Optimization of the Reliability Polynomial in Presence of Mediocre Elements

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Some connections between extremal set theory and the optimization of the reliability polynomial are shown. Then the concept of the reliability polynomial is generalized for the case when the elements can have three different states: good, mediocre and bad. The state of the device can be described by a 0,1,2 sequence. Such a state is called operative if the device operates when its elements are in the states described by the sequence. The maximum of the generalized reliability polynomial is studied under the condition that the set of operative states forms an antichain.

1. INTRODUCTION

Suppose that a complicated device is given, built from many components. Each component can go wrong with probability p ($0 < p < 1$). However, the breakdown of one component does not necessarily cause the malfunction of the whole device. The roles of the components in the operation of the device can be very different. Let X be the set of the n components. Call a subset $A \subset X$ an *operative set* if the device operates correctly whenever the elements of $X - A$ work but the elements of A do not work. The family of operative sets is the *operative family*. It will be denoted by $\mathcal{A}$. A typical example is a network of electrical transmission lines forming a graph. The system stops its operation if the graph becomes disconnected. In this example, the edges of the graph form the set X. The operative family consists of those sets of edges whose deletion does not disconnect the graph (i.e., the sets not containing a cut of the graph).

In the above example and in most practical cases, $\mathcal{A}$ has the following property:

$$A \in \mathcal{A} \quad \text{and} \quad B \subset A \quad \text{imply} \quad B \in \mathcal{A}.$$

Such families are called *ideals*. However, there are practical situations where $\mathcal{A}$ is not an ideal. Let us see now only a nonserious example. Let the device be a country. If one of the ministers goes crazy, the country stops its proper operation. However, if his phone goes wrong simultaneously, then nobody notices anything; that is, the two-element set {minister, its phone} is in $\mathcal{A}$, while the one-element set consisting only of the minister does not belong to $\mathcal{A}$.

If the elements fail with probability p, independently, then the probability of the event that the whole device is operating correctly is

$$\sum_{A \in \mathcal{A}} p^{|A|}(1-p)^{n-|A|}. \tag{1}$$

This is called the *reliability polynomial*. If p and $\mathcal{A}$ are fully known, then the reliability polynomial can be easily computed. In practical situations, however, $\mathcal{A}$

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is not known, only some of its properties or parameters. In such a situation, we can only give estimates on the reliability polynomial. The best estimates are the minimums and the maximums. Therefore, we try to study the minimum and maximum of (1) where n and p are fixed and $\mathcal{A}$ can be chosen under some given conditions.

Let $f_i(\mathcal{A})$ denote the number of i-element members of $\mathcal{A}$. Furthermore, use the notation $c(x) = p^x(1 - p)^{n-x}$. Then, (1) can be rewritten in the form

$$\sum_{i=0}^{n} f_i(\mathcal{A})c(i). \tag{2}$$

The conditions on $\mathcal{A}$ allow certain vectors $[f_0(\mathcal{A}), f_1(\mathcal{A}), \ldots, f_n(\mathcal{A})]$, only. The extreme values of (2) should be found for this set $\mathcal{V}$ of vectors. As (2) is a linear function of the vectors, it is enough to find the extreme points of the convex hull of $\mathcal{V}$. One of them will give the extreme value. These extreme points are determined for some classes of families $\mathcal{A}$ (i.e., for some sets $\mathcal{V}$ of vectors) in [7–10]. In Section 2, we will show how these types of results could be used in finding the maximums and minimums of (2). Unfortunately, most classes for which the extreme points are determined are not very practical from the point of view of reliability theory. Our aim is to emphasize the connection between the two areas only.

The main aim of the present paper is to generalize these ideas for the case when the components of the device may have three different states. They can be good, mediocre, or bad, with certain probabilities. In Section 3, the concept of the reliability polynomial is generalized for this case and its extreme values are determined for a very special class. This is done by proving a theorem determining the extreme points of the convex hull for a new class.

The present work was largely motivated by the papers of Van Slyke and Frank [17], Ball and Provan [2], and Colbourn and Harms [6] in which the traditional (two-state) model was considered and extremal combinatorial theorems were used.

