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Convex Hulls of More-Part Sperner Families

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Abstract. The convex hulls of more-part Sperner families is defined and studied. Corollaries of the results are some well-known theorems on 2 or 3-part Sperner families. Some methods are presented giving new theorems.

1. Introduction

In the last 25 years the well known Sperner Theorem [22] has been generalized in many directions. One of these directions are the so called more-part Sperner families.

The first result, proved independently by Katona [13] and Kleitman [18] is the next one.

2-part Sperner Theorem. Let $X = X_1 \cup X_2$, $X_1 \cap X_2 = \varnothing$ *and let* $\mathscr{F}$ *be a family of subsets of* X (i.e. $\mathscr{F} \subseteq 2^X$) *such that* $F_1, F_2 \in \mathscr{F}$ *and any of the next two conditions*

$$\begin{aligned} &F_1 \cap X_1 = F_2 \cap X_1 \quad \text{and} \quad F_1 \cap X_2 \subset F_2 \cap X_2, \\ &F_1 \cap X_1 \subset F_2 \cap X_1 \quad \text{and} \quad F_1 \cap X_2 = F_2 \cap X_2 \end{aligned} \tag{1}$$

imply $F_1 = F_2$. *Then* $|\mathscr{F}| \le \binom{n}{\lfloor n/2 \rfloor}$ *and this is the best upper bound.*

If $X_2 = \varnothing$ then this theorem gives the classic Sperner Theorem. Moreover, if X_1 and X_2 are non-empty then this theorem is a sharpening of it: the conditions of the 2-part Sperner theorem are weaker but the statement is the same.

Let $X = X_1 \cup \cdots \cup X_M$ be a partition of X. $\mathscr{F} \subseteq 2^X$ is called an M-part Sperner family if $\mathscr{F}$ does not contain two members E, F such that $E \cap X_i \subset F \cap X_i$ holds for some i while $E \cap X_j = F \cap X_j$ for all $j \ne i$. A simple example of [13] shows that a 3-part Sperner family may contain more than $\binom{n}{\lfloor n/2 \rfloor}$ members. Papers [17] and [11] found additional conditions to ensure this upper bound. The question of the maximum sized M-part Sperner family is not really solved. [8] and [12] give good asymtotic results. (See also [21].) The very recent [9] is a very good survey of M-part Sperner type problems.

If $\mathscr{F}$ is a family of subsets, its profile is the vector $(p_0, p_1, \ldots, p_n)$ where p_i is the number of i-element members of $\mathscr{F}$. Take the class of all families on X satisfying a certain property. The profiles of the families belonging to this class form a set of points in $\mathbb{R}^{n+1}$ having integer components. In [2], [3] and [7] the extreme points of this set are determined for several classes of families.

The main aim of the present paper is to obtain analogous results for M-part Sperner families and other families given on the M-parts. Here the profile will be, however, an M-dimensional matrix whose $i_1 i_2 \ldots i_M$th entry is the number of members of the family satisfying $|F \cap X_j| = i_j$ $(1 \leq j \leq M)$. Our Theorem 2.1 determines the extreme points of the class of the M-part Sperner families.

In Section 3 we prove this theorem in a much more general form (Theorem 3.2). Namely, we show that the extreme points (matrices) of many classes of families on M parts mimic the properties of the class. Griggs, Odlyzko and Shearer [12] proved that there are (sizewise) homogeneous families $\mathscr{F}$ maximizing $|\mathscr{F}|$ for M-part Sperner families. A consequence of Theorem 3.2 is that this is true for many other classes, too.

In Section 4 several known results are deduced as consequences of the theorems of Sections 2 and 3. Theorem 4.2 is the above mentioned 2-part Sperner theorem. Theorems 4.3 [20] and 4.6 [10] consider Sperner families of subsets meeting the first part (of 2 parts) in at least l elements.

Let us remark that the families themselves are really investigated only until the extreme points are determined. To deduce the above mentioned consequences in Section 4, we deal with "numbers", only. The families can be forgotten.

2. Definitions, Notations and the Main Result

In this section we introduce the necessary definitions, notations and formulate the main theorem. It will be analogous to [2] and [3].

