0, 1, 0), (1, 1, 1, 0, 0).

---

*The work was supported by the Hungarian National Foundation for Scientic Research, grant numbers 1090 and 2575.

SHEDDING SOME LIGHT ON SHADOWS

Theorem 1 (Kruskal [19] and Katona [14]).

$$\min_{n,k,|\mathcal{A}| = m \text{ are fixed}} = F(k,m)$$

is attained for the family $\mathcal{A}$ consisting of the lexicographically first m members of $\binom{X}{k}$.

Actually this form of the theorem is due to Clements and Lindström [3] who proved a more general theorem. Kruskal and Katona gave a complicated explicit formula for $F(k,m)$. The following lemma is needed to formulate it.

Lemma 1. For given positive integers m and k there is a unique expansion of the form

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_t}{t}$$

where $a_k > a_{k-1} > \ldots > a_t \geq t \geq 1$.

Then $F(k,m)$ can be expressed:

$$F(k,m) = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \ldots + \binom{a_t}{t-1}.$$

More recent, shorter proofs are due to Daykin [4], Eckhof and Wegner [6], Hilton [13], and Greene and Kleitman [11]. The (probably) shortest proof is given by Frankl [7].

2. The isoperimetric problem.

In the classical isoperimetric problem the volume of a body is given and the surface is to be minimized. It is known that the sphere is the optimum. In our case the space is the set of zero-one sequences of length n. This is called the Hamming space if the distance is defined as the number of different digits. Volume is replaced by the size $|\mathcal{A}|$ of the "body" $\mathcal{A}$. We only have to define the surface. It is the set of the sequences neighbouring (being of distance one of) an element of $\mathcal{A}$:

$$\delta(\mathcal{A}) = \{a : a \in \mathcal{A}, \exists\, b \notin \mathcal{A} \therefore a \text{ and } b \text{ differ in one position}\}.$$

The obvious analogue of the classical isoperimetric problem in this space is the following.

Discrete isoperimetric problem. Given n and the size $|\mathcal{A}|$ of a set $\mathcal{A}$ of zero-one sequences of length n, minimize $|\delta(\mathcal{A})|$.

The sphere can be expected to be the optimum here, too. But what is a sphere in this space? Of course, we choose a center and take the points of distance, say r, from this center. The most natural would be the all-zero sequence, but for some reason, to be made clear later, we choose the all-one sequence. If the size $|\mathcal{A}|$ is of the form

$$\binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{k+1} \tag{2}$$

for some integer k then the definition is clear, the *sphere* with center $(1, 1, \ldots, 1)$ and radius $r = n - k - 1$ should consist of all the zero-one sequences containing at least $k + 1$ ones. However, if $|\mathcal{A}|$ is between (2) and

$$\binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{k}$$

then the definition of the *sphere* is ambiguous: it consists of all zero-one sequences containing at least $k + 1$ ones and some, more exactly,

$$|\mathcal{A}| - \left[\binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{k+1}\right] \tag{3}$$

sequences containing exactly k ones.

Theorem 2 (Harper [12]). Given n and $|\mathcal{A}|$, the minimum of the surface $|\delta(\mathcal{A})|$ is attained for the sphere.

For a shorter proof see Frankl and Füredi [8]. It remained to determine the optimal choice of the sequences with k ones. The following notion helps in doing it. The *outer surface* is defined by

$$\Delta(\mathcal{A}) = \{b : b \notin \mathcal{A}, \exists a \in \mathcal{A} \ni a \text{ and } b \text{ differ in one position}\}.$$

It is easy to see that $\Delta(\mathcal{A}) = \delta(\bar{\mathcal{A}})$. Therefore, to find the minimum of $\Delta(\mathcal{A})$ for a given size $|\mathcal{A}|$ is equivalent to finding the minimum of $\delta(C)$ for the size $|C| = 2^n - |\mathcal{A}|$. That is, the two problems of minimization are equivalent.

