Linear Inequalities Describing the Class of Intersecting Sperner Families of Subsets, I

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1. Introduction

Let X be a finite set of n elements. We say that the family $\mathcal{A} = \{A_1, \ldots, A_m\}$ of its distinct subsets is *intersecting* if $A_i \cap A_j \neq \emptyset$ holds for any $1 \leq i \leq j \leq m$. A classic theorem of *Erdős Ko* and *Rado* [2] states that an intersecting family $\mathcal{A}$ consisting of k-element subsets, where $k \leq n/2$, has at most $\binom{n-1}{k-1}$ members.

Actually, they proved a more general theorem. The family $\mathcal{A}$ is called *Sperner* (see [9]) if $A_i \subseteq A_j$ never holds for $i \neq j$, that is, $\mathcal{A}$ is inclusion-free. It is proved in [2] that an intersecting Sperner family consisting of subsets of size at most $k (\leq n/2)$, has at most $\binom{n-1}{k-1}$ members. This paper of Erdős, Ko and Rado (together with [9]) started a new area of combinatorics. Here we present only the results significant from our point of view.

The next major step is due to *Bollobás* [1]. Let $p_i = p_i(\mathcal{A})$ denote the number of i-element members of $\mathcal{A}$.

BOLLOBÁS'S INEQUALITY. *If $\mathcal{A}$ is an intersecting Sperner family then the following holds:*

$$\sum_{i \leq n/2} \frac{p_i}{\binom{n-1}{i-1}} \leq 1. \tag{1}$$

Two subsets of size $k > n/2$, always intersect, so it is not interesting to investigate the maximally sized intersecting families consisting of such subsets. However, if all sizes are allowed then the question becomes interesting. *Milner* [7] proved that an intersecting Sperner family has maximally $\binom{n}{\lceil \frac{n+1}{2} \rceil}$ members. It is easy to see that the inequality

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of Bollobás implies both forms of the Erdős-Ko-Rado theorem, but it does nt imply Milner's one. This fact led *Greene, Katona* and *Kleitman* [6] to the following inequality:

$$\sum_{i\leq n/2}\frac{p_i}{\binom{n}{i-1}}+\sum_{n/2<j\leq n}\frac{p_j}{\binom{n}{j}}\leq 1. \qquad (2)$$

This inequality implies Milner's theorem, but it is too weak to imply Erdős-Ko-Rado. The reason of this annoying situation was shown in [3]. The vector $(p_0, p_1, \dots, p_n) = (p_0(\mathcal{A}), p_1(\mathcal{A}), \dots, p_n(\mathcal{A}))$ is called the *profile* of $\mathcal{A}$. The extreme points of the convex hull of the set of profiles of all intersecting Sperner families on n elements were determined in [3]. As a by-product, a large set of inequalities was determined for the intersecting Sperner families. Both (1) and (2) were particular cases in this class of inequalities. However, the class was too wide. Many of the inequalities were consequences of others. The authors of [3] did not notice that a minimal set of inequalities could have been easily deduced. This work is done in Section 2 of the present paper. These inequalities determine the hyperplanes bordering the convex hull of profiles of the set of all intersecting Sperner families.

2. Minimal set of inequalities

If a family $\mathcal{A}$ is intersecting then it does not contain the empty set as a member. Therefore p_0 is zero. We modify the definition of the *profile* omitting this superflous component: $(p_1, \dots, p_n)$. Consider the set of profiles of all intersecting Sperner families in the n-dimensional Euclidean space, where n is the size of the ground set of the families. Take the convex hull of this set of profiles. The vertices of this convex hull are called briefly the *extreme points* of the class of intersecting Sperner families. The starting point of our investigations is the following theorem:

THEOREM. [3] *The extreme points of the class of intersecting Sperner families are*

$$z = (0, \dots, 0),$$

$$v_j = \left(0, \dots, 0, \binom{n}{j}, 0, \dots, 0\right) \quad (n/2 < j \leq n),$$

$$w_i = \left(0, \dots, 0, \binom{n-1}{i-1}, 0, \dots, 0\right) \quad (1 \leq i \leq n/2),$$

$$w_{ij} = \left(0, \dots, 0, \binom{n-1}{i-1}, 0, \dots, 0, \binom{n-1}{j}, 0, \dots, 0\right) \quad (1 \leq i \leq n/2,\ n < i+j).$$

For two vectors x and y we write $x \leq y$ if the inequality holds componentwise. xy denotes the inner product.