2. EXTREME POINTS OF HYPERGRAPH CLASSES AND THEIR USE FOR THE RELIABILITY POLYNOMIAL

A family $\mathcal{A}$ of subsets is called a *Sperner* family if $A \not\subseteq B$ holds for any two members of it. As it was pointed out in the Introduction, the family of operative sets is usually an ideal. The defining condition of the Sperner families is just the opposite of the property of an ideal, so this is not a very practical case. However, this is the most known and most studied class. This is why we start the investigations with it.

Let us recall that $f_i(\mathcal{A})$ denotes the number of i-element members of $\mathcal{A}$. The vector $[f_0(\mathcal{A}), f_1(\mathcal{A}), \ldots, f_n(\mathcal{A})]$ is the *profile vector* of $\mathcal{A}$. The following theorem is formulated in [9] but it is actually only another form of the well-known LYM-inequality (more properly, YBLM-inequality, see [4, 15, 16, 18]).

Theorem A. *The extreme points of the convex hull of the profiles of Sperner families on n elements are*

$$(0, \ldots, 0) \text{ and } \left(0, \ldots, 0, \binom{n}{i}, 0, \ldots, 0\right) (0 \le i \le n).$$

Suppose now that the operative family is a Sperner family and determine the maximum and minimum of the reliability polynomial subject to this condition. The minimum is trivially 0. The maximum can be attained only for the nonzero extreme points in Theorem A. For these extreme points, (2) becomes

$$\binom{n}{i}c(i) = \binom{n}{i}p^{i}(1 - i)^{n-i} = P_i.$$

Comparing P_{i-1} and P_i, one can determine the maximum P_i:

Proposition 1. *The maximum of the reliability polynomial for Sperner operative families is*

$$\max_{0\le i\le n} \binom{n}{i}p^{i}(1 - p)^{n-i} = \binom{n}{i}p^{k}(1 - p)^{n-k},$$

where

$$k = max\left\{i: \quad \frac{i}{n - i + 1} \le \frac{p}{1 - p}\right\}.$$

A family $\mathcal{A}$ is called *intersecting* if $A \cap B \neq \emptyset$ holds for any two members of it. Taking the complements of the members of an intersecting family, the new family will satisfy $A \cup B \neq X$ for any two of its members. Such families are called *nonfilling*. The extreme points of the convex hull of the profiles of the intersecting families are determined in [9]. The intersecting families, however, are not ideals, so they are, again, far from the applications. The nonfilling families are not necessarily ideals, but they can be easily made ideals adding all the subsets of the members. On the other hand, the maximum of (1) and (2) will be attained for families containing all possible additional members. The following theorem can be easily obtained from the theorem describing the extreme points for the intersecting families turning the extreme vectors back [interchanging the ith and $(n - i)$th components].

Theorem B. *The extreme points of the convex hull of the profiles of nonfilling families on n elements are*

(1)

$$\left(1, \binom{n}{1}, \ldots, \binom{n}{k-1}, \binom{n-1}{k}, \ldots, \binom{n-1}{n-k-1}, \binom{n-1}{n-k}, 0, \ldots, 0\right),$$

where $1 \le k \le n/2$ and the 0th, 1st, ..., $(k-1)$th, kth, ..., $(n-k-1)$th, $(n-k)$th, $(n-k+1)$th, ..., nth components are shown;

(2)

$$\left(1, \binom{n}{1}, \ldots, \binom{n}{\frac{n-1}{2}}, 0, \ldots, 0\right)$$

if n is odd, where the 0th, 1st, ..., $(n-1)/2$th, $(n+1)/2$th, ..., nth components are shown;

(3) *All vectors obtained by replacing any set of components by 0 in the above vectors.*

The nonfilling families have some meaning for reliability theory. The condition expresses that the operative families cannot be too large, more precisely, the union of two of them cannot cover the whole device.

