Greedy Construction of Nearly Regular Graphs

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Dedicated to Bernt Lindström on his 60th birthday

1. Introduction

Let us try to build nearly regular graphs, starting with the empty graph on n vertices and adding one single edge at each step. If we are allowed to make a long-term plan this is very easy. First suppose that n is even. It is known [2] (a more elegant proof is due to von Walecki, see [1]) that the complete graph on n vertices can be decomposed into $n-1$ one-factors (= set of edges containing each vertex exactly once). Then take the edges in the first one-factor one by one, and after that the edges in the second one-factor, and so on. The differences of the degrees in the so obtained graphs are always at most one. The differences are zero for the graphs completing the first, second, ... one-factor, respectively.

Suppose now that our algorithm is 'on line' (the term is suggested by H. A. Kierstead) based on the momentary degrees: at each step some vertices with minimum degrees must be joined. We formulate this more precisely:

Algorithm. Put $G_0 = (V, \emptyset)$, where $|V| = n$. Denote the graph obtained after the mth step by $G_m = (V, E_m)$. The degrees in this graph are denoted by d_m. Consider all the pairs $(d_m(x), d_m(y))$, where $x, y \in V$, $x \neq y$, $(x, y) \notin E_m$, and $d(x) \leqslant d(y)$. Choose any pair (x_m, y_m) with the lexicographically smallest pair $(d_m(x), d_m(y))$. Then $E_{m+1} = E_m \cup \{(x_m, y_m)\}$.

The choice of the new edge is ambiguous. We have seen that there is a sequence of choices (best case) such that the difference of degrees does not exceed 1. Our question is how large the difference can be at the other (worst) choices:

$$f(n, m) = \max_{\text{possible choices}} \max_{|V|=n,\, x,y\in V} (d_m(x) - d_m(y)). \tag{1}$$

A graph G is called *feasible* if there is a sequence of graphs $G_0, G_1, \ldots, G_m = G$ following the rules of the algorithm. Thus $f(n, m)$ is the largest difference of degrees in feasible graphs with n vertices and m edges.

The following easy example shows that $f(n, m)$ may exceed 1: namely, $f(6, 6) > 1$. Start with the one-factor $(1, 2)$, $(3, 4)$, $(5, 6)$. Now all degrees are 1. Next join the vertices $(1, 3)$, $(2, 4)$. At this moment there are two vertices of degree 1, but they are joined. A vertex of degree 1 and a vertex of degree 2 must be jointed, say $(3, 5)$. Here $d_6(3) = 3$ and $d_6(6) = 1$. The difference is really 2.

This construction can be generalized for any $n > 4$. After having a cycle of length $n-2$ and a disjoint edge, the next edge creates difference 2: that is, n edges may cause the difference to be 2. On the other hand, one can see that this is the minimum. Consider a sequence of m edges yielding difference 2. Suppose that there is a vertex of degree 0 in the graph G_{m-1} formed by the first $m-1$ edges. All other vertices must be of degree at least 2, otherwise the new (mth) edge cannot make difference 2. On the

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other hand, if G_{m-1} contains no isolated vertex then at most two vertices may have degree 1. The sum of the degrees in G_m is at least $2n$; thus $m \geq n$.

To construct larger differences is not so trivial. In Section 2 we show that any difference may occur; that is, $f(n, m)$ can be arbitrarily large. In this construction the graph is almost complete. In Section 3 we give a better construction for the difference 3. We show that $f(4k+2, 4k^2+11k+3) \geq 3$ $(k \geq 4)$; that is, the difference can be 3 if the number of edges is about half of the total number of possible edges. In Section 4 the converse is proved; $f(4k+2, 4k^2+11k+2) \leq 2$. (The other cases mod 4 are also settled.) Finally, we list some open problems.

2. Constructing Large Differences

THEOREM 1. *For any positive integer d there are positive integers n and m such that*

$$f(n, m) \geq d.$$

PROOF. The following rooted trees will play a crucial role in the proof. They will be constructed recursively. Let T_1^d be a trivial tree consisting of a single vertex. Suppose that T_k^d has been constructed. Then T_{k+1}^d will contain its root r, the vertices $v_1, \ldots, v_d$ and d vertex-disjoint copies of each $T_1^d, \ldots, T_k^d$. r and v_i are joined for all $1 \leq i \leq d$ and there is an edge between v_i and the root of the ith copy of T_j^d for all $1 \leq j \leq k$, $1 \leq i \leq d$. Using these trees T_k^d, one more rooted tree, T^d will be constructed. Its root and each of the vertices $u_1, \ldots, u_{d+1}$ are joined. Moreover, there is an edge joining u_i and the root of the ith copy of T_j^d for $1 \leq j \leq d$, $1 \leq i \leq d+1$. Observe the similarity between T_{d+1}^d and T^d. T_{d+1}^d has d identical branches starting from the root, while T^d has $d+1$ such identical branches (Figure 1).

LEMMA 1. *The number $s^d(j)$ of vertices of degree j in T^d is*

$$(d+1)^d \qquad \text{if } j = 1, \tag{2}$$

$$d(d+1)^{d-j+1} \qquad \text{if } 1 < j \leq d, \tag{3}$$

$$\frac{d+1}{d}((d+1)^{d-1} - 1) + d + 2 \qquad \text{if } j = d+1. \tag{4}$$

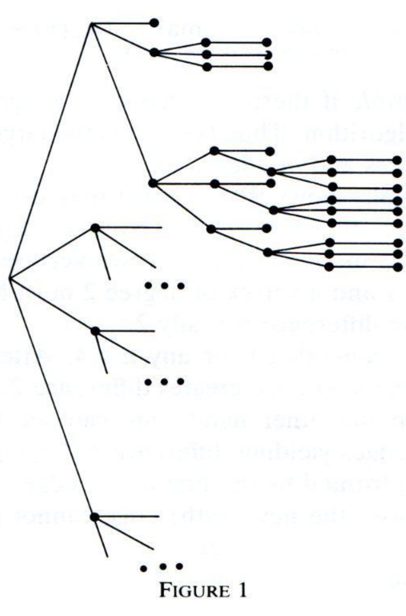

FIGURE 1

The total number of vertices in T^d is

$$s^d = \frac{(2d+1)(d+1)^d - 1}{d}. \tag{5}$$

PROOF. Denote by $s^d(j, k)$ the number of vertices of degree j in T^d_k. $s^d(1, 2) = d$ and $s^d(1, k+1) = s^d(1, k)(d+1)$ $(2 \leq k)$ are obvious. This implies $s^d(1, k) = d(d+1)^{k-2}$. Using the obvious similarity between T^d_{d+1} and T^d, one obtains

$$s^d(1) = \frac{d+1}{d} s^d(1, d+1) = (d+1)^d,$$

proving (2).

Let $1 < j \leq d$. The tree T^d_i contains no vertex of degree j if $i < j$, with the possible exception of the root when $j = d$. The first vertices of degree j appear in the construction of T^d_j $(v_1, \ldots, v_d)$. Thus, $s^d(j, j) = d$ hold. One can easily see that $s^d(j, j+1) = d^2$ and $s^d(j, k+1) = (d+1)s^d(j, k)$ if $j + 1 \leq k$. Therefore $s^d(j, k) = d^2(d+1)^{k-j-1}$ holds for $j + 1 \leq k$. Hence

$$s^d(j) = \frac{d+1}{d} s^d(j, d+1) = d(d+1)^{d-j+1}$$

follows, proving (3).

Finally, it is obvious that $s^d(d+1, 1) = s^d(d+1, 2) = 0$ and $s^d(d+1, 3) = d$. On the other hand, $s^d(d+1, k+1) = (d+1)s^d(d+1, k) + d$ if $3 \leq k \leq d$. This implies $s^d(d+1, k) = d(1 + (d+1) + \cdots + (d+1)^{k-3}) = (d+1)^{k-2} - 1$ for $3 \leq k$ and

$$s^d(d+1) = \frac{d+1}{d} s^d(d+1, d+1) + d + 2 = \frac{d+1}{d}((d+1)^{d-1} - 1) + d + 2,$$

completing the proof of (4).