Put $X = X_1 \cup X_2 \cup \cdots \cup X_M$, $X_i \cap X_j = \varnothing$ $(i \neq j)$, $|X_i| = n_i$, $|X| = n$. Let $\mathscr{H} \subset 2^X$ be a family of subsets. The M-dimensional matrix $P(\mathscr{H}) := (p_{i_1,\ldots,i_M})$ $i_j = 0, \ldots, n_j$, is called the *profile-matrix* of $\mathscr{H}$, where

$$p_{i_1,\ldots,i_M}(\mathscr{H}) := |\{H \in \mathscr{H} : \forall_j |H \cap X_j| = i_j\}|.$$

The matrix $P(\mathscr{H})$ can also be considered as a vector of $(n_1 + 1)(n_2 + 1)\ldots(n_M + 1)$ components. If $M = 1$ then we obtain the *profile* $P(\mathscr{H})$ of $\mathscr{H}$ (see [2] or [3]). In this way $P(\mathscr{H})$ is a point of the Euclidean space $\mathbb{R}^{(n_1+1)\ldots(n_M+1)} = \mathbb{R}^N$.

If α is a finite set in $\mathbb{R}^N$, the *convex hull* $\langle\alpha\rangle$ of α is the set of all convex linear combinations of the elements of α. We say that $e \in \alpha$ is an *extreme point* of α iff e is not a convex linear combination of elements of α different from e. The set of extreme points of α is $\varepsilon(\alpha)$. It is well-known that $\langle\alpha\rangle$ is equal to the set of all convex linear combinations of its extreme points. That is, the determination of the convex hull of a set is equivalent to finding its extreme points.

Suppose that the partition $X = X_1 \cup \cdots \cup X_M$ is fixed. Let $\mathbb{A} \subset 2^{2^X}$ be a class of families of subsets of X. Denote by $\mu(\mathbb{A})$ the set of all profile-matrices of the families belonging to $\mathbb{A}$. ($\mu(\mathbb{A}) \subset \mathbb{R}^N$ is obvious.)

A family $\mathscr{H}$ is called an *M-part Sperner family* if $E, F \in \mathscr{H}$, $E \cap X_i \subset F \cap X_i$, and $E \cap X_j = F \cap X_j$ for all $j \neq i$ (for all fixed i) imply $E = F$. Denote by $\mathbb{S}^M$ the family of all M-part Sperner families for the given partition. (The notation should contain $X_1, \ldots, X_M$; they are omitted for sake of brevity.)

Theorem 2.1. *The extreme points of* $\langle \mu(\mathbb{S}^M) \rangle$ *are the matrices* $S(I)$ *with the entries*

$$S_{i_1,\ldots,i_M}(I) = \begin{cases} 0 & \text{if } (i_1, \ldots, i_M) \notin I, \\ \binom{n_1}{i_1}\binom{n_2}{i_2}\cdots\binom{n_M}{i_M} & \text{otherwise,} \end{cases}$$

where I *is any set in* $\{0, \ldots, n_1\} \times \{0, \ldots, n_2\} \times \cdots \times \{0, \ldots, n_M\}$ *such that* $(i_1, \ldots, i_j, \ldots, i_M) \in I$, $(i_1, \ldots, i_j', \ldots, i_M) \in I$ *imply* $i_j = i_j'$.

3. Proof of the Main Theorem

The proof is analogous to those of [2] and [3]. First we formulate and prove a lemma similar to the cyclic permutation method (see [15]) then we prove our main theorem.

We say that $\mathscr{L}$ is a *product-chain* of $X = X_1 \cup \cdots \cup X_M$ if $\mathscr{L} = (x_1, \ldots, x_n)$ is a permutation of X and $\{x_i: i = n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j\} = X_j$, that is, if the elements of X_j are consecutive in $\mathscr{L}$ for every j. Furthermore, we say that a subset $H \subset X$ is *initial with respect to* $\mathscr{L}$ if

$$H \cap X_j = \{x_{n_1+\cdots+n_{j-1}+1}, \ldots, x_{n_1+\cdots+n_{j-1}+|H\cap X_j|}\},$$

that is, equal to the set of the first $|H \cap X_j|$ elements of X_j. If $\mathscr{H} \subset 2$ then $\mathscr{H}(\mathscr{L})$ denotes those members of $\mathscr{H}$ which are initial with respect to $\mathscr{L}$. It is easy to see that the profile-matrix $P(\mathscr{H}(\mathscr{L}))$ is a 0,1 matrix.