A polyhedron P in the n-dimensional Euclidean space is called ***anti-blocking type*** if $P \neq \emptyset$, $x \in P$ implies $0 \leq x$ and $0 \leq y \leq x \in P$ implies $y \in P$ (see [8]). P is ***full*** if it contains elements $(0,\dots,0,x_i,0,\dots,0)$ $(0 < x_i)$ for all i $(1 \leq i \leq n)$. It is easy to see that the polihedron determined by the points given in the Theorem (that is, the convex hall of the class of intersecting Sperner families) is full and anti-blocking type.

Define (see [8])

$$A(P) = \{z : 0 \leq z,\ zx \leq 1, \text{for all } x \in P\}.$$

By a theorem of ***Fulkerson*** [4], [5] (see Theorem 9.4 in [8]) $A(P)$ is full and anti-blocking type, again.

An extreme point x of a polyhedron P is called ***essential*** if there is no other extreme point $y \in P, x \leq y$. It is easy to see that an anti-blocking type polyhedron is uniquely determined by its essential extreme points.

A ***bordering hyperplane*** of an n-dimensional polyhedron P is an $n-1$-dimensional hyperplane, given by an equation $a_1x_1 + \dots + a_nx_n = 1$ or $a_1x_1 + \dots + a_nx_n = 0$, containing at least n extreme points of P and satisfying the inequality of the same direction for all points of P.

If P is full and anti-blocking type then $x_i = 0$ is a bordering hyperplane for each i and no other bordering hyperplane have 0 on the right hand side. The latter ones are the ***essential bordering hyperplanes***.

LEMMA 1. *$\sum a_ix_i = 1$ is an essential hyperplane of the full anti-blocking type polihedron P iff $a = (a_1,\dots,a_n)$ is an essential extreme point of $A(P)$.*

PROOF: First we verify that if $\sum a_ix_i = 1$ is a bordering hyperplane of P then $a \in A(P)$. As $0 \in P$, the inequality $ax \leq 1$ must hold for the elements x of P. To prove $a \in A(P)$ we need to show $0 \leq a$, only. The hyperplane contains at least n vertices of P. If all these vertices had 0 as a first component then they would be in the $(n-2)$-dimensional intersection of the hyperplanes $ax = 1$ and $x_1 = 0$. This contradiction ensures the existence of a vertex $x = (x_1,\dots,x_n)$ with a positive first component, on the hyperplane. $(0,x_2,\dots,x_n) \in P$ follows by the anti-blocking type property. However $ax = 1, a_1 < 0$ and $0 < x_1$ imply $0x_1 + a_2x_2 + \dots + a_nx_n > 1$. This contradiction proves $a_1 \geq 0$ and, in general, $a_i \geq 0$.

Let us show now that if $ax = 1$ is a bordering hyperplane then a is an extreme point of $A(P)$. Suppose that it is not true. Then there exists a non-zero n-dimensional vector b such that $a + b \in A(P), a - b \in A(P)$. Hence we have $(a+b)x \leq 1$ and $(a-b)x \leq 1$ for each vertex of P contained in the bordering hyperplane $ax = 1$. $bx \leq 0, -bx \leq 0$ and consequently $bx = 0$ follow for these vertices. This is a contradiction, since these vertices span a hyperplane not containing the origin.

Finish this part of the proof showing that if $ax = 1$ is a bordering hyperplane then a is an essential extreme point. Suppose the contrary: there is a $b \in A(P)$ such that $a \leq b$, $a \neq b$. Let $a_k < b_k$. We saw in the first section of this proof that there must exist a vertex $x \in P$ with equality in $ax = 1$ and satisfying $0 < x_k$. For this vertex $bx > ax = 1$ holds, contradicting $ax \leq 1$. Therefore, if $ax = 1$ is a bordering hyperplane of P, a must be an essential extreme point of $A(P)$.

Now we prove the other direction. Suppose that a is an essential extreme point of $A(P)$ and assume, in an indirect way, that $ax = 1$ is not a bordering hyperplane. By the definition of $A(P)$, a satisfies $ax \leq 1$ for all points x of P, thus the vertices v of P satisfying $av = 1$ are contained in a hyperplane H of dimension k less than $n-1$. Then all vertices v of P satisfying $av = 1$ are linear combinations of the vertices $v_1, \ldots, v_k$ where $av_i = 1$ $(1 \leq i \leq k)$. For the other vertices w of P $aw < 1$ holds. Two cases are distinguished.