(5) is a consequence of (2), (3) and (4). □

LEMMA 2. *If d is even then* (5) *is also even.*

PROOF. Since the term derived from $2d$ is obviously even, it suffices to observe that

$$\frac{(d+1)^d - 1}{d} = \binom{d}{1} + \sum_{i=2}^{d} \binom{d}{i} d^{i-1}$$

is even. □

We introduce some further notations. Let $R^d(i, j)$, $1 \leq i \leq j \leq d$ denote the set of roots of all copies of T^d_i in T^d_j. More formally, $R^d(i, i)$ is the one-element set of the root of T^d_i. If $R^d(i, k)$ is defined, then $R^d(i, k+1)$ consists of the roots of the d copies of T^d_i and of the union of all $R^d(i, j)$'s in all copies of T^d_j in the definition of T^d_{k+1} for all $i < j \leq k$. Finally, $R^d(i)$, $(1 \leq i \leq d)$ denotes the union of all $R^d(i, j)$'s for $i \leq j \leq d$ in all copies of T^d_j's $(i \leq j \leq d)$ in T^d: that is, $R^d(i)$ is the set of the roots of all copies of T^d_i in T^d. Let $R^d(d+1)$ denote the one-element set consisting of the root of T^d. Consider the edge 'coming' from the root to a vertex belonging to $R^d(i)$. The set of these edges is denoted by $E^d(i)$. The set of the other end points (the ones closer to the root) of the edges belonging to $E^d(i)$ is denoted by $A^d(i)$. (See Figure 2, where i circles mark the elements of $R^d(i)$ and i squares mark the elements of $\bigcup_{j=1}^{i} A^d(j)$.)

Now some simple but important properties of T^d will be collected.

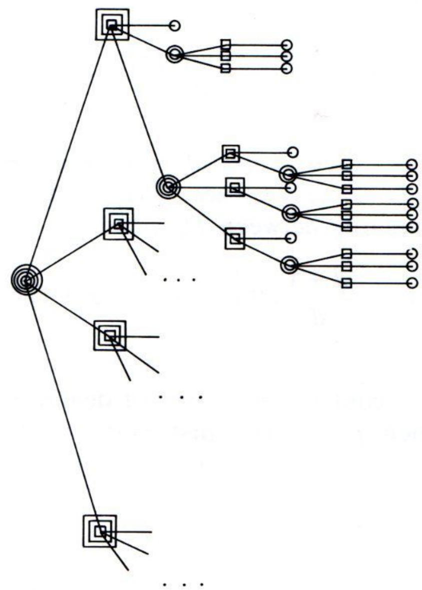

FIGURE 2

LEMMA 3. *Consider the unique coloring of T^d with two colors, pink and lavender, say, in which the root is pink. Then:*
(i) *a vertex is pink iff it is in the union of $R^d(i)$'s for $1 \leq i \leq d+1$;*
(ii) *the degrees of the pink vertices are either 1 or $d+1$;*
(iii) *each lavender vertex is incident to an edge from $E^d(1)$;*
(iv) *the elements of $A^d(i)$ $(1 \leq i \leq d)$ are all lavender and $A^d(i+1) \subset A^d(i)$ holds for $1 \leq i < d$;*
(v) *if $v \in \bigcup_{i=j}^{d+1} R^d(i)$, then all neighbors of v are in $A^d(j-1)$ $(2 \leq j \leq d+1)$.*

PROOF. Prove the analogous statements by induction for T_k^d. The coloring is hereditary in the recursion. □

Take another, vertex-disjoint copy T^{d*} of T^d. However, color T^{d*} with the two colors oppositely to T^d: the color of the root of T^{d*} is lavender. Add edges to this union V^d of the two trees to obtain a $d+1$-regular bipartite graph. The number of pink (lavender) vertices of degree j $(1 \leq j \leq d)$ is $s^d(j)$. Add one-factors formed from edges joining vertices of difference colors and of degree j until their degrees become $d+1$. This can be done since $s^d(j) \geq d+1-j$, by Lemma 1. In this way we obtain a $d+1$-regular bipartite graph G^d containing two vertex-disjoint copies of T^d. The size of each part is given by (5).

LEMMA 4. *Suppose that the sizes of the parts X and Y of the regular bipartite graph $G=(X, Y; E)$ are equal and even. Then the complement of G is feasible.*

PROOF. The complement of $G=(X, Y; E)$ is a union of two complete graphs on X and Y, respectively, and of a regular bipartite graph $H=(X, Y; F)$. Each of them can be decomposed into one-factors. Take the edges (one by one) of the first one-factor in the complete graph on X, and then the edges of the first one-factor in the complete graph on Y. Continue with the second, third, . . . one-factors. When all edges within X and Y, respectively, are there then choose the edges of the one-factors of H. □

By Lemmas 2 and 4, the complement C^d of G^d is feasible if d is even. Suppose that this is the case. C^d is also regular of degree $D = 2s^d - 1 - (d+1)$. The edges of T^d and T^{d*} are missing from C^d together with some other edges. Our algorithm allows us to add any missing edge at this moment. Add (one by one) all the edges belonging to $E^d(1)$ and $E^{d*}(1)$. The degrees of the vertices belonging to $R^d(1)$, $R^{d*}(1)$, $A^d(1)$ and $A^{d*}(1)$ are raised to $D+1$. By Lemma 3(iii), the vertices of degree D are either pink in T^d or lavender in T^{d*}. Their degrees in T^d and T^{d*}, respectively, are $d+1$, by (ii). Therefore all missing edges adjacent to them are edges of T^d or T^{d*}, respectively. In the rest of the proof there will always exist vertices of degree D; thus the algorithm allows us to join them to some other vertices, and so in the rest of the proof only edges of T^d and T^{d*} can be added. Moreover, the vertices of degree D (at this moment) are not adjacent either in T^d or in T^{d*} since they have the same color (pink in T^d, lavender in T^{d*}). We must join vertices of degrees D and $D+1$. Choose the edges in $E^d(2)$ and $E^{d*}(2)$. They are joining vertices of degree D (in $R^d(2)$ and $R^{d*}(2)$) and of degree $D+1$ (in $A^d(2)$ and $A^{d*}(2)$). Adding the new edges, these degrees increase by one. At this moment the set of vertices of degree D is equal to $\bigcup_{i=3}^{d+1} R^d(i) \cup \bigcup_{i=3}^{d+1} R^{d*}(i)$, by Lemma 3(i). (v) implies that their neighbors are in $A^d(2)$ and $A^{d*}(2)$. By (iv), these vertices have degree $D+2$. Now vertices of degree D and $D+2$ should be joined, etc. Continuing this procedure we arrive at the stage in which the edges belonging to $E^d(d)$ and $E^{d*}(d)$ are added. Then the degrees of the two vertices in $R^d(d+1)$ and $R^{d*}(d+1)$ are still D. On the other hand, (iv) implies that the degrees of the vertices in $A^d(d)$ and $A^{d*}(d)$ are $D+d$, proving the existence of difference d. □

3. Constructing Difference 3

The construction in Section 2 has a disadvantage: namely, it contains many edges. When the difference in degrees becomes 3, then the graph is almost complete. In this section, only the difference 3 will be considered. Given n, the number of vertices, we try to construct difference 3 with a low number of edges. It will be somewhat more than half of the total number of possible edges, and it will turn out in the next section that the constructions are best possible.

Before stating the theorem we prove a lemma which is a slight extension of [2].

Lemma 5. *If n is even then the complete graph on n vertices can be decomposed into $n-1$ one-factors such that the first two one-factors form one cycle.*

Proof. Choose the vertices v_n and $v_1, \ldots, v_{n-1}$ to be the center and the vertices (in this order) of a regular $n-1$-gon, respectively. Let the first one-factor consist of the 'radius' (v_n, v_1) and all the edges and diagonals orthogonal to this 'radius': $(v_2, v_{n-1}), (v_3, v_{n-2}), \ldots, (v_{n/2}, v_{(n/2)+1})$. The other one-factors are the $n-2$ different rotations of the first one. It is easy to see that these one-factors are disjoint and their union contains all edges of the complete graph. The second one-factor is $\{(v_n, v_{n-1}), (v_1, v_{n-2}), (v_2, v_{n-3}), \ldots, (v_{(n/2)-1}, v_{n/2})\}$. The first two one-factors form the following cycle if $n/2$ is even:

$$v_n, v_1, v_{n-2}, v_3, v_{n-4}, \ldots, v_{(n/2)-3}, v_{(n/2)+2}, v_{(n/2)-1}, v_{n/2}, v_{(n/2)+1}, v_{(n/2)-2}, v_{(n/2)+3}, \ldots, v_4, v_{n-3}, v_2, v_{n-1}, v_n.$$

If $n/2$ is odd, then this cycle is slightly modified:

$$v_n, v_1, v_{n-2}, v_3, v_{n-4}, \ldots, v_{(n/2)-2}, v_{(n/2)+1}, v_{n/2}, v_{(n/2)-1}, v_{(n/2)+2}, v_{(n/2)-3}, v_{(n/2)+4}, \ldots, v_4, v_{n-3}, v_2, v_{n-1}, v_n.$$ □

If n is odd then there is no one-factor. Denote by M_{v_i} a one-factor on the set $V - v_i$ ($v_i \in V$).

