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All Maximum 2-Part Sperner Families

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Let $X = X_1 \cup X_2$, $X_1 \cap X_2 = 0$ be a partition of an n-element set. Suppose that the family $\mathscr{F}$ of some subsets of X satisfy the following condition: if there is an inclusion $F_1 \not\subseteq F_2$ $(F_1, F_2 \in \mathscr{F})$ in $\mathscr{F}$, the difference $F_2 - F_1$ cannot be a subset of X_1 or X_2. Kleitman (*Math. Z.* **90** (1965), 251–259) and Katona (*Studia Sci. Math. Hungar.* **1** (1966), 59–63) proved 20 years ago that $|\mathscr{F}|$ is at most n choose $\lfloor n/2 \rfloor$. We determine all families giving equality in this theorem. © 1986 Academic Press, Inc.

1. INTRODUCTION

Let us start with a classic theorem of Sperner [9]:

If $\mathscr{F} \subseteq 2^X$ is a family of distinct subsets of an n-element set X sucht that $F_1 \not\subset F_2$ holds for all $F_1, F_2 \in \mathscr{F}$ then

$$|\mathscr{F}| \leqslant \binom{n}{\lfloor n/2 \rfloor}.$$

Kleitman [6] and Katona [5] independently discovered that the condition of this theorem can be weakened while its statement remains true:

Let $X = X_1 \cup X_2$, $X_1 \cap X_2 = \varnothing$ be a partition of X $(|X| = n)$. Suppose that the family $\mathscr{F} \subseteq 2^X$ satisfies the following condition:

$$F_1 \subset F_2, F_1, F_2 \in \mathscr{F} \textit{ imply } F_2 - F_1 \not\subset X_1 \textit{ and } F_2 - F_1 \not\subset X_2. \qquad (1)$$

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Then

$$|\mathscr{F}| \leqslant \binom{n}{\lfloor n/2 \rfloor}. \tag{2}$$

The families satisfying (1) are called *2-part Sperner families*. The main aim of the present paper is to determine all maximum 2-part Sperner families, that is, the ones with equality in (2). It is worth mentioning that all of them have the following *homogenity* property: if $F \in \mathscr{F}$ then $|F \cap X_1| = |G \cap X_1|$, $|F \cap X_2| = |G \cap X_2|$ imply $G \in \mathscr{F}$. This is not true for more than 2 parts. (See [4] for analogous questions.)

The proof is based on a theorem of [2]. We state it to make the paper self-contained.

Let $\mathscr{F}$ be a 2-part Sperner family, and let p_{ij} denote the number of members $F \in \mathscr{F}$ such that $|F \cap X_1| = i$, $|F \cap X_2| = j$ $(0 \leqslant i \leqslant n_1 = |X_1|$, $0 \leqslant j \leqslant n_2 = |X_2|)$. The *profile-matrix* $P(\mathscr{F})$ is defined by the entries p_{ij}. It can be considered as a point of the $(n_1+1)(n_2+1)$-dimensional space. Consider the set μ of all such points. The *extreme points* of μ are the ones which cannot be expressed as convex linear combinations of other points of μ. The next statement determines all extreme points of μ.

THEOREM A Particular case of Theorem 2.1 of [2]). *The extreme points of the set of profile-matrices of all 2-part Sperner families are the* $(n_1+1) \times (n_2+1)$ *matrices having either* 0 *or* $\binom{n_1}{i}\binom{n_2}{j}$ *as the* ijth *entry but having at most one non-zero entry in each row or column.*

For interested readers we also suggest the recent survey paper [3] on more-part Sperner theorems.

2. DETAILS

$|\mathscr{F}| = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} p_{ij}$ is a linear function of the variables p_{ij}. It follows that $|\mathscr{F}|$ will be maximum for some extreme points described in Theorem A and may be for some convex linear combinations of these maximum extreme points.

At first we determine the extreme points maximizing $|\mathscr{F}| = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} p_{ij}$. The non-zero entries of the extreme points (matrices) are in different rows and in different columns. The *partial transversals* are defined accordingly: $I \subset \{0,\ldots,n_1\} \times \{0,\ldots,n_2\}$ is a partial transversal iff (i_1, j_1), $(i_2, j_2) \in I$, $(i_1, j_1) \neq (i_2, j_2)$ imply $i_1 \neq i_2$, $j_1 \neq j_2$. So we have to maximize

$$\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} p_{ij} = \sum_{(i,j) \in I} \binom{n_1}{i}\binom{n_2}{j} \tag{3}$$

for partial transversals I.