Suppose that $\mathcal{A}$ is a sphere and let

$$\mathcal{A} = \{a : \text{the number of ones in } a \text{ is at least } k + 1\} \cup \mathcal{B}$$

where $\mathcal{B}$ consists of some sequences containing exactly k ones. It is easy to see that $\Delta(\mathcal{A})$ consists of all the sequences containing exactly k ones which are not in $\mathcal{B}$ and of $\sigma(\mathcal{B})$ where B is considered to be a family of k-element subsets rather than the set of their characteristic vectors. The first part has a fixed size: $\binom{n}{k}$ minus (3), that is,

$$\binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{k} - |\mathcal{A}|.$$

Hence the minimization of $\Delta(\mathcal{A})$ is equivalent to the minimization of $\sigma(\mathcal{B})$. By Theorem 1, this and the surface $\mathcal{A}$ will be minimum for the lexicographically first t sequences where t is given by (3). The definition of the sphere should be modified accordingly. This completes Theorem 2.

The minimum surface can be expressed by a formula. Let m denote the size of $|\mathcal{A}|$. Define k by

$$\binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{k+1} \leq m < \binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{k}.$$

Then

$$\min |\delta(\mathcal{A})| = \binom{n}{n} + \binom{n}{n-1} + \ldots + \binom{n}{k} - |\mathcal{A}| + F\left(k, |\mathcal{A}| - \binom{n}{n} - \binom{n}{n-1} - \ldots - \binom{n}{k+1}\right)$$

if n and $|\mathcal{A}|$ are fixed. [15] gives a direct proof resulting in this formula.

3. F(k,m) is a complicated function.

We will study the function F(k,m) only in the interval $0 \leq m \leq \binom{2k-1}{k}$. It is easy to see that $m \leq F(k,m)$ holds in this interval. Actually, we will consider the *excess function* $e(k,m) = F(k,m) - m$ expressing the shadow how much larger is than the original family. The *normalized excess function* is

$$s_k(x) = \frac{k}{\binom{2k-1}{k}} e\left(k, \left\lfloor \binom{2k-1}{k} x \right\rfloor\right) \qquad (0 \leq x \leq 1).$$

On the other hand, let us define the *Takagi function* by

$$t(x) = \sum_{i=1}^{\infty} \frac{\varphi_i(x)}{2^i}$$

where

$$\varphi(x) = \begin{cases} 2^i - 2j & \text{if } (2j/2^i) \leq x \leq (2j+1)/2^i \\ -2^i + 2j + 2 & \text{if } (2j+1)/2^i \leq x \leq (2j+2)/2^i \end{cases} \qquad \text{for } 0 \leq j < 2^{i-1}.$$

This function is continuous, nowhere differentiable and self-similar (Takagi [22]).

Theorem 2 (Frankl, Matsumoto, and Tokushige [9]). The sequence $s_k(x)$ converges uniformly to $t(x)$ when $k \to \infty$.

This theorem shows that F(k,m) as a function of m jumps rather strangely. It is not easy to compute and use it. Lovász [21] has a very nice suggestion to overcome this difficulty. Generalize the bionomial coefficient for non-integer values:

$$\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!}$$

is a polynomial of x defined for any real $x \geq k$. Then the following theorem can be proved.

Theorem 4 (Lovász [21]). Let $\mathcal{A}$ be a family of k-element sets where $|\mathcal{A}| = \binom{x}{k}$. Then

$$\binom{x}{k-1} \leq |\sigma(\mathcal{A})|.$$

This theorem gives only a lower estimate on $|\sigma(\mathcal{A})|$ in contrast to Theorem 1 which determines the exact minimum. However, it is much easier to calculate and use this lower estimate than F(k,m). On the other hand, this lower estimate is sharp for the integer xs, that is, it gives the right order of magnitude.