LEMMA 6. *If n is odd then the complete graph on $V = \{v_1, \ldots, v_n\}$ can be decomposed into n matchings $M_{v_1}, \ldots, M_{v_n}$.*

PROOF. Use the idea of the previous proof. The vertices of a regular n-gon are considered, without a center. Let M_{v_1} consist of the edges $(v_2, v_n), (v_3, v_{n-1}), \ldots, (v_{(n+1)/2}, v_{(n+3)/2})$. The other one-factors are the rotations of this one. □

LEMMA 7. *If n is odd then the complete graph on $V = \{v_1, \ldots, v_n\}$ can be decomposed into $(n-1)/2$ two-factors, each containing exactly one odd cycle. One of the two-factors is a cycle of length n.*

PROOF. Use the notations of the proof of Lemma 6. M_{v_i}, $M_{v_{n-i+1}}$ and the edge (v_i, v_{n-i+1}) $(1 \leqslant i \leqslant (n-1)/2)$ form a two-factor. Consider a cycle of it, not containing the edge (v_i, v_{n-i+1}). The edges must alternate between M_{v_i} and $M_{v_{n-i+1}}$; therefore its length is even. Only the cycle containing (v_i, v_{n-i+1}) can (and must) be of odd length. It is easy to see that M_{v_1}, M_{v_n} and the edge (v_1, v_n) form a cycle of length n. □

THEOREM 2.

$$\begin{aligned}
&f(4k, \quad 4k^2 + 9k - 3) \geqslant 3 && \textit{for } 11 \leqslant k,\\
&f(4k+1, 4k^2 + 10k) \geqslant 3 && \textit{for } 9 \leqslant k,\\
&f(4k+2, 4k^2 + 11k + 3) \geqslant 3 && \textit{for } 4 \leqslant k,\\
&f(4k+3, 4k^2 + 14k + 6) \geqslant 3 && \textit{for } 10 \leqslant k.
\end{aligned}$$

PROOF. (1) $n = 4k + 2$. We will give a sequence of choices of edges in accordance with the Algorithm giving difference 3 after the $(4k^2 + 11k + 3)$rd edge. Partition the vertex set: $V = A \cup B$, where $A = \{a_1, \ldots, a_{2k}\}$ and $B = \{b_1, \ldots, b_{2k+2}\}$.

(1.1) The complete graphs on A and B can be partitioned into one-factors. Let these factors be $F_1, \ldots, F_{2k-1}$ and $H_1, \ldots, H_{2k+1}$, respectively. Suppose that the H's are ordered in such a way that the last two ones form one cycle. This can be done by Lemma 5. Let the Algorithm first choose the edges in F_1 one by one then the edges in H_1. After this all the degrees are 1. Continue with the edges of F_2 and then H_2, and so on. We finish this part of the construction with the edges in F_{2k-1} and H_{2k-1}. We obtained G_m, where $m = (2k-1)(2k+1)$. The graph is regular, $d_m(x) = 2k - 1$. It forms a complete graph in A, while the restriction of G_m in B is a complete graph minus a cycle of length $2k + 2$. Suppose that this missing cycle is $(b_1, b_2, \ldots, b_{2k+2}, b_1)$.

(1.2) Continue the Algorithm by choosing the edges of the following four one-factors one by one:

$$(b_1, b_2), (a_1, b_3), (a_2, b_4), (a_3, b_{2k+2}), (a_4, b_5), (a_5, b_6), \ldots, (a_{2k}, b_{2k+1});$$

$$(a_1, b_1), (a_2, b_2), (b_3, b_4), (a_3, b_5), (a_4, b_6), \ldots, (a_{2k}, b_{2k+2});$$

$$(a_1, b_2), (a_2, b_3), (a_3, b_1), (a_4, b_4), (b_5, b_6), (a_5, b_8), (a_6, b_9), \ldots,\\ (a_{2k-1}, b_{2k+2}), (a_{2k}, b_7);$$

$$(a_1, b_4), (a_2, b_1), (a_3, b_2), (a_4, b_3), (b_7, b_8), (a_5, b_9), (a_6, b_{10}), \ldots,\\ (a_{2k-2}, b_{2k+2}), (a_{2k-1}, b_5), (a_{2k}, b_6).$$

These one-factors are disjoint if $k \geq 4$. The graph $G_{(2k+3)(2k+1)}$ thus obtained is regular, again, with the uniform degree $2k+3$. Two more properties of this graph will be used in the later steps. First, it constains a complete bipartite graph between $\{a_1, a_2\}$ and $\{b_1, b_2, b_3, b_4\}$. The second property is that the new edges within B are chosen from the 'missing cycle' in such a way that there is still a 'missing one-factor'. (That is, the complement of $G_{(2k+3)(2k+1)}$ contains a complete one-factor in B.)

(1.3) Add the edges of the 'missing one-factor' in B one by one: $(b_2, b_3), (b_4, b_5), \ldots, (b_{2k}, b_{2k+1}), (b_{2k+2}, b_1)$. The new graph $G_{(2k+3)(2k+1)+k+1}$ is not regular. $d_{(2k+3)(2k+1)+k+1}(x) = 2k+3$ if $x \in A$ and $2k+4$ if $x \in B$. It is important that the graph is complete in A and between $\{a_1, a_2\}$ and $\{b_1, b_2, b_3, b_4\}$.

(1.4) Since all the vertices of degree $2k+3$ are adjacent to each other, we have to join a vertex of degree $2k+3$ and a vertex of degree $2k+4$. Add the following one-factor between $\{a_3, \ldots, a_{2k}\}$ and $\{b_5, \ldots, b_{2k+2}\}$:

$$(a_3, b_{2k+1}), (a_4, b_{2k+2}), (a_5, b_5), (a_6, b_6), \ldots, (a_{2k}, b_{2k}).$$

It is easy to see that this set of edges is disjoint to the one-factors given in (1.2) if $k \geq 4$. Now choose these edges one by one. They are new edges and the degrees of the end points are $2k+3$ and $2k+4$, respectively. The graph becomes $G_{(2k+3)(2k+1)+3k-1}$. The degrees in $\{a_1, a_2\}$ are $2k+3$, they are $2k+4$ in $\{a_3, \ldots, a_{2k}\}$ and $\{b_1, \ldots, b_4\}$ and, finally, they are $2k+5$ in $\{b_5, \ldots, b_{2k+2}\}$.

(1.5) Observe that the only elements of degree $2k+3$ (a_1 and a_2) are adjacent to each other and to all the vertices with degree $2k+4$. We have to join a vertex of degree $2k+3$ and a vertex of degree $2k+5$. Join a_1 and b_5. The new graph has $(2k+3)(2k+1)+3k = 4k^2+11k+3$ edges. Here $d_{4k^2+11k+3}(a_2) = 2k+3$ and $d_{4k^2+11k+3}(b_5) = 2k+6$; that is, the difference is really 3.

(2) $n = 4k$. Partition the vertex set: $V = A \cup B$, where $A = \{a_1, \ldots, a_{2k-1}\}$ and $B = \{b_1, \ldots, b_{2k+1}\}$.

(2.1) Use Lemma 6 for both A and B, and denote the one-factor of $A - \{a_i\}$ by M_i and the one-factor of $B - \{b_i\}$ by N_i. Suppose that the order of the vertices in B is chosen in such a way that $N_{2k+1} = \{b_1, b_2), (b_3, b_4), \ldots, (b_{2k-1}, b_{2k})\}$. Start the Algorithm with the edges of M_1. Then take (a_1, b_1) and the edges of N_1. Continue in the same way: $M_2, (a_2, b_2), N_2, M_3, (a_3, b_3), N_3, \ldots, M_{2k-1}, (a_{2k-1}, b_{2k-1}), N_{2k-1}$. The graph $G_{2k(2k-1)}$ thus obtained is regular of degree $2k-1$.