Sometimes it is easier to use the set of places with entry 1 rather than the 0,1 matrix itself. Let I be a subset of $\{0, \ldots, n_1\} \times \{0, \ldots, n_2\} \times \cdots \times \{0, \ldots, n_M\}$. Then $T(I)$ denotes the matrix having 1's in the place belonging to I and 0's otherwise:

$$t_{i_1,\ldots,i_M}(I) = \begin{cases} 1 & \text{if } (i_1, \ldots, i_M) \in I, \\ 0 & \text{if } (i_1, \ldots, i_M) \notin I. \end{cases}$$

The definition of $S(I)$ given in Theorem 2.1 is analogous, but 1 is replaced by $\binom{n_1}{i_1}\binom{n_2}{i_2}\cdots\binom{n_M}{i_M}$.

If $\mathbb{A} \subset 2^{2^X}$, introduce the notation $\mathbb{A}(\mathscr{L}) = \{\mathscr{H}(\mathscr{L}): \mathscr{H} \in \mathbb{A}\}$ for any product-chain $\mathscr{L}$. Then, by definition, $\mu(\mathbb{A}(\mathscr{L})) = \{P(\mathscr{H}(\mathscr{L})): \mathscr{H} \in \mathbb{A}\}$. As $P(\mathscr{H}(\mathscr{L}))$ are 0,1 matrices, the set of extreme points $\varepsilon(\mathbb{A}(\mathscr{L})) = \mu(\mathbb{A}(\mathscr{L}))$.

The next lemma plays a fundamental role in the proof of Theorem 2.1 and is analogous to Theorem 4 (Blowing up the circle) of [3]. It shows an important connection between $\varepsilon(\mathbb{A})$ and $\varepsilon(\mathbb{A}(\mathscr{L}))$.

Lemma 3.1. (*Blowing up the product-chain*). *Suppose that* $\varepsilon(\mathbb{A}(\mathscr{L})) = \mu(\mathbb{A}(\mathscr{L}))$ *does not depend on* $\mathscr{L}$. *Then*

$$\mu(\mathbb{A}) \subseteq \langle \{S(I): T(I) \in \varepsilon(\mathbb{A}(\mathscr{L}))\} \rangle \tag{2}$$

holds.

Proof. To prove (2) it is enough to show that for every $\mathscr{H} \in \mathbb{A}$ there are coefficients $\mu(I)$ $(\mu(I) \geq 0, \sum_{T(I) \in \varepsilon(\mathbb{A}(\mathscr{L}))} \mu(I) = 1)$ such that $P(\mathscr{H}) = \sum_{T(I) \in \varepsilon(\mathbb{A}(\mathscr{L}))} \mu(I) S(I)$.

Consider the sum

$$\sum P(\{H\})/n_1!n_2!\dots n_M! \tag{3}$$

for all pairs $(\mathscr{L}, H)$ where $\mathscr{L}$ is a product-chain, $H \in \mathscr{H}$ and H is initial with respect the product-chain $\mathscr{L}$. We use double counting for this sum.

$$\begin{aligned}\sum_{(\mathscr{L},H)} P(\{H\})/n_1!\dots n_M! &= \sum_{\mathscr{L}} \frac{1}{n_1!\dots n_M!} \sum_{\substack{H \text{ is initial with} \\ \text{respect to } \mathscr{L}}} P(\{H\}) \\ &= \sum_{\mathscr{L}} \frac{1}{n_1!\dots n_M!} P(\mathscr{H}(\mathscr{L})).\end{aligned} \tag{4}$$

Observe that $P(\mathscr{H}(\mathscr{L})) \in \mu(\mathbb{A}(\mathscr{L})) = \varepsilon(\mathbb{A}(\mathscr{L}))$, therefore for any $\mathscr{L}$ there is a unique I such that $T(I) = P(\mathscr{H}(\mathscr{L}))$. Collect the equal terms on the right hand side of (4):

$$\sum_{\mathscr{L}} \frac{1}{n_1!\dots n_M!} P(\mathscr{H}(\mathscr{L})) = \sum_{T(I) \in \varepsilon(\mathbb{A}(\mathscr{L}))} \lambda(I) T(I) \tag{5}$$

where $\lambda(I)$ is the proportion of the $n_1!\dots n_M!$ permutations such that $P(\mathscr{H}(\mathscr{L})) = T(I)$. Therefore the sum of the coefficients $\lambda(I)$ is 1. Hence we have

$$\sum_{(\mathscr{L},H)} P(\{H\})/n_1!\dots n_M! \tag{6}$$

is a convex linear combination of the elements of $\varepsilon(\mathbb{A}(\mathscr{L}))$.