It is intuitively clear that (3) is maximum if the great numbers $\binom{n_1}{i}$ are paired with great $\binom{n_2}{j}$'s and the little ones with little ones. Some easy lemmas leading in this direction are:

LEMMA 1. *Let* $a_1,\ldots, a_u$ *and* $b_1,\ldots, b_v$ *be integers and I a partial transversal. Suppose that* (i_1, j_2), $(i_2, j_1) \in I$ *and* $a_{i_1} > a_{i_2}$, $b_{j_1} > b_{j_2}$ *hold and define* $I' = (I - \{(i_1, j_2), (i_2, j_1)\}) \cup \{(i_1, j_1), (i_2, j_2)\}$. *Then*

$$\sum_{(i,j)\in I} a_i b_j < \sum_{(i,j)\in I'} a_i b_j. \tag{4}$$

Proof. We have $\sum_{(i,j)\in I'} a_i b_j - \sum_{(i,j)\in I} a_i b_j = a_{i_1} b_{j_1} + a_{i_2} b_{j_2} - a_{i_1} b_{j_2} - a_{i_2} b_{j_1} = (a_{i_1} - a_{i_2})(b_{j_1} - b_{j_2}) > 0$, proving (4). ∎

LEMMA 2. *Let* $a_1 > a_2 \geqslant a_3 \geqslant \cdots \geqslant a_u > 0$ *and* $b_1 > b_2 \geqslant b_3 \geqslant \cdots \geqslant b_v > 0$ *be integers. If I is a partial transversal maximizing*

$$\sum_{(i,j)\in I} a_i b_j \tag{5}$$

then $(1, 1) \in I$.

Proof. Suppose, on the contrary, that $(1, 1) \notin I$. We will find contradictions distinguishing several cases. If there is no other pair with component 1 in I then any element (i, j) can be replaced by $(1, 1)$ and this is a contradiction by $a_i b_j < a_1 b_1$ and the maximality of I.

If $(i, 1) \in I$ $(i \neq 1)$ but $(1, j) \notin I$ for any j then $(i, 1)$ can be replaced by $(1, 1)$. This is a contradiction by $a_1 b_j < a_1 b_1$. The case when $(1, j) \in I$ $(j \neq 1)$ but $(i, 1) \in I$ holds for no i can be settled in the same way.

Finally suppose that $(i, 1) \in I$ and $(1, j) \in I$ $(i \neq 1 \neq j)$. Then $(i, j) \notin I$, because I is a partial transversal. Replacing $(i, 1)$ and $(1, j)$ by $(1, 1)$ and (i, j), Lemma 1 gives the contradiction. ∎

LEMMA 3. *Let* $a_1 > a_2 \geqslant a_3 \geqslant \cdots \geqslant a_u > 0$ *and* $b_1 = b_2 > b_3 \geqslant \cdots \geqslant b_v > 0$ *be integers. If I is a partial transversal maximizing* (5) *then either* $(1, 1) \in I$ *or* $(1, 2) \in I$ *holds.*

Proof. Suppose, on the contrary, that none of them holds. The proof of Lemma 2 can be repeated, since it does not lead here to a contradiction only if $(1, 2)$ is involved. However, it is not in I by the indirect assumption. ∎

LEMMA 4. *Let* $a_1 = a_2 > a_3 \geqslant \cdots \geqslant a_u > 0$ *and* $b_1 = b_2 > b_3 \geqslant \cdots \geqslant b_v > 0$ *be integers. If I is a partial transversal maximizing* (5) *then either* $(1, 1), (2, 2) \in I$ *or* $(1, 2), (2, 1) \in I$ *hold.*

Proof. Suppose that none of (1, 1), (2, 2), (1, 2), (2, 1) is in I. Then the proof of Lemma 1 leads to a contradiction, since (2, 2), (1, 2), and (2, 1) are not involved in the changes. Hence at least one of (1, 1) (2, 2), (1, 2), and (2, 1), say (i, j), is in I. Delete a_i and b_j from the numbers. The remaining numbers satisfy the conditions of Lemma 2, thus $(3-i, 3-j) \in I$. ∎

Now we are able to determine all partial transversals I maximizing (3); however, we have to distinguish cases according to the parity of n_1 and n_2.