(2.2). Add the following three one-factors to $G_{2k(2k-1)}$:

$$(a_1, b_2), (a_2, b_3), (a_3, b_4), (a_4, b_1), (a_5, b_6), (a_6, b_7), (a_7, b_8), (a_8, b_9), (a_9, b_5),$$
$$(a_{10}, b_{13}), (a_{11}, b_{10}), (b_{11}, b_{12}), (a_{12}, b_{14}), (a_{13}, b_{15}), \ldots, (a_{2k-1}, b_{2k+1});$$

$$(a_1, b_3), (a_2, b_4), (a_3, b_1), (a_4, b_2), (a_5, b_7), (a_6, b_8), (a_7, b_9), (a_8, b_5), (a_9, b_6),$$
$$(a_{10}, b_{11}), (a_{11}, b_{12}), (a_{12}, b_{10}), (b_{13}, b_{14}), (a_{13}, b_{16}), (a_{14}, b_{17}), \ldots,$$
$$(a_{2k-2}, b_{2k+1}), (a_{2k-1}, b_{15});$$

$$(a_1, b_4), (a_2, b_1), (a_3, b_2), (a_4, b_3), (a_5, b_8), (a_6, b_9), (a_7, b_5), (a_8, b_6), (a_9, b_7),$$
$$(a_{10}, b_{12}), (a_{11}, b_{13}), (a_{12}, b_{11}), (a_{13}, b_{14}), (a_{14}, b_{10}), (b_{15}, b_{16}), (a_{15}, b_{19}),$$
$$(a_{16}, b_{20}), \ldots, (a_{2k-3}, b_{2k+1}), (a_{2k-2}, b_{17}), (a_{2k-1}, b_{18}).$$

We obtain the graph G_{4k^2+4k}, which is regular of degree $2k+2$. Observe that there is still a missing one-factor between $\{a_5, \ldots, a_9\}$ and $\{b_5, \ldots, b_9\}$.

(2.3) The edges of N_{2k} and then (a_{2k-1}, b_{2k}) come. In the present graph, G_{4k^2+5k+1} the degrees are as follows: $d_{4k^2+5k+1}(a_i) = 2k+2$ for $1 \leq i \leq 2k-2$, $d_{4k^2+5k+1}(a_{2k-1}) = 2k+3 = d_{4k^2+5k+1}(b_i)$ for all $1 \leq i \leq 2k+1$.

(2.4) The vertices of degree $2k+2$ are adjacent; the Algorithm forces us to join vertices of degree $2k+2$ and $2k+3$. Add the following one-factor between $A-\{a_1, a_{2k-1}\}$ and $B-\{b_1, b_2, b_3, b_4\}$:

$$(a_2, b_{10}), (a_3, b_{11}), (a_4, b_{12}), (a_5, b_9), (a_6, b_5), (a_7, b_6), (a_8, b_7), (a_9, b_8),$$
$$(a_{10}, b_{15}), (a_{11}, b_{16}), \ldots, (a_{2k-4}, b_{2k+1}), (a_{2k-3}, b_{13}), (a_{2k-2}, b_{14}).$$

Observe that this one-factor completes the complete bipartite graph between $\{a_5, \ldots, a_9\}$ and $\{b_5, \ldots, b_9\}$. The only vertex of degree $2k+2$, a_1, is adjacent to all the other a's and to b_1, b_2, b_3 and b_4. Finish this step by adding 3 more edges: (a_1, b_{10}), (b_1, b_2) and (b_3, b_4). The degrees at this moment are $d_{4k^2+7k+1}(a_i)=2k+3$ $(1 \le i \le 2k-1)$, $d_{4k^2+7k+1}(b_i)=2k+4$ $(1 \le i \le 2k+1,\ i \ne 10)$ and $d_{4k^2+7k+1}(b_{10})=2k+5$.

(2.5) Vertices of degree $2k+3$ and $2k+4$ should be joined. The following one-factor between $A-\{a_5, \ldots, a_8\}$ and $B-\{b_5, \ldots, b_{10}\}$ satisfies this condition:

$$(a_1, b_{11}), (a_2, b_{12}), (a_3, b_{13}), (a_4, b_{14}), (a_9, b_1), (a_{10}, b_2), (a_{11}, b_3), (a_{12}, b_4),$$
$$(a_{13}, b_{2k-1}), (a_{14}, b_{2k}), (a_{15}, b_{2k+1}), (a_{16}, b_{15}), (a_{17}, b_{16}), \ldots, (a_{2k-1}, b_{2k-2}).$$

Here $d_{4k^2+9k-4}(a_i)=2k+4$, except for $i=5, 6, 7, 8$ when it is $2k+3$, and $d_{4k^2+9k-4}(b_i)=2k+5$, except for $i=5, 6, 7, 8, 9$ when it is $2k+4$. The vertices of degree $2k+3$ are all adjacent to all vertices of degree $2k+4$: therefore, following the rule of the Algorithm, a vertex of degree $2k+3$ and a vertex of degree $2k+5$ should be joined, making the difference 3.

(3) $n=4k+3$. Partition the vertex set: $V=A \cup B$, where $A=\{a_1, \ldots, a_{2k}\}$ and $B=\{b_1, \ldots, b_{2k+3}\}$.

(3.1) Lemmas 5 and 7 will be used for A and B, respectively. Let $M_1, \ldots, M_{2k-1}$ be the one-factors in A. Similarly, let $R_1, \ldots, R_{k+1}$ be the two-factors in B, where the last one is a cycle of length $2k+3$. The symmetric 'half' of R_k completed with the only symmetric edge of R_k is denoted by R_k^*. The degrees of R_k^* are ones at each but one vertex where the degree is 2. Rename the vertices in such a way that R_{k+1} is the cycle $(b_1, \ldots, b_{2k+3}, b_1)$ and the degree of b_{2k+3} is 2 in R_k^*.

Start the Algorithm with the edges of M_1; then add the edges of R_1, following the rules of the Algorithm. Now take M_2 and then M_3. Continue in the same way: R_2, M_4, M_5, $R_3, \ldots, R_{k-1}$, M_{2k-2}, M_{2k-1}, R_k^*. The graph G_{4k^2+k-1} thus obtained is almost regular of degree $2k-1$. The only exception is $d_{4k^2+k-1}(b_{2k+3})=2k$.

(3.2) Roughly speaking, we add 4 one-factors to G_{4k^2+k-1} between A and B:

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots, (a_{2k-4}, b_{2k-4}), (b_{2k-3}, b_{2k-2}),$$
$$(a_{2k-3}, b_{2k-1}), (a_{2k-2}, b_{2k}), (a_{2k-1}, b_{2k+1}), (a_{2k}, b_{2k+2});$$

$$(a_1, b_2), (a_2, b_3), (a_3, b_4), (a_4, b_1), (a_5, b_6), (a_6, b_7), (a_7, b_8), (a_8, b_9), (a_9, b_5),$$
$$(a_{10}, b_{11}), (a_{11}, b_{12}), \ldots, (a_{2k-3}, b_{2k-2}), (a_{2k-2}, b_{10}),$$
$$(b_{2k}, b_{2k+1}), (a_{2k-1}, b_{2k+2}), (a_{2k}, b_{2k+3});$$

The degree of b_{2k-1} is smaller; we have to correct it with, say, (b_{2k-2}, b_{2k-1}).

$$(a_1, b_3), (a_2, b_4), (a_3, b_1), (a_4, b_2), (a_5, b_7), (a_6, b_8), (a_7, b_9), (a_8, b_5), (a_9, b_6),$$
$$(a_{10}, b_{2k-1}), (a_{11}, b_{10}), (a_{12}, b_{11}), \ldots, (a_{2k-2}, b_{2k-3}), (b_{2k+1}, b_{2k+2}),$$
$$(a_{2k-1}, b_{2k+3}), (a_{2k}, b_{2k});$$

$$(a_1, b_4), (a_2, b_1), (a_3, b_2), (a_4, b_3), (a_5, b_8), (a_6, b_9), (a_7, b_5), (a_8, b_6), (a_9, b_7),$$
$$(a_{10}, b_{2k-4}), (a_{11}, b_{2k-3}), (a_{12}, b_{10}), (a_{13}, b_{11}), \ldots, (a_{2k-3}, b_{2k-5}),$$
$$(b_{2k-1}, b_{2k}), (a_{2k-2}, b_{2k+3}), (a_{2k-1}, b_{2k-2}), (a_{2k}, b_{2k+1}).$$

Finish this part by adding the edge (b_{2k+2}, b_{2k+3}). We obtain the graph G_{4k^2+9k+5}, which is almost regular of degree $2k+3$. The only exception is $d_{4k^2+9k+5}(b_{2k+3}) = 2k+4$. Observe that there is still a missing one-factor between $\{a_5, \ldots, a_9\}$ and $\{b_5, \ldots, b_9\}$.

(3.3) The edges of $R_k - R_k^*$ come. In the present graph, $G_{4k^2+10k+6}$, the degrees are as follows: $d_{4k^2+10k+6}(a_i) = 2k+3$ for $1 \leq i \leq 2k$ and $d_{4k^2+10k+6}(b_i) = 2k+4$ for $1 \leq i \leq 2k+3$.