On the other hand, summing the other way around we obtain

$$\begin{aligned}\sum_{(\mathscr{L},H)} P(\{H\})/n_1!\dots n_M! &= \sum_{H}\sum_{\mathscr{L}} P(\{H\})/n_1!\dots n_M! \\ &= \sum_{H \in \mathscr{H}} \frac{i_1!(n_1 - i_1)!\dots i_M!(n_M - i_M)!}{n_1!\dots n_M!} P(\{H\})\end{aligned} \tag{7}$$

where $|H \cap X_j| = i_j$ $(1 \leq j \leq M)$. $i_1!(n_1 - i_1)!\dots i_M!(n_M - i_M)!$ is the number of product-chains satisfying this condition and H is initial with respect to these $\mathscr{L}$. The last sum is equal to

$$\sum_{H \in \mathscr{H}} \frac{P(\{H\})}{\binom{n_1}{i_1}\cdots\binom{n_M}{i_M}} = \left(\frac{p_{i_1,\dots,i_M}(\mathscr{H})}{\binom{n_1}{i_1}\cdots\binom{n_M}{i_M}}\right)_{\forall j: i_j = 0,\dots,n_j} \tag{8}$$

It follows by (6), (7) and (8) that the right hand side of (8) is a convex linear combination of the elements of $\varepsilon(\mathbb{A}(\mathscr{L}))$. This implies that $P(\mathscr{H})$ is a convex linear combination of the matrices $S(I)$ such that $T(I) \in \varepsilon(\mathbb{A}(\mathscr{L}))$. □

Lemma 3.1 becomes really useful if $S(I) \in \mu(\mathbb{A})$ holds for all $T(I) \in \varepsilon(\mathbb{A}(\mathscr{L}))$. Then, by (2), any element of $\mu(\mathbb{A})$ is a convex linear combination of them. We use this idea to prove Theorem 2.1.

Proof of Theorem 2.1. Lemma 3.1 will be applied with $\mathbb{A} = \mathbb{S}^M$.

The elements of $\mu(\mathbb{S}^M(\mathscr{L}))$ are 0,1 matrices. $I \subset \{0,\dots,n_1\} \times \{0,\dots,n_2\} \times \dots \times \{0,\dots,n_M\}$ is called a *partial transversal* if $(i_1,\dots,i_j,\dots,i_M) \in I$ and $(i_1,\dots,$

$i'_j, \ldots, i_M) \in I$ imply $i_j = i'_j$. It is easy to see that $\mu(\mathbb{S}^M(\mathscr{L})) = \{T(I)$: I is a partial transversal$\}$.

It does not depend on $\mathscr{L}$, therefore the lemma can be applied. It follows by (2) that any element of $\mu(\mathbb{S}^M)$ is a convex linear combination of the matrices $S(I)$ where I are partial transversals.

For a partial transversal I take the family

$$\mathscr{F} = \bigcup_{(i_1, \ldots, i_M) \in I} \{F \colon |F \cap X_j| = i_j \text{ for all } 1 \leq j \leq M\}.$$

which is obviously an M-Sperner family, that is, the matrices $S(I)$ are all in $\mu(\mathbb{S}^M)$. On the other hand, none of them is a convex linear combination of the other ones. Hence they are the extreme points of $\mu(\mathbb{S}^M)$. □

Lemma 3.1 and the above (trivial) proof allow a much more general form:

Theorem 3.2. *Suppose that* $\mathbb{A} \subseteq 2^{2^X}$ *satisfies the following two conditions*:

$$\varepsilon(\mathbb{A}(\mathscr{L})) = \mu(\mathbb{A}(\mathscr{L})) \text{ does not depend on } \mathscr{L}, \tag{9}$$

$$T(I) \in \mu(\mathbb{A}(\mathscr{L})) \text{ implies } S(I) \in \mu(\mathbb{A}). \tag{10}$$

Then

$$\varepsilon(\mathbb{A}) = \{S(I) \colon T(I) \in \mu(\mathbb{A}(\mathscr{L})), \tag{11}$$

that is, the extreme points of $\mu(\mathbb{A})$ *are obtained by replacing the* 1's *by* $\binom{n_1}{i_1} \cdots \binom{n_M}{i_M}$ *in the* $i_1, \ldots, i_M$*th entries of the elements of* $\mu(\mathbb{A}(\mathscr{L}))$.