The minimum of the reliability polynomial for nonfilling families is 0. Its maximum is attained for extremal points of type (1) and (2) in Theorem B. Thus, the possible candidates for the maximum of (2) are

$$P_k = \sum_{i=0}^{k-1} \binom{n}{i} p^i (1-p)^{n-i} + \sum_{j=k}^{n-k} \binom{n-1}{j} p^j (1-p)^{n-j} (1 \le k \le n/2)$$

and

$$P_{\frac{n+1}{2}} = \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{i} p^i (1-p)^{n-i}.$$

It is easy to see that

$$P_k \le P_{k+1} \quad \text{iff} \quad p \le \frac{1}{2}\left(1 \le k \le \frac{n-1}{2}\right).$$

This proves the following theorem:

Theorem 2. *The maximum of the reliability polynomial for nonfitting operative families is*

$$1 - p \text{ if } \frac{1}{2} \le p,$$

$$P_{\frac{n}{2}} = \sum_{i=0}^{\frac{n}{2}-1} \binom{n}{i} p^i (1-p)^{n-i} + \binom{n-1}{\frac{n}{2}} p^{\frac{n}{2}} (1-p)^{\frac{n}{2}} \quad \text{if } p \le \frac{1}{2}$$

and n *is even*

and

$$P_{\frac{n+1}{2}} = \sum_{i=0}^{\frac{n-1}{2}} \binom{n}{i} p^i (1-p)^{n-i}$$

$\text{if } p \le \frac{1}{2}$ *and* n *is odd.*

Let us see some more practical conditions on the operative family. In [17], the following one was considered. Let $\mathcal{A}$ be an ideal, suppose that $f_i(\mathcal{A}) = 0$ for all $i > k$, where k and the value $f_k\mathcal{A}) \ne 0$ is also known. The maximum of (2) under this condition is trivial; one has to take $f_i(\mathcal{A}) = \binom{n}{i}$ for all $i < k$. To describe the minimum, some definitions and statements are needed.

Suppose that the groundset X is ordered in some way. Then, the subsets of X can be described by their *characteristic vectors*.The characteristic vector of the set $A \subset X$ is an n-dimensional 0,1-vector, its ith component is 1 iff the ith element of X is in A. We say that the subset A lexicographically precedes the subset B if the characteristic vector of A lexicographically precedes the characteristic vector of B. Let $\mathcal{A}_k$ be a family of k-element subsets. Introduce the notation

$$s_i(\mathcal{A}_k) = \{B : |B| = i, \exists A \in \mathcal{A}_k \text{ such that } B \subset A\} \ (i < k).$$

Theorem C [12, 13], for a shorter proof see[11].). *Suppose that the size of the family $\mathcal{A}_k$ of k-element sets is fixed. Then, $|s_i(\mathcal{A}_k)|$ $(i < k)$ is minimum for the $\mathcal{A}_k$ consisting of the lexicographically first k-element sets.*

This theorem makes it easy to minimize (2) for the case when the largest nonzero $f_k(\mathcal{A})$ is given. Denote the subfamily consisting of the k-element members of $\mathcal{A}$ by $\mathcal{A}_k$. As $\mathcal{A}$ is an ideal, we have

$$|s_i(\mathcal{A})| \le f_i(\mathcal{A}) \ (i < k). \tag{3}$$

Hence,

$$\sum_{i=0}^{k} |s_i(\mathcal{A})| c(i) \le \sum_{i=0}^{n} f_i(\mathcal{A}) c(i) \tag{4}$$

follows. The left-hand side is achievable, by Theorem C; $\mathcal{A}_k$ should be chosen to be the family of the lexicographically first k-element subsets in X. By this, one of the extremal constructions is found. (It is not always unique.) This minimum can be calculated from (4) using another version of the theorem. However, this result is a complicated formula expressed by binomial coefficients. If, incidently, $|\mathcal{A}_k| = f_k(\mathcal{A})$ is of the form $\binom{a}{k}$, then the minimum of (2) [left-hand side of (4)] is

$$\sum_{i=0}^{k} \binom{a}{i} p^i(1-p)^{n-i}.$$

This line is continued in the works of Ball and Provan [2] and Colbourn and Harms [6] for the so-called *shallable* families.

The next condition is, again, fairly practical. Suppose that $\mathcal{A}$ is an ideal and $|\mathcal{A}| = m$ is known. Both the minimum and the maximum of (2) were determined for this case in [1] under the condition that $c(x)$ is monotone. (However, we have to mention that [5] solved the problem earlier for a more general structure when $c(i) = i$ but their method works for any monotone weight. [14] also formulated the analogous statement for another structure and monotone weights. [1] and [3] have extensions for the nonmonotone case.) It is inconvenient to give the exact minimum and maximum; instead, we show the extreme constructions only.