(3.4) The vertices of degree $2k+3$ are adjacent, the Algorithm forces us to join vertices of degree $2k+3$ and $2k+4$. Add the following one-factor between $A - \{a_1\}$ and $B - \{b_1, b_2, b_3, b_4\}$:

$$(a_2, b_{10}), (a_3, b_{11}), (a_4, b_{12}), (a_5, b_9), (a_6, b_5), (a_7, b_6),$$
$$(a_8, b_7), (a_9, b_8), (a_{10}, b_{13}), (a_{11}, b_{14}), \ldots, (a_{2k-5}, b_{2k-2}),$$
$$(a_{2k-4}, b_{2k+1}), (a_{2k-3}, b_{2k+3}), (a_{2k-2}, b_{2k+2}), (a_{2k-1}, b_{2k}), (a_{2k}, b_{2k-1}).$$

Observe that this one-factor completes the complete bipartite graph between $\{a_5, \ldots, a_9\}$ and $\{b_5, \ldots, b_9\}$. The only vertex of degree $2k+3$, a_1, is adjacent to all the other a's and to b_1, b_2, b_3 and b_4. Finish this step by adding 3 more edges: (a_1, b_{10}), (b_1, b_2) and (b_3, b_4). The degrees at this moment are $d_{4k^2+12k+8}(a_i) = 2k+4$ $(1 \leq i \leq 2k)$, $d_{4k^2+12k+8}(b_i) = 2k+5$ $(1 \leq i \leq 2k+3,\ i \neq 10)$ and $d_{4k^2+12k+8}(b_{10}) = 2k+6$.

(3.5) Vertices of degree $2k+4$ and $2k+5$ should be joined. The following one-factor between $A - \{a_5, a_6, a_7\}$ and $B - \{b_5, \ldots, b_{10}\}$ satisfies this condition:

$$(a_1, b_{11}), (a_2, b_{12}), (a_3, b_{13}), (a_4, b_{14}), (a_8, b_1), (a_9, b_2), (a_{10}, b_3), (a_{11}, b_4),$$
$$(a_{12}, b_{2k+3}), (a_{13}, b_{15}), (a_{14}, b_{16}), \ldots, (a_{2k-5}, b_{2k-3}),$$
$$(a_{2k-4}, b_{2k}), (a_{2k-3}, b_{2k+2}), (a_{2k-2}, b_{2k+1}), (a_{2k-1}\ b_{2k-1}), (a_{2k}, b_{2k-2}).$$

Here $d_{4k^2+14k+5}(a_i) = 2k+5$, except for $i = 5, 6, 7$ when it is $2k+4$, and $d_{4k^2+14k+5}(b_i) = 2k+6$, except for $i = 5, 6, 7, 8, 9$ when it is $2k+5$. The vertices of degree $2k+4$ are all adjacent to all vertices of degree $2k+5$; therefore, following the rule of the Algorithm, a vertex of degree $2k+4$ and a vertex of degree $2k+6$ should be joined, making the difference 3.

(4) $n = 4k+1$. This case is very similar to the previous one. $V = A \cup B$, where $A = \{a_1, \ldots, a_{2k-1}\}$ and $B = \{b_1, \ldots, b_{2k+2}\}$.

(4.1) Lemmas 5 and 7 will be used for B and A, respectively. Let $N_1, \ldots, N_{2k+1}$ be the one-factors in B. Similarly, let $S_1, \ldots, S_{k-1}$ be the two-factors in A. Rename the vertices in such a way that $N_{2k} \cup N_{2k+1}$ is the cycle $(b_1, b_2, \ldots, b_{2k+2}, b_1)$.

Start the Algorithm with the edges of N_1; then add the edges of S_1, following the rules of the Algorithm. Now take N_2 and then N_3. Continue in the same way: $S_2, N_4, N_5, S_3, \ldots, S_{k-1}, N_{2k-2}$. The graph G_{4k^2-3k-1} thus obtained is regular of degree $2k-2$.

(4.2) Roughly speaking, we add 4 one-factors to G_{4k^2-3k-1} between A and B:

$$(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots, (a_{2k-1}, b_{2k-1}), (b_{2k}, b_{2k+1}), (b_{2k+1}, b_{2k+2});$$

$$(a_1, b_2), (a_2, b_3), (a_3, b_4), (a_4, b_1), (a_5, b_6), (a_6, b_7), (a_7, b_8), (a_8, b_9), (a_9, b_5),$$
$$(a_{10}, b_{11}), (a_{11}, b_{12}), \ldots, (a_{2k-3}, b_{2k-2}), (a_{2k-2}, b_{10}), (b_{2k-1}, b_{2k}), (a_{2k-1}, b_{2k+2});$$

$$(a_1, b_3), (a_2, b_4), (a_3, b_1), (a_4, b_2), (a_5, b_7), (a_6, b_8), (a_7, b_9), (a_8, b_5), (a_9, b_6),$$
$$(a_{10}, b_{2k+2}), (a_{11}, b_{10}), (a_{12}, b_{11}), \ldots, (a_{2k-3}, b_{2k-4}),$$
$$(b_{2k-3}, b_{2k-2}), (a_{2k-2}, b_{2k}), (a_{2k-1}, b_{2k+1}), (b_{2k-2}, b_{2k-1});$$

(a_1, b_4), (a_2, b_1), (a_3, b_2), (a_4, b_3) (a_5, b_8), (a_6, b_9), (a_7, b_5), (a_8, b_6), (a_9, b_7),
(a_{10}, b_{2k-1}), (a_{11}, b_{2k+1}), (a_{12}, b_{10}), (a_{13}, b_{11}), ..., (a_{2k-3}, b_{2k-5}),
(b_{2k-4}, b_{2k-3}), (a_{2k-2}, b_{2k+2}), (a_{2k-1}, b_{2k}).

We obtained the graph G_{4k^2+5k+1}, which is regular of degree $2k+2$.

(4.3) The edges of N_{2k-1} come. In the resulting graph G_{4k^2+6k+2}, the degrees are as follows: $d_{4k^2+6k+2}(a_i)=2k+2$ for $1\leqslant i\leqslant 2k-1$ and $d_{4k^2+6k+2}(b_i)=2k+3$ for $1\leqslant i\leqslant 2k+2$.

(4.4) The vertices of degree $2k+2$ are adjacent; so the Algorithm forces us to join vertices of degree $2k+2$ and $2k+3$. Add the following one-factor between $A-\{a_1\}$ and $B-\{b_1, b_2, b_3, b_4\}$:

(a_2, b_{10}), (a_3, b_{11}), (a_4, b_{12}), (a_5, b_9), (a_6, b_5), (a_7, b_6), (a_8, b_7),
(a_9, b_8), (a_{10}, b_{13}), (a_{11}, b_{14}), ..., (a_{2k-6}, b_{2k-3}), (a_{2k-5}, b_{2k}),
(a_{2k-4}, b_{2k+1}), (a_{2k-3}, b_{2k+2}), (a_{2k-2}, b_{2k-1}), (a_{2k-1}, b_{2k-2}).

This one-factor completes the complete bipartite graph between $\{a_5, \ldots, a_9\}$ and $\{b_5, \ldots, b_9\}$, again. The only vertex of degree $2k+2$, a_1, is adjacent to all the other a's and to b_1, b_2, b_3 and b_4. Finish this step by adding 3 more edges: (a_1, b_{10}), (b_1, b_2) and (b_3, b_4). The degrees at this moment are $d_{4k^2+8k+3}(a_i)=2k+3$ $(1\leqslant i\leqslant 2k-1)$, $d_{4k^2+8k+3}(b_i)=2k+4$ $(1\leqslant i\leqslant 2k+2,\ i\neq 10)$ and $d_{4k^2+8k+3}(b_{10})=2k+5$.

(4.5) Vertices of degree $2k+3$ and $2k+4$ should be joined. The following one-factor between $A-\{a_5, a_6, a_7\}$ and $B-\{b_5, \ldots, b_{10}\}$ satisfies this condition:

(a_1, b_{11}), (a_2, b_{12}), (a_3, b_{13}), (a_4, b_{14}), (a_8, b_1), (a_9, b_2), (a_{10}, b_3), (a_{11}, b_4),
(a_{12}, b_{2k+2}), (a_{13}, b_{15}), (a_{14}, b_{16}), ..., (a_{2k-6}, b_{2k-4}),
(a_{2k-5}, b_{2k-2}), (a_{2k-4}, b_{2k}), (a_{2k-3}, b_{2k-1}), (a_{2k-2}, b_{2k+1}), (a_{2k-1}, b_{2k-3}).

Here $d_{4k^2+10k-1}(a_i)=2k+4$, except for $i=5, 6, 7$ when it is $2k+3$, and $d_{4k^2+10k-1}(b_i)=2k+5$, except for $i=5, 6, 7, 8, 9$ when it is $2k+4$. The vertices of degrees $2k+3$ are all adjacent to all vertices of degree $2k+4$; therefore, following the rule of the Algorithm, a vertex of degree $2k+3$ and a vertex of degree $2k+5$ should be joined, making the difference 3. □

4. Difference 3 Can Occur Only When the Degrees are at Least $(n+4)/2$

In this section we prove that our constructions are the best possible. In the proof we need the following lemmas.