4. Oriented, coutinuous version.

The notions and statements of this section will be formulated only for $k = 3$, since it is easier to visualize this case, bur everything holds for larger ks, too. The word oriented in the title of this section means that we consider ordered 3-tuples, or 3-dimensional vectors rather than 3-element subsets of the ground set X. Thus $\mathcal{A}$ is a collection of vectors (x_1, x_2, x_3) where $(x_1, x_2, x_3) \in X$. To be closer to our traditional case we suppose that all oriented variants of the set $\{x_1, x_2, x_3\}$ are chosen, that is, $\mathcal{A}$ is closed under the permutation of the coordinates. However, an essential difference is that the xs are not necessarily different here. Another relaxation is that X can be any measurable set. The traditional case can be considered as a discrete space with equal probabilities $\frac{1}{n}$. To emphasize the other extreme case, suppose that X is the [0,1] interval with the usual Lebesque measure μ. Then the number of (subsets) sequences in $\mathcal{A}$ can be replaced by the measure of them. Therefore in this section $\mathcal{A}$ is a measurable set of the unit cube with the property that $(x_1, x_2, x_3) \in \mathcal{A}$ implies that any vector obtained by permutation of the coordinates of (x_1, x_2, x_3) is in $\mathcal{A}$. In our traditional case we deleted elements from the subsets to obtain the members of the shadow. It can be easily repeated here. Delete one coordinate. Since $\mathcal{A}$ is closed under permutation, it does not matter which coordinate. Thus the analogue of the shadow in this case is the good old projection:

$$\pi(\mathcal{A}) = \{(x_1, x_2) : (x_1, x_2, x_3) \in \mathcal{A}\}.$$

Theorem 5. The minimum area (measure) of the projection $\pi(\mathcal{A})$ for all such measurable subsets $\mathcal{A}$ of the unit (3-dimensional) cube which are closed for the permutations of the coordinates, have a measure μ and have a measurable projection is $\mu^{2/3}$.

It is easy to see that this estimate is sharp. the "small" cube of size $\mu^{2/3}$ serves as a construction.

This theorem can be deduced from Theorem 1 as it is shown in [16]. (This analogy was independently discovered by Daykin [5].) On the other hand, it has been proved earlier by Loomis and Whitney [20]. Unfortunately, the elegant analytic proof of Loomis and Whitney (using Hölder;s inequality) does not help in proving Theorem 1. The exclusion of the repetition of the coordinates changes the situation.

5. Applications in reliability theorey.

Suppoe that a complicated device is given, built from many components. Each component can go wrong with probability p $(0 < p < 1)$. However, the breakdown of one component does not necessarily cause the malfunction of the whole device. The roles of the components in the operation of the device can be very different. Let X be the set of the n components. Call a subset $A \subset X$ an *operative set* if the device operates correctly whenever the elements of $X - A$ work but the elements of A do not work. The family of operative sets is the *operative family.* It will be denoted by $\mathcal{A}$. A typical example is a network of electrical transmission lines forming a graph. The system stops its operation if the graph becomes disconnected. In this example the edges of the graph form the set X. The operative family consists of those sets of edges whose deletion do not disconnect the graph (that is, the sets not containing a cut of the graph).

In the above example and in most practical cases $\mathcal{A}$ has the following property:

$$A \in \mathcal{A} \text{ and } \mathcal{B} \subset \mathcal{A} \text{ imply } \mathcal{B} \in \mathcal{A}.$$

Such families are called *ideals.* However there are practical situations where $\mathcal{A}$ is not an ideal. Let us see now only a non-serious example. Let the device be a country. If one of the ministers goes crazy the country stops its proper operation. However, if his phone goes wrong simultaneously, then nobody notices anything. That is, the two-element set {minister, its phone} is in $\mathcal{A}$ while the one-element set consisting only of the minister does not belong to $\mathcal{A}$.

The probability of the event that the whole device is operating correctly is

$$\sum_{A\in\mathcal{A}} p^{|A|}(1-p)^{n-|A|}. \tag{4}$$

This is called the *reliability polynomial.* If p and $\mathcal{A}$ are fully known then the reliability polynomial can be easily computed. In practical situations, however, $\mathcal{A}$ is not known, only some of its properties or parameters. In such a situation we can only give estimates on the reliability polynomial. The best estimates are the minimums and the maximums. Therefore we try to study the minimum and maximum of (4) where n and p are fixed and $\mathcal{A}$ can be chosen under some given conditions.