LEMMA 5. *If n_1 and n_2 are both even then I is a partial transversal* (3) *iff*

$$\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \in I \tag{6}$$

and exactly one of the following two rows hold for each $k = 1, 2, \ldots, \min\{\frac{n_1}{2}, \frac{n_2}{2}\}$:

$$\left(\frac{n_1}{2}-k, \frac{n_2}{2}-k\right) \in I, \qquad \left(\frac{n_1}{2}+k, \frac{n_2}{2}+k\right) \in I, \tag{7}$$

$$\left(\frac{n_1}{2}-k, \frac{n_2}{2}+k\right) \in I, \qquad \left(\frac{n_1}{2}+k, \frac{n_2}{2}-k\right) \in I. \tag{8}$$

Proof. We use first Lemma 2 with the numbers $\binom{n_1}{0}, \ldots, \binom{n_1}{n_1}$ and $\binom{n_2}{0}, \ldots, \binom{n_2}{n_2}$ ordered decreasingly, respectively. This proves (6). Delete $\binom{n_1}{n_1/2}$ and $\binom{n_2}{n_2/2}$ from the numbers. The remaining numbers satisfy the conditions of Lemma 4, therefore either (7) or (8) holds with $k=1$. The proof of the necessity of (6)–(8) can be completed by induction.

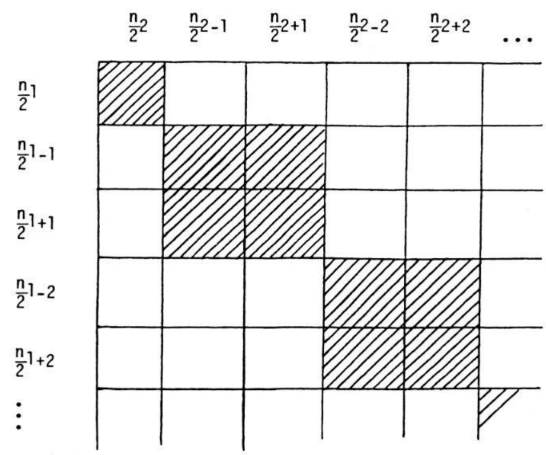

FIGURE 1

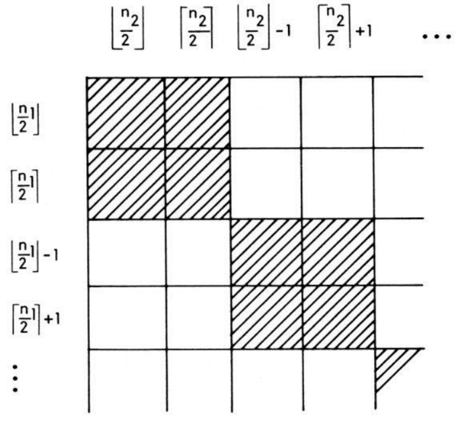

FIGURE 2

On the other hand, it is easy to see that all such I's give the same value for (3), maximizing it. ∎

This result can be better visualized if the rows and the columns of the matrix are ordered according to the decreasing order of the binomial coefficients (Fig. 1). I has to contain two oppositive corners of each 2×2 shaded block and the 1×1 shaded one.

The proof of the next lemma is analogous.

LEMMA 6. *If n_1 and n_2 are both odd then I is a partial transversal minimizing* (3) *iff exactly one of the following two rows holds for each* $k = 0, 1, \ldots, \min\{\lfloor n_1/2 \rfloor, \lfloor n_2/2 \rfloor\}$ (Fig. 2):

$$\left(\left\lfloor \frac{n_1}{2} \right\rfloor - k, \left\lfloor \frac{n_2}{2} \right\rfloor - k\right) \in I, \qquad \left(\left\lceil \frac{n_1}{2} \right\rceil + k, \left\lceil \frac{n_2}{2} \right\rceil + k\right) \in I$$

$$\left(\left\lceil \frac{n_1}{2} \right\rceil + k, \left\lfloor \frac{n_2}{2} \right\rfloor - k\right) \in I, \qquad \left(\left\lfloor \frac{n_1}{2} \right\rfloor - k, \left\lceil \frac{n_2}{2} \right\rceil + k\right) \in I.$$

The proof of the remaining case, when the parities are different, is again analogous. However the formulation of the statement is less convenient.