Lemma 8. *Suppose that the graph $G(V, E)$ possesses the following properties for a fixed positive integer d. The vertex set has a partition $V=V_1\cup V_2$ such that V_1 induces a complete graph and the degrees satisfy the following inequalities: $d(x)\leqslant d$ for each $x\in V_1$, $d(x)\geqslant d$ for each $x\in V_2$, with strict inequality for at least one $x\in V_2$. Then $|V_1|<|V_2|$.*

Proof. Let us count the number e^* of edges $(x, y)\in E$ such that $x\in V_1$, $y\in V_2$ in two different ways, considering the degrees in V_1 and V_2, respectively:

$$\sum_{x\in V_1}|\{y: y\in V_2, (x, y)\in E\}|=e^*=\sum_{y\in V_2}|\{x: x\in V_1, (x, y)\in E\}|. \tag{6}$$

On the left-hand side we have

$$|\{y: y\in V_2, (x, y)\in E\}|=d(x)-|\{y: y\in V_1, (x, y)\in E\}|$$

and hence

$$\sum_{x \in V_1} |\{y : y \in V_2, (x, y) \in E\}| = \sum_{x \in V_1} d(x) - \sum_{x \in V_1} |\{y : y \in V_1, (x, y) \in E\}|.$$

Let us use the assumptions of the lemma in the latter formula. The following lower estimate is obtained:

$$e^* \leq |V_1|\, d - |V_1|\,(|V_1| - 1). \tag{7}$$

Similarly, the right-hand side of (6) implies

$$e^* > |V_2|\, d - |V_2|\,(|V_2| - 1). \tag{8}$$

The conclusion of all these calculations is obtained from (7) and (8):

$$|V_1|\,(d - |V_1| + 1) > |V_2|\,(d - |V_2| + 1). \tag{9}$$

There is a vertex of degree $\geq dH$ and this is at most $|V| - 1$. This gives rise to our other useful inequality:

$$|V_1| + |V_2| > d + 1. \tag{10}$$

The function $x(d + 1 - x)$ is a parabola with maximum at $(d + 1)/2$. (9) implies that $|V_1|$ is closer to $(d + 1)/2$ than $|V_2|$. On the other hand, (10) implies that the smaller of $|V_1|$ and $|V_2|$ is closer to (the middle) $(d + 1)/2$. Hence we have $|V_1| < |V_2|$. □

This statement would be sufficient to prove a somewhat weaker theorem: for our exact result a stronger variant is needed.

LEMMA 9. *Suppose that the graph $G = (V, E)$ possesses the following properties with a fixed integer d:*
(i) *$d(x) = d$ for all $x \in V_1$;*
(ii) *$d(x) \geq d + 1$ for all $x \in V_2$;*
(iii) *V_1 spans a complete graph;*
(iv) *there is at least one edge between V_1 and V_2.*
Then $|V_1| + 2 \leq |V_2|$ holds.

PROOF. In view of Lemma 8, we only have to exclude the case $|V_2| = |V_1| + 1$. Now (7), (8) (with $d + 1$ and $\geq$ instead of d and $>$ in (8)) and $|V_2| = |V_1| + 1$ give $|V_1| \geq d + 1$, which contradicts (i), (iii) and (iv). □

(This proof, shorter than our original one, is due to one of the referees.)

THEOREM 3.

$$\begin{aligned} f(4k,\quad & 4k^2 + 9k - 4) \leq 2,\\ f(4k + 1,\ & 4k^2 + 10k - 1) \leq 2,\\ f(4k + 2,\ & 4k^2 + 11k + 2) \leq 2,\\ f(4k + 3,\ & 4k^2 + 14k + 5) \leq 2. \end{aligned}$$

PROOF. Let m be the minimal value satisfying $f(n, m) \geq 3$. Suppose that the sequence $G_0, G_1, \ldots, G_m$ verifies it; that is, these graphs are constructed in accordance with the Algorithm and there are two vertices in G_m the degrees of which differ by at least 3. The difference cannot be larger since the maximal difference in G_{m-1} is at most 2 and the addition of one edge may increase the difference by at most one.

Denote the possible degrees in G_m by d, $d+1$, $d+2$ and $d+3$. $G_0, G_1, \ldots, G_{m-1}$ cannot contain vertices with difference 3 by the definition of m.

Suppose that the degree $d+2$ appears first in G_{v+1}: that is, all degrees in G_v are smaller than $d+2$ but G_{v+1} contains a vertex of degree $d+2$. G_v must contain a vertex of degree $d+1$, so the possible degrees are $d-1$, d and $d+1$. We distinguish cases according to the number of vertices of degree $d-1$ in G_v.

(α) There are two different vertices x and y such that their degrees in G_v are $d_v(x)=d_v(y)=d-1$. As G_{v+1} contains a vertex of degree $d+2$, one of the end points of the new edge $E_{v+1}-E_v$ should have degree $d+1$ in G_v, therefore it cannot be (x, y). The consequence is that one of x and y has degree $d-1$ in G_{v+1}. Thus, there are vertices in G_{v+1} with difference 3 in degrees. This is a contradiction, since $v+1<m$. This case is impossible.

(β) All degrees in G_v are at least d. We introduce the following notation. $V_1=\{x: d_v(x)=d\}$ and $V_2=\{x: d_v(x)=d+1\}$. As G_{v+1} contains a vertex of degree $d+2$, one of the end points of the new edge $E_{v+1}-E_v$ should have degree $d+1$ in G_v; therefore it cannot be totally in V_1. By the Algorithm all vertices in V_1 must be connected in G_v. If (iv) holds then Lemma 9 can be applied for G_v:

$$|V_1|+2\leq|V_2|. \tag{12}$$

On the other hand, if (iv) is not valid then $|V_1|$ must be equal to $d+1$ and $|V_2|$ must be at least $d+2$. If it is more, then (12) holds. In the only remaining case $|V_1|=d+1$, $|V_2|=d+2$ and consequently

$$|V| \text{ is odd. } G_v \text{ is a union of two complete graphs.} \tag{13}$$

The Algorithm joins now vertices in V_1 to vertices in V_2 for a while: that is, let $E_{v+1}-E_v, E_{v+2}-E_{v+1}, \ldots, E_{v+r}-E_{v+r-1}$ be edges joining a vertex in V_1 to a vertex in V_2. (The end points of these edges are distinct.) But suppose that $E_{v+r}-E_{v+r-1}$ is the last such edge. The reason for that can only be that all vertices having degree d in G_{v+r} (there must be one, otherwise degree d disappears too early) are adjacent to all vertices having degree $d+1$. Hence these vertices in V_1 are adjacent to all other elements of V_1 and with $|V_2|-r$ elements of V_2. This implies the inequality

$$d\geq|V_1|-1+|V_2|-r. \tag{14}$$

Under the assumptions of the present case, G_{m-1} may contain vertices of degrees d, $d+1$ and $d+2$. There must be at least one of degree d since G_m also has one. Moreover, there are at least two vertices of degree d in G_{m-1}. Otherwise, if x were the only such vertex, the Algorithm would require x to be joined to some other vertex in the mth step. However, this is a contradiction since there is a vertex of degree d in G_m. Also, since $v+r\leq m-1$, there are at least two vertices of degree d in G_{v+r}. This implies

$$|V_1|-r\geq 2. \tag{15}$$

The sum of the degrees in G_{v+r} is

$$\begin{aligned} d\,|V|+|V_2|+2r &\geq (|V_1|-r+|V_2|-1)n+|V_2|+2r \\ &= (|V_1|-r)(n-2)+(n-1)\,|V_2|+n\geq 3n-4+(n-1)\,|V_2|, \end{aligned}$$

where the inequalities (14) and (15) were used. The number of edges in G_{v+r} is at least half of the above number. By $v+r\leq m-1$ we have

$$\tfrac{1}{2}(3n-4+(n-1)\,|V_2|)\leq|E_{m-1}|. \tag{16}$$

($\beta 1$) $n=4k+2$. (13) cannot; therefore (12) must hold. This implies $|V_2|\geq 2k+2$.

The substitution of it into (16) yields $4k^2+11k+2 \leqslant |E_{m-1}|$, proving the theorem in this case.

($\beta 2$) $n = 4k$. As in the previous case, (12) implies that $|V_2| \geqslant 2k+1$. However, the sum of the degrees in G_v is $d\,|V_1| + (d+1)\,|V_2|$ and this is odd when $|V_1| = 2k-1$ and $|V_2| = 2k+1$. This contradiction proves that $|V_2| \geqslant 2k+2$. We obtain $4k^2+9k-3 \leqslant |E_{m-1}|$, which is better than we need.