Let $a_i(\mathcal{A})$ denote the numbers of i-element members of $\mathcal{A}$. Then (4) can be rewritten in the form

$$\sum_{i=0}^{n} a_i(\mathcal{A})p^i(1-p)^{n-i}. \tag{5}$$

$p^i(1-p)^{n-i}$ is a decreasing (increasing) function of i when $p < \frac{1}{2}$ $(p > \frac{1}{2})$. (5) behaves very differently in these two cases. Consider only the first one: $p < \frac{1}{2}$. Find the minimum of (5) (the maximum is easy in this case) for ideals $\mathcal{A}$ such that

$$|\mathcal{A}| = \sum_{i=0}^{n} a_i = m$$

is given.

If a member of $\mathcal{A}$ is replaced by a member of larger size, (5) will be smaller. However, this "pushing up" cannot be done without limitations, the condition that $\mathcal{A}$ is an ideal is an obstacle.

$$F(i,a_i) \le a_{i-1} \tag{6}$$

is a consequence of Theorem 1 and the fact that $\mathcal{A}$ is an ideal. Heuristically, the "pushing up" procedure can be continued until $F(i,a_i)$ is nearly equal to a_{i-1}. If m is a power of 2, say 2^b, then equality can be attained at each level. The family $\mathcal{A}$ of all subsets of a b-element subset of X show it. In general, a *cylinder* of size $m = 2^{b_1} + 2^{b_2} + \ldots + 2^{b_r}$ $(b_1 > b_2 > \ldots > b_r \ge 0)$ is defined by a sequence of sets $X \supset B_1 \supset B_2 \supset \ldots \supset B_r$, elements $a_1 \in X - B_1$, $a_2 \in B_1 - B_2, \ldots, a_{r-1} \in B_{r-2} - B_{r-1}$:

$$\mathcal{C} = \{A : A \subseteq B_1\} \cup \{A \cup \{a_1\} : A \subseteq B_2\} \cup \ldots \cup \{A \cup \{a_1, \ldots a_{r-1}\} : A \subseteq B_r\}.$$

One can prove, using some lemmas giving inequalities on F F(k,m), that the "pushing up" procedure ends up in a cylinder under constraint (6).

Theorem 6 (Ahlswede and Katona [1]).

$$\max_{\mathcal{A} \text{ is an ideal, } |\mathcal{A}| = m} \sum_{i=0}^{n} a_i(\mathcal{A}) p^i (1-p)^{n-i}.$$

is attained for the cylinder.

For more practical and deeper applications, see the monograph of Ball, Colbourn, and Provan [2].

6. Reliabability with mediocre states.

In this section we suppose that each element of the device may have three different states. It may be *operative, mediocre* or *failing* with probability p_0, p_1, or p_2, respectively. ($p_0 + p_1 + p_2 = 1$.) A *state of the device* is a 0, 1, 2-vector of dimension n where the i-th component is 0, 1, or 2 if the i-th element of the device is good, mediocre, or failing, respectively. A state $(s_1, s_2, \ldots, s_n)$ is call *operative* if the device operates when its i-th element is in state s_i $(1 \le i \le n)$. The set of operative states of the device will be denote by $\mathcal{A} \subseteq \{0, 1, 2\}^X$. $(s_1, s_2, \ldots, s_n) \le (r_1, r_2, \ldots, r_n)$ is the abbreviation for $s_i \le r_i$ for $1 \le r \le n$. As in the trditional case, $\mathcal{A}$ is an ideal if $A \in \mathcal{A}$ and $\mathcal{B} \le \mathcal{A}$ imply $\mathcal{B} \in \mathcal{A}$. In most parctical cases this property is satisfied.