LEMMA 7. *If n_1 is even and n_2 is odd, then I is a partial transversal maximizing* (3) *iff I contains exactly one element of the following sets for each k*:

$$\left\{\left(\frac{n_1}{2}, \left\lfloor \frac{n_2}{2} \right\rfloor\right), \left(\frac{n_1}{2}, \left\lceil \frac{n_2}{2} \right\rceil\right)\right\},$$

$$\left\{\left(\frac{n_1}{2}-k, \left\lfloor \frac{n_2}{2} \right\rfloor - k + 1\right), \left(\frac{n_1}{2}-k, \left\lceil \frac{n_2}{2} \right\rceil + k - 1\right),\right.$$

$$\left.\left(\frac{n_1}{2}+k, \left\lfloor \frac{n_2}{2} \right\rfloor - k + 1\right), \left(\frac{n_1}{2}+k, \left\lceil \frac{n_2}{2} \right\rceil + k - 1\right)\right\},$$

$$k = 1, 2, \ldots, \min\left\{\frac{n_1}{2}, \left\lceil \frac{n_2}{2} \right\rceil\right\},$$

$$\left\{\left(\frac{n_1}{2}-k, \left\lfloor \frac{n_2}{2} \right\rfloor - k, \left(\frac{n_1}{2}-k, \left\lceil \frac{n_2}{2} \right\rceil + k\right),\right.\right.$$

$$\left.\left(\frac{n_1}{2}+k, \left\lfloor \frac{n_2}{2} \right\rfloor - k\right), \left(\frac{n_1}{2}+k, \left\lceil \frac{n_2}{2} \right\rceil + k\right)\right\}$$

$$k = 1, 2, \ldots, \min\left\{\frac{n_1}{2}, \left\lfloor \frac{n_2}{2} \right\rfloor\right\}.$$

The statement is visualized in Fig. 3. I has to contain exactly one element of each shaded block (1×2 or 2×2). And it has to be a partial transversal, of course. A typical example is shown in Fig. 4. The case when n_1 is odd and n_2 is even can be formulated and proved analogously.

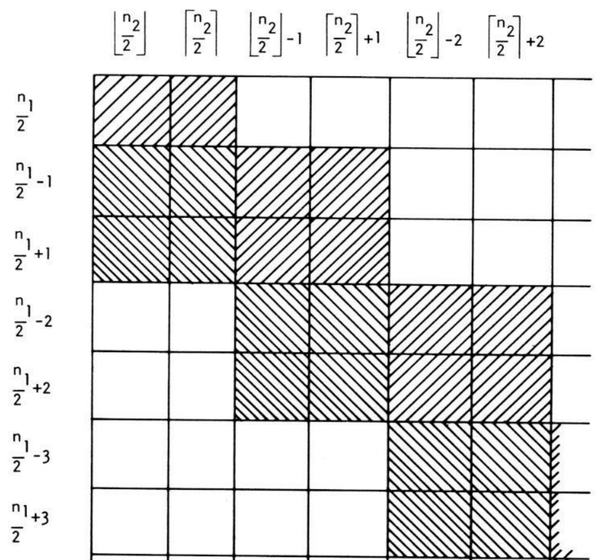

FIGURE 3

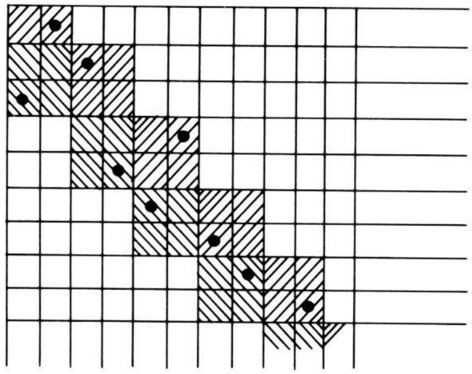

FIGURE 4

By this we have finished the first part of our work; the extreme points maximizing $|\mathscr{F}|$ are determined. In the rest of the paper we show that there are no other 2-part Sperner families with equality in (2).