($\beta 3$) $n = 4k+3$. In case of (13) nothing can stop the procedure of taking edges between V_1 and V_2 until all vertices become of degree at least $d+1$. This contradicts (15). Therefore (12) is true and $|V_2| \geqslant 2k+3$ follows.

In case of equality, $|V_1| = 2k$ and $|V_2| = 2k+3$ hold. By (14) and (15) we have $d \geqslant |V_2| + 1 = 2k+4$. Two cases are distinguished. If $d = 2k+4$ then the sum of the degrees in G_v is $d\,|V_1| + (d+1)\,|V_2| = (2k+4)2k + (2k+5)(2k+3)$, an odd number. This contradiction shows that the other case might be considered: $d \geqslant 2k+5$. The sum of the degrees in G_{v+r} is at least $d\,|V| + |V_2| + 2r \geqslant (2k+5)(4k+3) + 2k+3$; that is, $|E_{m-1}| \geqslant 4k^2+14k+9$, which is more than what is needed.

Thus $|V_2| \geqslant 2k+4$ can be supposed. (16) implies $|E_{m-1}| \geqslant 4k^2+16k+\frac{13}{2}$, which is larger than the desired lower estimate.

($\beta 4$) $n = 4k+1$. As in the previous case, one can assume (12) and thus obtain $|V_2| \geqslant 2k+2$.

In the case of equality, $|V_1| = 2k-1$ and $|V_2| = 2k+2$ hold. By (14) and (15) we have $d \geqslant |V_2| + 1 = 2k+3$. Two cases are distinguished. If $d = 2k+3$ then the sum of the degrees in G_v is $d\,|V_1| + (d+1)\,|V_2| = (2k+3)(2k-1) + (2k+4)(2k+2)$ an odd number. This contradiction shows that the other case might be considered: $d \geqslant 2k+4$. The sum of the degrees in G_{v+r} is at least $d\,|V| + |V_2| + 2r \geqslant (2k+4)(4k+1) + 2k+2$; that is, $|E_{m-1}| \geqslant 4k^2+10k+3$, which is more than what is needed.

Thus $|V_2| \geqslant 2k+3$ can be supposed. (16) implies $|E_{m-1}| \geqslant 4k^2+12k-\frac{1}{2}$, which is again larger than the desired lower estimate.

(γ) There is exactly one vertex of degree $d-1$ in G_v. The following lemma is needed in this case.

LEMMA 10. *Let δ be the smallest integer such that, for some G_i:*
(i) *there are vertices of degrees $\delta-1$, δ and $\delta+1$ in G_i, only;*
(ii) *the number of vertices of degree $\delta-1$ is exactly one, the number of vertices of degree $\delta+1$ is at least one; and, finally,*
(iii) *the vertex of degree $\delta-1$ is adjacent to all vertices of degree δ.*
Further, let $u < i$ be such that G_u is the last graph not containing vertices of degree $\delta+1$.
Then $(n+2)/2 \leqslant |U_2| = |\{x: d_u(x) = \delta\}|$ and $|U_2| + 1 \leqslant \delta \leqslant d$.

PROOF. $\delta \leqslant d$ is obvious. G_u cannot have vertices of degree $\delta-3$ or less, since the difference cannot be 3. Suppose that there is a vertex of degree $\delta-2$. $E_{u+1} - E_u$ must join vertices of degrees $\delta-2$ and δ. Then G_{u+1} contains a vertex of degree $\delta+1$; therefore it cannot contain one of degree $\delta-2$. Hence G_u has exactly one vertex of degree $\delta-2$. Moreover, (iii) also holds, contradicting the definition of δ. We proved that G_u has only two different degrees. Let $U_1 = \{x: d_u(x) = \delta-1\}$ and $U_2 = \{x: d_u(x) = \delta\}$. U_1 spans a complete graph so Lemma 8 implies $|U_1| + 1 \leqslant |U_2|$. Moreover, we claim that (iv) of Lemma 9 holds so Lemma 9 can be applied:

$$|U_1| + 2 \leqslant |U_2|. \tag{17}$$

Really, if there were no edge between U_1 and U_2, then the forthcoming edges, $E_{u+1} - E_u$, $E_{u+2} - E_{u+1}, \ldots, E_{u+|u_1|-1} - E_{u+|u_1|-2}$, would be between U_1 and U_2, creating G_i. By the definition of G_i (that is, of δ), the only remaining vertex of degree

$\delta-1$ has to be adjacent to the vertices of degree δ in U_2 (in G_i). These edges existed also in G_u. (iv) and (17) are proved. $(n+2)/2 \leqslant |U_2|$ easily follows. Finally, the vertex of degree $\delta-1$ (in G_i) is adjacent to all other $|U_1|-1$ vertices of U_1 and some $|U_2|-(|U_1|-1)$ vertices in U_2. This proves $|U_2| \leqslant \delta-1$ and the lemma. □

(γ1) $n=4k+2$. Lemma 10 implies $2k+3 \leqslant d$. Suppose first that we have a strict inequality: $2k+4 \leqslant d$. After adding the edge $E_{v+1}-E_v$, the graph G_{v+1} contains vertices of degrees d, $d+1$ and $d+2$. The forthcoming edges join vertices of degree d until the set $V_1=\{x: d_j(x)=d\}$ spans a complete graph. It is obvious that $j \leqslant m-1$. By Lemma 8 we know that the set $V_2=\{x: d_j(x) \geqslant d+1\}$ satisfies $2k+2 \leqslant |V_2|$. Counting the sum of the degrees in G_j gives the following lower estimate:

$$\tfrac{1}{2}(d\,|V|+|V_2|) \leqslant |E_{m-1}|, \tag{18}$$

yielding $\frac{1}{2}((2k+4)(4k+2)+(2k+2)) \leqslant |E_{m-1}|$, which is stronger than what we need.

Suppose now that $d=2k+3$. Then $\delta=d$ also holds; consequently $i=v$. Again, by Lemma 10, $|U_2|=2k+2$ and $|U_1|=2k$ holds in G_u. The edges $E_{u+1}-E_u$, $E_{u+2}-E_{u+1}, \ldots, E_{u+|U_1|-1}-E_{u+|U_1|-2}$ must join vertices of degrees $d-1$ and d. We obtained the graph G_v. It contains $2k+2$ vertices of degree d. The next edge $E_{u+|U_1|}-E_{u+|U_1|-1}$ (or $E_{v+1}-E_v$) joins the only vertex of degree $d-1$ and a vertex of degree $d+1$. G_{v+1} now contains $2k+3$ vertices of degree d. By the Algorithm, we have to join vertices of degree d. Adding such an edge preserves the property that the number of vertices of degree d is odd. This should be done until the set of vertices of degree d spans a complete graph. Let G_w be this graph. Introduce the notation $W_1=\{x: d_w(x)=d\}$ and $W_2=\{x: d_w(x)>d\}$, where $|W_1|$ is odd. By Lemma 8 we have $2k+2 \leqslant |W_2|$, but the equality is impossible since $|W_1|$ is odd. Therefore,

$$|W_1| \leqslant 2k-1. \tag{19}$$

The forthcoming edges $E_{w+1}-E_w$, $E_{w+2}-E_{w+1}, \ldots, E_{w+r}-E_{w+r-1}$ join vertices of degrees d and $d+1$. The integer r is chosen in such a way that $w+r=m-1$; that is, all pairs of vertices of degrees d and $d+1$ are adjacent in $G_{w+r}=G_{m-1}$. The number of vertices of degrees d, $d+1$ and $d+2$ are $|W_1|-r$, $|W_2|-1$ and $r+1$, respectively. Any given vertex of degree d in G_{m-1} is adjacent to all other vertices of degree d and all vertices of degree $d+1$. This implies the inequality $|W_1|-r-1+|W_2|-1 \leqslant d$. Since $|W_1|+|W_2|=4k+2$ and $d=2k+3$, the former inequality results in

$$2k-3 \leqslant r. \tag{20}$$

The sum of the degrees in G_{m-1} is

$$(2k+3)(|W_1|-r)+(2k+4)(|W_2|-1)+(2k+5)(r+1)=(2k+3)(|W_1|-r)$$
$$+(2k+4)(4k+1-|W_1|)+(2k+5)(r+1)=(2k+4)(4k+1)-|W_1|+2r+2k+5.$$

Using (19) and (20) we obtain a lower estimate:

$$|E_{m-1}| \geqslant \tfrac{1}{2}((2k+4)(4k+1)-(2k-1)+2(2k-3)+2k+5)=4k^2+11k+2.$$

(γ2) $n=4k$. Lemma 10 implies that $2k+2 \leqslant d$, only. However, equality here would imply $\delta=d=2k+2$, $i=v$, $|U_1|=2k-1$ and $|U_2|=2k+1$. Count the sum of the degrees in G_u: $(2k+1)(2k-1)+(2k+2)(2k+1)$. This is odd; the contradiction proves that $2k+3 \leqslant d$.