1)Suppose that $a_j = 0$ implies that there is a vertex $u = (u_1, \ldots, u_n) \in P$ satisfying $au = 1$ and $0 < u_j$ for all j. Then $u' = (u_1, \ldots, u_{j-1}, 0, u_{j+1}, \ldots, u_n) \in P$ holds by the properties of P. The equation $au' = 1$ implies $u' \in H$. Let b be a non-zero solution of the system $bv_i = 0$ $(1 \leq i \leq k)$. Since both u and u' are linear combinations of $v_1, \ldots, v_k$, we have $bu = bu' = 0$. These two equations imply $b_j u_j = 0$ and, by our assumption, $b_j = 0$ is true. Then $a \pm \varepsilon b$ consists of non-negative components if $\varepsilon > 0$ is small enough. The equation $(a \pm \varepsilon b)v = 1$ is obvious for all elements of H and $(a \pm \varepsilon b)w < 1$ for all other vertices of P if ε is small enough. Thus $a + \varepsilon b$ and $a - \varepsilon b$ are in $A(P)$, a contradiction.

2)Let $a_j = 0$ and let all vertices $u = (u_1, \ldots, u_n)$ of P satisfying $au = 1$ have $u_j = 0$. Then $a' = a + (0, \ldots, 0, \varepsilon, 0, \ldots, 0)$ (ε is in the jth component) satisfies $a'u = 1$ for these vertices and $a'w < 1$ for other vertices, if ε is small enough. $a' \in A(P)$ is a contradiction. ■

Let P be the convex hull of the points given in the theorem. Then v_j $(n/2 < j \leq n)$ and w_{ij} $(1 \leq i \leq n/2,\ n < i + j)$ are then essential extreme points of P. $A(P)$ consists of the points $x = (x_1, \ldots, x_n)$ satisfying

$$\binom{n}{j} x_j \leq 1 \quad (n/2 < j \leq n), \tag{3}$$

$$x_1 \leq 1, \tag{4}$$

$$\binom{n-1}{i-1} x_i + \binom{n-1}{j} x_j \leq 1 \quad (2 \leq i \leq n/2,\ j < n,\ n < i + j). \tag{5}$$

By Lemma 1 we have to determine only the essential extreme points of $A(P)$, that is, the set of vectors x determined by (3),(4) and (5).

The following easy lemma enables us to make a reduction. Let $\operatorname{diag}(c_1,\dots,c_n)$ denote an $n \times n$ matrix with these entries in the main diagonal and 0s otherwise. For a polyhedron P, define

$$\operatorname{diag}(c_1,\dots,c_n)P = \{\operatorname{diag}(c_1,\dots,c_n)x : x \in P\}.$$

LEMMA 2. *Suppose that $c_1,\dots,c_n > 0$. If P is anti-blocking type then $\operatorname{diag}(c_1,\dots,c_n)P$ has the same property. x is an essential extreme point of P iff $\operatorname{diag}(c_1,\dots,c_n)x$ is an essential extreme point of $\operatorname{diag}(c_1,\dots,c_n)P$.*

PROOF: It is trivial. ■

Let $c_i = \binom{n-1}{i-1}$ $(1 \le i \le n/2)$, $c_j = \binom{n-1}{j}$ $(n/2 < j < n)$, $c_n = 1$. By (3),(4) and (5) $S = \operatorname{diag}(c_1,\dots,c_n)A(P)$ consists of the points y satisfying

$$y_j \le 1 - \frac{j}{n} \quad (n/2 < j < n), \tag{6}$$

$$y_1 \le 1, \quad y_n \le 1, \tag{7}$$

$$y_i + y_j \le 1 \quad (2 \le i \le n/2,\ j < n,\ n < i + j). \tag{8}$$

Now we try to find the essential extreme points of S (as it is anti-blocking type by Lemma 2).