If $A \in \mathcal{A}$ then $1(A)$ denotes the number of 1's in A while $2(A)$ denotes the number of 2's in A. The probability of the event that the device operates properly is expressed by the formula

$$\sum_{A \in \mathcal{A}} p_0^{\,n-1(A)-2(A)} p_1^{\,1(A)} p_2^{\,2(A)}. \tag{7}$$

In this case this is the (generalized) *reliability polynomial*. Introduce the notation $a(i,j)$ for the number of elements $A \in \mathcal{A}$ satisfying $i = 1(a)$ and $j = 2(a)$. This gives rise to another form of the reliability polynomial:

$$\sum_{i \text{ and } j} a(i,j) p_0^{\,n-i-j} p_1^{\,i} p_2^{\,j}. \tag{8}$$

The aim of the investigations, again, is to find the minimum and the maximum of the reliability polynomial for certain classes of $\mathcal{A}$'s. Obviously, in most practical cases $\mathcal{A}$ should be an ideal. So the conditions containing "ideal" are the most important. However, the only result [17] treats only the least important case when $B < A$ is forbidden in $\mathcal{A}$. In the rest of the paper we will try to find the minimum of (7) (and (8)) under the conditions that (1) $\mathcal{A}$ is an ideal and (2) $|\mathcal{A}| = m$ is fixed. "Try" means that we are unable to reach our goal, we only show the difficulties. The key idea in proving Theorem 6 was the lower estimate on a_{i-1} in terms of a_i, using Theorem 1. Here we need similar lower estimates on $a(i-1,j)$ and $a(i,j-1)$ when $a(i,j)$ is given.

A sequence containing 0, 1, and 2 can be described by the subsets of positions containing 1 and 2, respectively. This description is more similar to the situation given in Theorem 1. Therefore, consider a family $\mathcal{A}_{ij}$ of pairs (B, C) where $B, C \subset X$, $B \cap C = \emptyset$, $|B| = i$, $|C| = j$. Introduce the notions of the *left shadow*:

$$\sigma_1(\mathcal{A}_{ij}) = \{(B',C) : B' = B - \{x\},\ x \in B,\ (B, C) \in \mathcal{A}_{ij}\},$$

and the *right shadow*:

$$\sigma_2(\mathcal{A}_{ij}) = \{(B,C') : C' = C - \{x\},\ x \in C,\ (B, C) \in \mathcal{A}_{ij}\}.$$

Suppose that $\mathcal{A}$ is an ideal of 0, 1, 2-sequences of length n. Consider the sequences $\mathcal{A}$ containing i 1's and j 2's. Let $\mathcal{A}_{ij}$ denote the family of pairs defined by these sequences. It is easy to see that $a(i-1, j) \geq |\sigma_1(\mathcal{A}_{ij})|$ and $a(i, j-1) \geq |\sigma_2(\mathcal{A}_{ij})|$. This pair of inequalities shows that our problem is different from the problem of minimizing the size of the shadow. The left shadow can be very small, however, this would imply that the right shadow is large. Therefore, our aim could be to determine the pairs in the right hand sides of the above inequalities which cannot be decreased.

Problem. Find the minimal pairs (as two-dimensional vectors) $(|\sigma_1(\mathcal{A}_{ij})|, |\sigma_2(\mathcal{A}_{ij})|)$ where n, i, j and $|\mathcal{A}_{ij}|$ are fixed.

We do not even have a nice conjecture. However, there is a theorem in a special case. Namely, when the positions of the 1's and 2's are separated. Suppose that X can be partitioned ($X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$) in such a way that $(B, C) \in \mathcal{A}_{ij}$ implies that $B \subset X_1$, $C \subset X_2$. Such families are called *two part* (i,j)-*families*.