Two cases are distinguished: $2k+3<d$ and $2k+3=d$. In the former case add the edge $E_{v+1}-E_v$; the graph G_{v+1} contains vertices of degrees d, $d+1$ and $d+2$. The forthcoming edges join vertices of degree d until the set $V_1=\{x: d_j(x)=d\}$ spans a complete graph. It is obvious that $j \leqslant m-1$. By Lemma 8 we know that the set

$V_2 = \{x: d_j(x) \geq d+1\}$ satisfies $2k+1 \leq |V_2|$. (18) yields $|E_{m-1}| \geq \frac{1}{2}((2k+4)4k + (2k+1)) = 4k^2 + 9k + \frac{1}{2}$, better than what is needed.

We may suppose that $d = 2k+3$. The edge $E_{v+1} - E_v$ joins the only vertex of degree $d-1$ and a vertex of degree $d+1$. By the Algorithm, we have to join vertices of degree d. This should be done until the set of vertices of degree d spans a complete graph. Let G_w be this graph. Use the notations W_1 and W_2 of the previous (γ1) case. By Lemma 8 we have (19).

The forthcoming edges $E_{w+1} - E_w$, $E_{w+2} - E_{w+1}, \ldots, E_{w+r} - E_{w+r-1}$ join vertices of degrees d and $d+1$. The integer r is chosen in such a way that $w + r = m-1$; that is, all pairs of vertices of degrees d and $d+1$ are adjacent in $G_{w+r} = G_{m-1}$. The number of vertices of degrees d, $d+1$ and $d+2$ are $|W_1| - r$, $|W_2| - 1$ and $r+1$, respectively. Any given vertex of degree d in G_{m-1} is adjacent to all other vertices of degree d and all vertices of degree $d+1$. This implies the inequality $|W_1| - r - 1 + |W_2| - 1 \leq d$. Since $|W_1| + |W_2| = 4k$ and $d = 2k+3$, the former inequality results in

$$2k - 5 \leq r. \tag{21}$$

The sum of the degrees in G_{m-1} is

$$(2k+3)(|W_1| - r) + (2k+4)(|W_2| - 1) + (2k+5)(r+1) = (2k+3)(|W_1| - r)$$
$$+ (2k+4)(4k - 1 - |W_1|) + (2k+5)(r+1) = (2k+4)(4k-1) - |W_1| + 2r + 2k + 5.$$

Use (19) and (21) to obtain a lower estimate:

$$|E_{m-1}| \geq \tfrac{1}{2}((2k+4)(4k-1) - (2k-1) + 2(2k-5) + 2k + 5) = 4k^2 + 9k - 4,$$

as desired.

(γ3) $n = 4k+3$. Lemma 10 imples that $2k+4 \leq d$. Suppose first that we have a strict inequality: $2k+5 \leq d$. After adding the edge $E_{v+1} - E_v$, the graph G_{v+1} contains vertices of degrees d, $d+1$ and $d+2$. The forthcoming edges join vertices of degree d until the set $V_1 = \{x: d_j(x) = d\}$ spans a complete graph. It is obvious that $j \leq m-1$. By Lemma 8 we know that the set $V_2 = \{x: d_j(x) \geq d+1\}$ satisfies $2k+2 \leq |V_2|$. The lower estimate (18) gives

$$4k^2 + 14k + \tfrac{17}{2} \leq |E_{m-1}|,$$

more than our need.

We may suppose that $d = 2k+4$. The edge $E_{v+1} - E_v$ joins the only vertex of degree $d-1$ and a vertex of degree $d+1$. By the Algorithm, we have to join vertices of degree d. This should be done until the set of vertices of degree d spans a complete graph. Let G_w be this graph. Use the notations W_1 and W_2 of the previous (γ1) and (γ2) cases. By Lemma 9 (the existence of $E_{v+1} - E_v$ ensures (iv)) we have

$$|W_1| \leq 2k. \tag{22}$$

The forthcoming edges $E_{w+1} - E_w$, $E_{w+2} - E_{w+1}, \ldots, E_{w+r} - E_{w+r-1}$ join vertices of degrees d and $d+1$. The integer r is chosen in such a way that $w + r = m-1$; that is, all pairs of vertices of degrees d and $d+1$ are adjacent in $G_{w+r} = G_{m-1}$. The number of vertices of degrees d, $d+1$ and $d+2$ are $|W_1| - r$, $|W_2| - 1$ and $r+1$, respectively. Any given vertex of degree d in G_{m-1} is adjacent to all other vertices of degree d and all vertices of degree $d+1$. This implies the inequality $|W_1| - r - 1 + |W_2| - 1 \leq d$. Since $|W_1| + |W_2| = 4k+3$ and $d = 2k+4$, the former inequality results in

$$2k - 3 \leq r. \tag{23}$$

The sum of the degrees in G_{m-1} is

$$\begin{aligned}(2k+4)(|W_1|-r)+(2k+5)(|W_2|-1)+(2k+6)(r+1)\\ =(2k+4)(|W_1|-r)+(2k+5)(4k+2-|W_1|)+(2k+6)(r+1)\\ =(2k+5)(4k+2)-|W_1|+2r+2k+6.\end{aligned}$$

Use (22) and (23) to obtain a lower estimate:

$$|E_{m-1}|\geq \tfrac{1}{2}((2k+5)(4k+2)-2k+2(2k-3)+2k+6)=4k^2+14k+5,$$

as desired.

(γ4) $n=4k+1$. This case can be settled in the same way as the previous one. Only the calculations are different. □

5. Further Results and Problems

1. Let $g(n, d)$ denote the minimum m such that $f(n, m)\geq d$. Theorems 2 and 3 have determined $g(n, 3)$. In a forthcoming paper we will publish the following results:

$$\lim_{n\to\infty} g(n, 4)\Big/\binom{n}{2}=g(n, 5)\Big/\binom{n}{2}=\tfrac{2}{3}.$$

We conjecture that

$$\lim_{n\to\infty} g(n, d)\Big/\binom{n}{2}=\begin{cases}(3k-1)/3k, & \text{for } d=4k,\\ (3k-1)/3k, & \text{for } d=4k+1,\\ 3k/(3k+1), & \text{for } 4k+2,\\ (3k+1)/(3k+2), & \text{for } d=4k+3.\end{cases}$$

We are able to prove the upper estimates by appropriate constructions.

2. Probably the most natural question here is to determine $h(d)$ as the minimum n such that $f(n, m)\geq d$ holds for some m. The solution of this problem is urged by Paul Erdős. The construction in Section 2 gives only

$$h(d)\leq 2\frac{(2d+1)(d+1)^d-1}{d}.$$

For $d=3$: $h(3)\leq 298$. However, the construction in Section 3 shows that $h(3)\leq 18$.

3. There are other ways to use the degrees in an 'on line' algorithm to obtain nearly regular graphs. For instance, one could join the pairs (of degrees) (4, 4) before (3, 5). It seems that it does not make the graph more regular. The same is very likely to be true if only two degrees are used to determine the new adjacent pair. However, this might be changed if 3 or more degrees are used.

4. Suppose that, in an ambiguous case of the Algorithm, not the worst choice is made, but all possibilities are chosen randomly with equal probabilities. We conjecture that the probability of the event that the difference in degrees is ≥ 3 tends to 0 with $n\to\infty$. Computer experience due to K. Balińska supports this belief. W. Imrich thinks, however, that (for sufficiently large n) the difference in degrees should be at least 2 with high probability.

5. Which are the feasible graphs? Are all regular graphs feasible? Obviously not, as F. Hoffman and K. Kriegel pointed out; a feasible graph must contain a one-factor (one-factor neglecting one vertex) if n is even (odd). If a graph can be decomposed into one-factors then it is clearly feasible. The converse is not true. Together with M. Aigner, we found that the famous Petersen graph is feasible, and it is well known that

it cannot be decomposed into one-factors. The feasibility is shown by the following sequence of edges: $(1,6)$, $(2,7)$, $(3,8)$, $(4,9)$, $(5,10)$, $(2,3)$, $(4,5)$, $(7,9)$, $(8,10)$, $(1,5)$, $(6,8)$, $(1,2)$, $(3,4)$, $(6,9)$ and $(7,10)$. Thus feasibility is a generalization of the one-factor-decomposition. Is it true that all regular graphs containing a one-factor are feasible?

6. S. Poljak asked if the problem of feasibility is NP-complete.

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