A *quasi-sphere* (*with center* 0) is a family containing all i-element subsets $(0 \le i \le r)$ for some $0 \le r \le n$ and the lexicographically first some $r + 1$-element subsets of X. A *quasi-cylinder* of size N is the family of the lexicographically first N subsets of X.

Theorem 3 [1, 5]. *If $\frac{1}{2} \le p$, then the minimum of* (2) *is assumed for the quasi-sphere and the maximum is assumed for the quasi-cylinder. However, if $p \le \frac{1}{2}$, then the minimum and the maximum are assumed for the quasi-cylinder and the quasi-sphere, respectively.*

Of course, these four statements are really only one. The minimum for an increasing $c(i)$ and the maximum for a decreasing $c(i)$ are trivial. On the other hand, the minimum for a decreasing weight and the maximum for an increasing one can be obtained from each other by multiplying the weight by -1.

3. RELIABILITY POLYNOMIAL IN PRESENCE OF MEDIOCRE ELEMENTS

Here we suppose that each element of the device may have three different states: It may be *operative, mediocre,* or *failing* with probability p_0, p_1, or p_2, respectively ($p_0 + p_1 + p_2 = 1$.) A *state of the device* is a 0, 1, 2-vector of dimension n where the ith component is 0, 1, or 2 if the ith element of the device is good, mediocre, or failing, respectively. A state $(s_1, s_2, \ldots, s_n)$ is called *operative* if the device operates when its ith element is in state s_i $(1 \le i \le n)$. The set of operative states of the device will be denoted by $\mathcal{A} \subseteq \{0, 1, 2\}^X$.

Some more notations are needed: If $A \in \mathcal{A}$, then $0(A)$ denotes the number of 0's in A while $1(A)$ denotes the number of 0's and 1's in A. The probability of the event that the device operates properly is expressed by the formula

$$\sum_{A\in\mathcal{A}} p_0^{0(A)} p_1^{1(A)-0(A)} p_2^{n-1(A)}. \tag{5}$$

In this case, this is the (generalized) reliability polynomial. Introduce the notation $f_{ij}(\mathcal{A})$ for the number of elements $A \in \mathcal{A}$ satisfying $i = 0(A)$ and $j = 1(A)$. Also, for the sake of brevity, use the notation $c(i, j) = p_0^i p_1^{j-i} p_2^{n-j}$. Then, another form of the reliability polynomial is obtained:

$$\sum_{0\le i\le j\le n} f_{ij}(\mathcal{A}) c(i, j). \tag{6}$$

We write that $A = (s_1, s_2, \ldots, s_n) \le B = (t_1, t_2, \ldots, t_n)$ iff $s_i \le t_i$ holds for all $1 \le i \le n$. A set of operative states $\mathcal{A}$ is an *ideal* if $A \le B$ and $B \in \mathcal{A}$ imply $A \in \mathcal{A}$. On the other hand, if $A \not\le B$ holds for any two distinct members of $\mathcal{A}$, then it is called an *antichain*.

The aim of the investigations, again, is to find the minimum and the maximum of the reliability polynomial for certain classes of $\mathcal{A}$'s. Obviously, in most practical cases, $\mathcal{A}$ should be an ideal. So the conditions containing "ideal" are the most important. However, we were able to treat only the least important case: the case of antichains. In the rest of the paper, we will try to determine the maximum of (5) (and (6) under the condition that $\mathcal{A}$ is an antichain. The method of the paper [10] will be used.

Introduce the concept of the *profile-matrix* of $\mathcal{A}$:

$$F(\mathcal{A}) = \begin{pmatrix} f_{0,0}(\mathcal{A}) & f_{0,1}(\mathcal{A}) & \cdot & \cdot & \cdot & f_{0,1}(\mathcal{A}) \\ 0 & f_{1,1}(\mathcal{A}) & \cdot & \cdot & \cdot & f_{1,n}(\mathcal{A}) \\ 0 & 0 & \cdot & \cdot & \cdot & f_{2,n}(\mathcal{A}) \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & f_{n,n}(\mathcal{A}) \end{pmatrix}.$$

This matrix may be considered as a vector in the $[(n + 2)(n + 1)]/2$-dimensional Euclidean space. Each antichain $\mathcal{A}$ determines one such profile-matrix. Denote their set by $\mathcal{V}$(antichain). (6) is a linear function of these variables; thus, the maximum of (6) is attained for one of the extreme points of the convex hull of $\mathcal{V}$(antichain). Therefore, it is sufficient to determine these extreme martrices.