The following lemma is a part of the folklore, a sharpening of the so-called LYM-inequality ([7, 8, 10]) (which was proved independently by Bollobás [1] in a more general form).

LEMMA 8. *Let $\mathscr{H}$ be a Sperner family on an n-element set. The number of i-element members is p_i. Then*

$$\sum_{i=0}^{n} \frac{p_i}{\binom{n}{i}} \leqslant 1$$

with equality only when $p_j = \binom{n}{j}$ for some $0 \leqslant j \leqslant n$.

The next lemma is a similar statement for 2-part Sperner families.

LEMMA 9. *Let $\mathscr{F}$ be a 2-part Sperner family on $X = X_1 \cup X_2$ $(X_1 \cap X_2 = \varnothing,\ |X_1| = n_1,\ n_2 = |X_2|)$, and let p_{ij} denote the number of its members F such that $|F \cap X_1| = i$, $|F \cap X_2| = j$. Suppose that the following conditions hold for some indices u, v $(0 \leqslant u \leqslant n_1,\ 0 \leqslant v \leqslant n_2)$:*

$$p_{uv} > 0, \tag{9}$$

$$\sum_{j=0}^{n_2} \frac{p_{uj}}{\binom{n_1}{u}\binom{n_2}{j}} = 1, \tag{10}$$

$$\sum_{i=0}^{n_1} \frac{p_{iv}}{\binom{n_1}{i}\binom{n_2}{v}} = 1. \tag{11}$$

Then

$$p_{uv} = \binom{n_1}{u}\binom{n_2}{v}. \tag{12}$$

Proof. Introduce the following notations:

$$\mathscr{F}_1(A) = \{F: F \subset X_2, A \cup F \in \mathscr{F}\} \ (A \subset X_1),$$
$$\mathscr{F}_2(B) = \{F: F \subset X_1, F \cup B \in \mathscr{F}\} \ (B \subset X_2),$$
$$p_j(A) = |\{F: |F| = j, F \in \mathscr{F}_1(A)\}|,$$
$$q_i(B) = |\{F: |F| = i, F \in \mathscr{F}_2(B)\}|.$$

Observe that the above families are Sperner families for each $A \subset X_1$, $B \subset X_2$. Therefore

$$\sum_{j=0}^{n_2} \frac{p_j(A)}{\binom{n_2}{j}} \leqslant 1 \tag{13}$$

holds for any $A \subset X_1$. Summing up for all sets $A \subset X_1$ with $|A| = u$ we obtain

$$\sum_{\substack{A \subset X_1 \\ |A| = u}} \sum_{j=0}^{n_2} \frac{p_j(A)}{\binom{n_2}{j}} \leqslant \binom{n_1}{u}. \tag{14}$$

As $\sum_{A \subset X_1, |A| = u} p_j(A) = p_{uj}$, (14) is equivalent to

$$\sum_{j=0}^{n_2} \frac{p_{uj}}{\binom{n_1}{u}\binom{n_2}{j}} \leqslant 1.$$

By (10) we have equality here and in (14), consequently (13) hold with equality for all $A \subset X_1$, $|A| = u$. By Lemma 8, one of the numbers $p_j(A)$, say $p_{j(A)}(A)$, is equal to $\binom{n_2}{j(A)}$, the other ones are zero. $\sum_{A \subset X_1, |A| = u} p_v(A) = p_{uv}$ and (9) imply the existence of an $A^* \subset X_1$, $|A^*| = u$ such that $p_v(A^*) > 0$. This means that $j(A^*) = v$ for this A^*: $p_v(A^*) = \binom{n_2}{v}$.