Theorem 7. Suppose that $|X_1|$, $|X_2|$, i, j, and the size of a two part (i,j)-family $\mathcal{A}$ are given. If $(|\sigma_1(\mathcal{A})|, |\sigma_2(\mathcal{A})|)$ is a minimal pair under these conditions, then it can be attained by a family $\mathcal{A}$ satisfying the following conditions: If $(B, C) \in \mathcal{A}$, B' precedes B and C' precedes C, respectively, in the lexicographic ordering, then $(B', C') \in \mathcal{A}$.

Unfortunately, this is not a full solution even in this special case since the condition given in the theorem allows a lot of freedom. However, it does not seem to be very hard to finish this part of the problem. Let us mention that Theorem 7 was independently discovered by Kleitman [18] trying to prove an old conjecture of Erdös. Finally, let us mention that the interested reader should see the survey paper of Frankl and Tokushige [10] about the same subject.

References

[1] R. Ahlswede and G. O. H. Katona, "Contributions to the geometry of Hamming spaces", *Discrete Matn.* 17 (1977), 1-22.

[2] M. O. Ball, C. J. Colbourn, and J. S. Provan, *Network Reliability*, manuscript of a book.

[3] G. F. Clements and B. LindstrUom, "A generalization of a combinatorial theorem of Macaulay", *J. Combin. Theory* 7 (1969), 230-238.

[4] D. E. Daykin, "A simple proof of the Kruskal-Katona theorem", *J. Combin. Theory* 17 (1974), 252-253.

[5] D. E. Daykin, "Inequalities between the volume of a set and the product of its projection area", *Nanta Math.* 10 (1977), 106-108.

[6] J. Eckhof and G. Wegner, "Über einer Satz von Kruskal", *Periodica Math. Hungar.* 6 (1975), 137-142.

[7] P. Frankl, "A new short proof of the Kruskal-Katona theorem", *Discrete Math.* 48 (1984), 327-329.

[8] P. Frankl and Z. Füredi, "A short proof for a theorem of Harper about Hamming-spheres", *Discrete Math.* 34 (1981), 311-313.

[9] P. Frankl, M. Matsumoto, and N. Tokushige, "The shadow function and the Takagi function", *J. Combin. Theorey A*, submitted.

[10] P. Frankl and N. Tokushige, "The Kruskal-Katona theorem, some of its analogues and applications" *Extremal Problems for Finite Sets*, János Bolyai Math. Soc., Budapest, 1994.

[11] C. Greens and D. J. Kleitman, "Proof techniques in the theorey of finite sets, *Studies in Mathematics* 17, Math. Assoc. America, Washington, DC, 1978.

[12] K. H. Harper, "Optimal numberings and isoperimetric problems for graphs", *J. Combin. Theory* 1 (1966), 385-393.

[13] A. J. W. Hilton, "A simple proof of the Kruskal-Katona theorem and of some related binomial inequalities", *Periodica Math. Hungar.* 10 (1979), 25-30.

[14] G. O. H. Katona, "A theorem of finite sets", *Theory of Graphs*, Akadémiai Kiadó, Budapest, 1968.

[15] G. O. H. Katona, "The Hamming-sphere has minimum boundary", *Studia Sci. Math. Hungar.* 10 (1975), 131-140.

[16] G. O. H. Katona, "Continuous versions of some extremal hypergraph problems, II", *Acta Math. Acad. Sci. Hungar.* 35 (1980), 67-77.

[17] G. O. H. Katona and W. Woyczynski, "Optimization of the reliability polynomial in presence of mediocre elements", *Networks* 23 (1993), 327-332.

[18] D. J. Kleitman, personal communication.

[19] J. B. Kruskal, "The number of simplices in a complex", *Math. Optimization Technics*, Univ. California Press, Berkeley, 1963.

[20] L. M. Loomis and H. Whitney, "An inequality related to the isoperimetric inequality", *Bull. Amer. Math. Soc.* 55 (1949), 961-962.

[21] L. Lovász, "Problem [3.3]", *Combinatorial Problems and Exercises*, North Holland, 1979.

[22] T. Takagi, "A simple example of a continuous function without derivative" *Proc. Japan Phys. Math. Soc.* 1 (1903), 176-177.