Theorem 4. *The extreme matrices (extreme points) of the convex hull of the profile-matrices of the antichains in* $\{0, 1, 2\}^X$ ($|X| = n$) *are the matrices* $E = (e_{ij})$, *where*

$$e_{ij} = 0 \quad \text{if} \quad i > j,$$

$$e_{ij} \quad \text{is either} \quad 0 \quad \text{or} \quad \binom{n}{j}\binom{j}{i} \quad \text{if} \quad i \leq j, \quad (7)$$

and

$$e_{ij} \neq 0, \quad e_{kl} \neq 0 \quad \textit{imply} \quad (i, j) \nless (k, l). \quad (8)$$

Proof. (1) First we introduce some notations convenient to the proof. Let I denote a set of pairs (i, j) satisfying

$$i \leq j \quad \text{and} \quad (i, j) \nless (k, l) \quad \text{for all} \quad (i, j), (k, l) \in \mathcal{I} \quad (9)$$

(i.e., it is an antichain). The set of all such sets I is denoted by $\mathcal{I}$. The $(n + 1) \times (n + 1)$ matrix $T(I)$ contains entries 0's and 1's. The entry of the ith row and the jth column is 1 iff $(i, j) \in I$. Replace the 1 by $\binom{n}{j}\binom{j}{i}$ in the ith row and jth column for all $(i, j) \in I$. The so-obtained matrix is denoted by $E(I)$.

(2) Now solve the problem analogous to the theorem for the monotone elements of $\{0, 1, 2\}^X$. It is obvious that there is exactly one monotone element of $\{0, 1, 2\}^X$ containing i zeros and $j - i$ ones. Therefore, if $\mathcal{B}$ is an antichain in $\{0, 1, 2\}^X$ and consists of monotone elements, then $f_{ij}(\mathcal{B})$ is either 0 or 1. $f_{ij}(\mathcal{B})$ is obviously 0 if $i > j$. Moreover, if $e_{ij} = 1$, $e_{kl} = 1$, then $(i, j) \nless (k, l)$, i.e., the profile-matrix $F(\mathcal{B})$ of any such $\mathcal{B}$ is $T(I)$ for some $I \in \mathcal{I}$. It is easy to see that they all are extreme points as well.

(3) We have to prove that the profile-matrix of any antichain $\mathcal{A} \in \{0, 1, 2\}^X$ is a convex linear combination (the coefficients are nonnegative and their sum is 1) of the matrices $E(I)$, $I \in \mathcal{I}$. Fix $\mathcal{A}$ and consider the sum

$$\sum \frac{F(\{A\})}{n!} \quad (10)$$

for all pairs (P, A), where P is a permutation of the groundset X, A is a member of $\mathcal{A}$, and A is monotone in the permutation P.

One way of summing (10) is the following:

$$\sum_P \sum_A \frac{F(\{A\})}{n!} = \frac{1}{n!} \sum_P \sum_A F(\{A\}). \quad (11)$$

On the right-hand side, $\Sigma_A F(\{A\})$ is equal to $F(\mathcal{B})$, where $\mathcal{B}$ is the set of monotone (in P) elements of $\mathcal{A}$. By Section 2 of this proof, this is equal to $T(I)$ for some $I \in \mathcal{I}$. Denote its coefficient on the right-hand side of (11) by $\lambda(I)$. Their sum is obviously 1, since the number of permutations is $n!$. Hence, (10) is a convex linear combination of $T(I)$'s:

$$\sum \frac{F(\{A\})}{n!} = \sum_{I \in \mathcal{I}} \lambda(I) T(I). \quad (12)$$