All sets F satisfying $F \cap X_1 = A^*$, $|F \cap X_2| = v$ are in $\mathscr{F}$. Choose one of them, its intersection with X_2 will be denoted by B^*. Therefore $B^* \subset X_2$, $|B^*| = v$, $A^* \cup B^* \in \mathscr{F}$, $A^* \in \mathscr{F}_2(B^*)$, and

$$q_u(B^*) > 0 \tag{15}$$

all hold. $\mathscr{F}_2(B)$ is a Spencer family; it satisfies

$$\sum_{i=0}^{n_1} \frac{q_i(B)}{\binom{n_1}{i}} \leqslant 1. \tag{16}$$

The sum of these inequalities for all $B \subset X_2$, $|B| = v$ leads to

$$\sum_{i=0}^{n_1} \frac{p_{iv}}{\binom{n_1}{i}\binom{n_2}{j}} \leqslant 1$$

because $\sum_{B \subset X_2, |B| = v} q_i(B) = p_{iv}$. The equality in (11) implies that we must have equality in (16) for all $B \subset X_2$, $|B| = v$, including B^*. By Lemma 8, exactly one of $q_i(B^*)$ is non-zero, and by (15) this is $q_u(B^*) = \binom{n_1}{u}$. Therefore all sets $A \subset X_1$, $|A| = u$ are in $\mathscr{F}_2(B^*)$, that is, $A \cup B^* \in \mathscr{F}$ holds for them. But this holds for all $B^* \subset X_2$, $|B^*| = v$, therefore $\mathscr{F}$ includes all sets $A \cup B$, where $A \subset X_1$, $B \subset X_2$, $|A| = u$, $|B| = v$. Hence $p_{uv} = \binom{n_1}{u}\binom{n_2}{v}$. ∎

THEOREM. *Let $X = X_1 \cup X_2$, $X_1 \cap X_2 = \varnothing$, $|X_1| = n_1$, $|X_2| = n_2$. The maximally sized 2-part Sperner families are of the form*

$$\mathscr{F} = \{F: |F \cap X_1| = i,\ |F \cap X_2| = j,\ (i, j) \in I\}$$

where I is a partial transversal described in Lemmas 5–7 (Figs. 1–4.).

Proof. Lemmas 5–7 determined the extreme points maximizing $|\mathscr{F}|$ for the 2-part Sperner families. To prove the theorem we only have to show that no proper convex linear combination of these maximum extreme points can be the profile matrix of a 2-part Sperner family.

Suppose that M is the profile matrix of a 2-part Sperner family and M is a convex linear combination of extreme points described in Lemmas 5–7:

$$M(m_{ij}) = \sum_{k=1}^{m} \lambda_k S(I_k) \qquad \left(\lambda_1, \ldots, \lambda_m \geqslant 0,\ \sum_{k=1}^{m} \lambda_k = 1\right), \tag{17}$$

where $S(I_k)$ is the extreme point determined by the partial transversal I_k:

$$S(I_k) = (s_{ij}^k)_{1 \leqslant i \leqslant n_1, 1 \leqslant j \leqslant n_2}$$

and

$$s_{ij}^k = \begin{cases} \binom{n_1}{i}\binom{n_2}{j} & \text{if } (i, j) \in I_k, \\ 0 & \text{otherwise.} \end{cases}$$

Consider first the case when n_1 and n_2 are of equal parity. By symmetry we may suppose that $n_1 \leqslant n_2$. It is obvious from Lemmas 5 and 6 that all of these $S(I_k)$'s contain exactly one non-zero entry in each row and each column with index j such that $(n_2 - n_1)/2 \leqslant j \leqslant (n_2 + n_1)/2$ (the first $n_1 + 1$ columns in the ordering of the figures). Hence we have

$$\sum_{j=0}^{n_2} \frac{s_{ij}^k}{\binom{n_1}{i}\binom{n_2}{j}} = 1, \qquad (0 \leqslant i \leqslant n_1) \tag{18}$$

and

$$\sum_{i=0}^{n_1} \frac{s_{ij}^k}{\binom{n_1}{i}\binom{n_2}{j}} = 1, \qquad \left(\frac{n_2 - n_1}{2} \leqslant j \leqslant \frac{n_2 + n_1}{2}\right). \tag{19}$$