LEMMA 3. *If $(y_1,\dots,y_n)$ is an essential extreme point of S then*

$$y_1 = y_n = 1, \tag{9}$$

for every i $(2 \le i \le n/2)$ there is a j such that
$n - i < j < n$ and $y_i + y_j = 1$ (10)

for every j $(n/2 < j < n)$ either $y_j = 1 - j/n$
or there is an $i > n - j$ such that $y_i + y_j = 1$. (11)

PROOF: y_1 and y_n occur in (7) but not in (6) and (8). If their values were < 1, y could be increased, consequently it would not be an essential extreme point. (9) is proved. (10) and (11) can be proved by the same argument: if there is a strict inequality at all occurences of a variable y_i in (6) and (8) then its value can be increased letting the other components unchanged. This contradicts the assumption that y is an essential extreme point. ■

LEMMA 4. *If $y = (y_1, \dots, y_n)$ is an essential extreme point of S then $y_{i-1} \ge y_i$ $(2 \le i < n)$.*

PROOF: Suppose that $y = (y_1, \dots, y_n) \in S$ and $y_{j-1} < y_j$ $(n/2 + 1 < j < n)$. Then we can show that $(y_1, \dots, y_{j-2}, y_j, y_j, y_{j+1}, \dots, y_n) \in S$. (6) holds for the new $(j-1)$th component since $y_j \le 1 - j/n < 1 - (j-1)/n$. (8) holds for it because $i + j - 1 > n$ implies $i + j > n$. The modified vector is larger than y, so y cannot be an essential extreme point of S.

If $y_{j-1} < y_j$ $(2 \le j \le n/2)$ then the same argument is used to ensure the validity of (8). The missing cases are $j = \frac{n+1}{2}$ and $j = \frac{n}{2} + 1$. In the first one of these cases $j = \frac{n-1}{2}$ holds and $y_{\frac{n+1}{2}} + y_k \le 1$ should be proved for $\frac{n-1}{2} + k > n$. This is, however, a consequence of (6). The case $j = \frac{n}{2} + 1$ can be settled in the same way. ■

LEMMA 5. *If $y = (y_1, \dots, y_n)$ is an essential extreme point of S then*

$$y_{n-i+1} = 1 - y_i \quad (2 \le i \le n/2).$$

PROOF: Suppose the contrary: $y_i + y_{n-i+1} < 1$ holds for some $2 \le i \le n/2$. By Lemma 4 we have

$$y_{n-1} \le y_{n-2} \le \dots \le y_{n-i+1} < 1 - y_i,$$

that is, (10) is violated for this i. ■

LEMMA 6. *$(y_1, \dots, y_n)$ is an essential extreme point of S iff $y_1 = y_n = 1$, and there are some integers $1 = i_1 < i_2 < \dots < i_{r+1} = \lceil \frac{n+1}{2} \rceil$ $(1 \le r)$ such that*

$$y_{i_k} = 1 - \frac{i_k - 1}{n} \quad (1 \le k \le r),$$

$$y_i = y_{i_k} \quad (i_k \le i < i_{k+1},\ 1 \le k \le r),$$

$$y_{n-i+1} = 1 - y_i, \quad (2 \le i \le n/2) \tag{12}$$

and

$$y_{\frac{n+1}{2}} = \frac{1}{2} - \frac{1}{2n}.$$

PROOF: First, let us prove that the essential extreme points $(y_1, \dots, y_n)$ must have this form. Lemma 5 and (6) imply

$$y_i \ge 1 - \frac{i-1}{n} \quad (2 \le i \le \frac{n}{2}). \tag{13}$$

If n is even, then $y_{\frac{n}{2}} \ge \frac{1}{2} + \frac{1}{n}$ follows by (13) while $y_{\frac{n}{2}+1} \le \frac{1}{2} - \frac{1}{n}$ is a consequence of (6). Similarly, $y_{\frac{n-1}{2}} \ge \frac{1}{2} + \frac{3}{2n}$ and $y_{\frac{n+1}{2}} \le \frac{1}{2} - \frac{1}{2n}$ hold for the odd ns. Our conclusion is that these neighbouring ys must be different.

So, if m is defined by $y_2 = y_{m-1} > y_m$ then the inequality $m - 1 \le \frac{n}{2}$ is obvious.