The other way of summing (10) is

$$\sum_A \sum_P \frac{F(\{A\})}{n!}. \quad (13)$$

The number of permutations in which a given sequence A is monotone is $0(A)!(1(A) - 0(A))!(n - 1(A))!$. Therefore, (13) is equal to

$$\sum_{A \in \mathcal{A}} \frac{0(A)!(1(A) - 0(A))!(n - 1(A))!}{n!} F(\{A\})$$

$$= \sum_{A \in \mathcal{A}} \frac{F(\{A\})}{\binom{n}{1(A)}\binom{1(A)}{0(A)}} = \left(\frac{f_{i,j}(\mathcal{A})}{\binom{n}{j}\binom{j}{i}} \right)_{0 \leq i \leq j \leq n}$$

Applying (12), we have

$$\left(\frac{f_{i,j}(\mathcal{A})}{\binom{n}{j}\binom{j}{i}} \right)_{0 \leq i \leq j \leq n} = \sum_{I \in \mathcal{I}} \lambda(I) T(I).$$

But this equation is equivalent to

$$(f_{i,j}(\mathcal{A}))_{0 \leq i \leq j \leq n} = \sum_{I \in \mathcal{I}} \lambda(I) E(I). \quad \blacksquare$$

As a consequence, the maximum of the reliability polynomial (6) for antichains is equal to

$$\max \sum_{(i,j) \in \mathcal{I}} \binom{n}{j}\binom{n}{i} c(i, j), \quad (14)$$

where the maximum is taken over all antichains $\mathcal{I}$ satisfying (9). One might have the objection that the number of extreme matrices in Theorem 4 is exponential. However, the original number of candidate extreme families is double exponential; thus, the theorem gives a real reduction.

In what follows we further reduce the number of possibilities for the case $c(i, j) = p_0^i p_1^{j-i} p_2^{n-j}$. Suppose that $\mathcal{I}$ maximizes (14). Some properties of $\mathcal{I}$ under this assumption will be proved.

Compare two neighboring possible terms of (14):

$$\binom{n}{j}\binom{j}{i-1} p_0^{i-1} p_1^{j-i+1} p_2^{n-j}$$

$$>(<) \binom{n}{j}\binom{j}{i} p_0^i p_1^{j-i} p_2^{n-j} \quad \text{iff} \quad \frac{i}{j-i+1} >(<) \frac{p_0}{p_1}. \qquad (15)$$

We will show that there is no "jump in i" in $\mathcal{I}$, i.e., $(i, j) \in \mathcal{I}$, $0 < i$, and $j < n$ imply the existence of an $(i-1, j') \in \mathcal{I}$ satisfying $j < j'$. Indeed, if there is no element $(i', j') \in \mathcal{I}$ such that $i > i'$ and $j < j'$, then $\mathcal{I}$ can be enlarged by $(i-1, j+1)$, a contradiction. So, we may suppose that there is such an element of $\mathcal{I}$ where $i', \leq i-2$. The inequality

$$\frac{i}{j-i+1} > \frac{i'+1}{j'-i'}$$

is an obvious consequence of the assumptions. Consequently, p_0/p_1 is either smaller than the left-hand side or greater than the right-hand side. In the first case, replacing (i, j) by $(i-1, j)$ in $\mathcal{I}$, it enlarges the sum, by (15), contradicting the optimality of $\mathcal{I}$. In the second case, the replacement of (i', j') by $(i'+1, j')$ gives the contradiction.

The same argument shows that there is no "jump in j" in $\mathcal{I}$ either. Summarizing the two directions: $(i, j) \in \mathcal{I}$, $0 < i$ and $j < n$ implies $(i-1, j+1) \in \mathcal{I}$. On the other hand, if $(i, j) \in \mathcal{I}$, where $i + 2 \leq j$, then it cannot be the element with the smallest j. Therefore, the elements of $\mathcal{I}$ can be described with $i + j = k$ with a fixed k. The following version of Theorem 4 is proved.

Theorem 5. *The maximum of the reliability polynomial* (5) *under the assumption that the operative set is an antichain is*

$$\max_{0 \leq k \leq 2n} \sum_{i=\max\{0,k-n\}}^{\lceil \frac{k}{2} \rceil} \binom{n}{k-i}\binom{k-i}{i} p_0^i p_1^{k-2i} p_2^{n-k+i}.$$

It is easy to give limitations on the best k, but we were not able to determine it.

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