These inequalities imply

$$\begin{aligned} \sum_{j=0}^{n_2} \frac{m_{ij}}{\binom{n_1}{i}\binom{n_2}{j}} &= \sum_{j=0}^{n_2} \sum_{k=1}^{m} \lambda_k \frac{s_{ij}^k}{\binom{n_1}{i}\binom{n_2}{j}} \\ &= \sum_{k=1}^{m} \lambda_k \sum_{j=0}^{n_2} \frac{s_{ij}^k}{\binom{n_1}{i}\binom{n_2}{j}} \\ &= \sum_{k=1}^{m} \lambda_k = 1, \qquad (0 \leqslant i \leqslant n_1) \end{aligned}$$

and

$$\sum_{i=0}^{n_1} \frac{m_{ij}}{\binom{n_1}{i}\binom{n_2}{j}} = 1 \qquad \left(\frac{n_2 - n_1}{2} \leqslant j \leqslant \frac{n_2 + n_1}{2}\right). \tag{21}$$

On the other hand, all entries m_{ij} with $j < (n_2 - n_1)/2$ or $(n_2 + n_1)/2 < j$ are 0. Therefore, for any u $(0 \leqslant u \leqslant n_1)$ there is a v $((n_2 - n_1)/2 \leqslant v \leqslant (n_2 + n_1)/2)$ satisfying $m_{uv} > 0$. The entries m_{ij} satisfy conditions (9)–(11) of Lemma 9 by (20) and (21). We obtain $m_{uv} = \binom{n_1}{u}\binom{n_2}{v}$. So, in each row i of M there is an entry such that $m_{i,v(i)} = \binom{n_1}{i}\binom{n_2}{v(i)}$. By (20) $v(i)$ are distinct, that is, M is equal to $S(I)$ for some partial transversal I, having exactly one non-zero value in each row and each column between $(n_2 - n_1)/2$ and $(n_2 + n_1)/2$. So $M = S(I_k)$ for some k $(1 \leqslant k \leqslant m)$.

The situation is somewhat different if n_1 and n_2 have different parities. Suppose first that n_1 is even, n_2 is odd, and $n_1 < n_2$. The other cases can be treated analogously.

In this case (as it is easy to see by Lemma 7) $S(I_k)$'s again contain exactly one non-zero entry in each row. It is also true for the columns j such that $(n_2 - n_1 + 1)/2 \leqslant j \leqslant (n_2 + n_1 - 1)/2$. However, columns $(n_2 - n_1 - 1)/2$ and $(n_2 + n_1 + 1)/2$ are exceptional. Exactly one of them contains a non-zero entry of $S(I_k)$. Therefore (18) remains valid, but (19) holds only from $(n_2 - n_1 + 1)/2$ to $(n_2 + n_1 - 1)/2$. The same can be said about (20) and (21).

For any $(0 \leqslant u \leqslant n_1)$ there is a $v = v(u)$ satisfying $m_{uv} > 0$. If $1 \leqslant u \leqslant n_1 - 1$ then $(n_2 - n_1 + 1)/2 \leqslant v(u) \leqslant (n_2 + n_1 - 1)/2$ must hold because no $S(I_k)$ has a non-zero entry with indices $1 \leqslant u \leqslant n_1 - 1$ and $v < (n_2 - n_1 + 1)/2$ or $(n_2 + n_1 - 1)/2 < v$, by Lemma 7. Lemma 9 can be applied for $m_{u,v(u)}$ if $1 \leqslant u \leqslant n_1 - 1$:

$$m_{u,v(u)} = \binom{n_1}{u} \binom{n_2}{v(u)}.$$

A particular case of (20) is the following equality:

$$\sum_{j=0}^{n_2} \frac{m_{0j}}{\binom{n_2}{j}} = 1. \tag{22}$$

Here m_{0j} is the number of members $F \in \mathscr{F}$ such that $F \cap X_1 = \varnothing$, $|F \cap X_2| = j$. Using the notations of the proof of Lemma 9, $m_{0j} = p_j(\varnothing)$. Since $\mathscr{F}_1(\varnothing)$ is a Sperner family, (22) and Lemma 8 lead to $m_{0j} = \binom{n_1}{0}\binom{n_2}{j}$ for some $j = v(0)$. The existence of a $v(n_1)$ such that $m_{n_1, v(n_1)} = \binom{n_1}{n_1}\binom{n_2}{v(n_1)}$ can be proved similarly. (21) implies that $v(0), v(1), \ldots, v(n_1)$ are all distinct. Therefore $M = S(I)$ for some partial transversal I. It must be one of the I_k's. ∎

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