(13) implies $1 - \frac{1}{n} \le y_2$. Suppose that $1 - \frac{1}{n} < y_2 < 1$. Then there is an $\varepsilon > 0$ such that both

$$(1, y_2 + \varepsilon, \ldots, y_{m-1} + \varepsilon, y_m, \ldots, 1 - y_{n-m+1}, 1 - y_{n-m+2} - \varepsilon, \ldots, 1 - y_2 - \varepsilon, 1)$$

$$(1, y_2 - \varepsilon, \ldots, y_{m-1} - \varepsilon, y_m, \ldots, 1 - y_{n-m+1}, 1 - y_{n-m+2} + \varepsilon, \ldots, 1 - y_2 + \varepsilon, 1)$$

are in S. This contradiction shows that $(y_1, \ldots, y_n)$ cannot be an essential extreme point. Consequently, y_2 is either $1 - \frac{1}{n}$ or 1. In the first case $i_2 = 2$, in the latter one $i_2 = m$. Repeating this procedure we obtain that the essential extreme points satisfy the conditions of the lemma. (If n is odd then $y_{\frac{n+1}{2}} \le 1/2 - 1/2n$. Choosing equality, the vector is in S and is larger than the vectors with smaller $y_{\frac{n+1}{2}}$.)

It remained to prove that all these vectors are essential extreme points. Denote their set by Y. No two of them are comparable (in the sense of $\le$) so it is sufficient to see that they are extreme points, that is, they are not convex linear combinations of other extreme points. In other words, if $y \in Y$, $z^1, \ldots, z^t \ne y$ are different extreme points of S and $\alpha_1, \ldots, \alpha_k$ are positive numbers ($\sum \alpha_i = 1$) then

$$y = \sum_{i=1}^{t} \alpha_i z^i \tag{14}$$

is impossible.

First prove this statement for the case if one of the z^is is a non-essential extreme point. Then this z^i satisfies $z^i \le y', z^i \ne y'$ for some $y' = (y'_1, \ldots, y'_n) \in Y$. The inequality $y'_i + y'_{n-i+1} < 1$ (or $y'_{\frac{n+1}{2}} < 1/2 - 1/2n$) is a consequence for some i. (14) implies that the same holds for y, that is, $y \notin Y$.

Therefore we may suppose that $z^1, \ldots, z^t \in Y$. An element z of Y is uniquely determined by the sequence $i_1(z) < i_2(z) < \ldots < i_r(z)$ $(r = r(y))$. If $z^1 \ne z^2$ then these sequences are different, so there is an $i_j = i_j(z^1)$ which is not equal to any $i_s(z^2)$ (or the other way around). Clearly we have

$$1 - \frac{i_j - 1}{n} = z^1_{i_j} < z^1_{i_j - 1}$$

and

$$z^1_{i_j} < z^2_{i_j} = z^2_{i_j - 1}$$

by the definition of the sequence i_j. (14) results in $1 - \frac{i_j - 1}{n} < y^1_{i_j} < y^1_{i_j - 1}$, in contradiction with $y \in Y$. ■

To obtain $A(P)$ Lemma 2 is used with $\mathrm{diag}^{-1}(c_1, \ldots, c_n)$. (See the lines after Lemma 2.) The application Lemma 1 gives the final result.

THEOREM. *The essential bordering hyperplanes of the convex hull of the class of intersecting Sperner families are the following ones. Let $1 = i_1 < \ldots < i_{r+1} = \lceil \frac{n+1}{2} \rceil$ $(1 \le r)$ be some integers.*

$$\sum_{k=1}^{r} \left(\sum_{i=i_k}^{i_{k+1}-1} \frac{p_i}{\binom{n-1}{i-1}\frac{n}{n-i_k+1}} \right) + \frac{p_{\frac{n+1}{2}}}{\binom{n}{\frac{n+1}{2}}} + \sum_{k=1}^{r} \left(\sum_{n-i_{k+1}+1<j\le n-i_k+1} \frac{p_j}{\binom{n-1}{j}\frac{n}{i_k-1}} \right) \le 1,$$

where the middle term appears only for odd ns, the terms with $i_1 (= 1)$ should be taken to be 0 and the term with $j = n$ is simply p_n.

Observe that $i_2 = \lceil \frac{n+1}{2} \rceil$ leads to Bollobás's inequality (with a slight modification) while the case $i_k = k$ $(1 \le k < \lceil \frac{n+1}{2} \rceil)$ gives the inequality of Greene, Katona and Kleitman. The total number of inequalities in the theorem is exponential.

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