In what follows, we show many consequences of this general theorem.

A family $\mathscr{F}$ on $X = X_1 \cup \cdots \cup X_M$ is called *homogeneous* if $F \in \mathscr{F}$ implies $G \in \mathscr{F}$ for all $G \subset X$ satisfying $|F \cap X_j| = |G \cap X_j|$ (for all $1 \leq j \leq M$). Observe that the matrices $S(I)$ in the theorem are profile-matrices of homogeneous families belonging to $\mathbb{A}$.

$$|\mathscr{F}| = \sum_{i_1=0}^{n_1} \cdots \sum_{i_M=0}^{n_M} p_{i_1, \ldots, i_M}$$

where $p_{i_1, \ldots, i_M}$ are the entries of the profile-matrix $P(\mathscr{F})$. This is a linear function of $p_{i_1, \ldots, i_M}$ so it assumes its maximum in one of the extreme points given in Theorem 3.2. As we mentioned, these points are profile-matrices of homogeneous families. Therefore the following statement is true:

Corollary 3.3. *Suppose that* $\mathbb{A}$ *satisfies* (9) *and* (10). *Then there is a homogeneous* $\mathscr{F} \subset \mathbb{A}$ *maximizing* $|\mathscr{F}|$ *in* $\mathbb{A}$.

Of course, the proof of the corollary above shows that it is true for any linear function of the variables $p_{i_1, \ldots, i_M}$.

Griggs, Odlyzko and Shearer [12] proved Corollary 3.3 for the special case when $\mathbb{A}$ is $\mathbb{S}^M$, the class of all M-part Sperner families. In [9], jointly with Füredi, they independently proved it for several other special cases. Of course, they used other means.

The next theorem presents the extreme points of the set of the profile-matrices of the Sperner families ($= 1 -$ Sperner families). That is, we consider the Sperner-condition on X, but the profile-matrices are defined by the partition $X = X_1 \cup \cdots \cup X_M$. $\mathbb{S}^1 = \mathbb{S}$ denotes the set of all Sperner families on X. We call the set $I \subset \{0, \ldots, n_1\} \times \{0, \ldots, n_2\} \times \cdots \times \{0, \ldots, n_M\}$ an *antichain* iff $(i_1, \ldots, i_M)$, $(j_1, \ldots, j_M) \in I$ and $i_1 \le j_1, \ldots, i_M \le j_M$ imply $i_1 = j_1, \ldots, i_M = j_M$.

Theorem 3.4.

$$\varepsilon(\mathbb{S}) = \{S(I)\colon I \text{ is an antichain}\}.$$

The *proof* is analogous to that of Theorem 2.1. □

This theorem can be generalized toward a theorem of Paul Erdös [1]. A family $\mathscr{F} \subset 2^X$ is called *k-Sperner* iff it contains no $k + 1$ distinct members satisfying $F_1 \subset F_2 \subset \cdots \subset F_{k+1}$. Similarly, a set $I \subset \{0, \ldots, n_1\} \times \cdots \times \{0, \ldots, n_M\}$ is a *k-antichain* iff it contains no $k + 1$ distinct elements $(i_1^1, \ldots, i_M^1), \ldots, (i_1^{k+1}, \ldots, i_M^{k+1})$ such that $i_j^1 \le i_j^2 \le \cdots \le i_j^{k+1}$ holds for all j $(1 \le j \le M)$.

Theorem 3.5.

$$\mu(k\text{-Sperner families}) = \{S(I)\colon I \text{ is a } k\text{-antichain}\}. \qquad \square$$

We need some more definitions for another variant of these theorems. The family $\mathscr{F} \subset 2^X$ is an *M-part k-Sperner family* iff there are no $k + 1$ distinct members such that $F_1 \subset F_2 \subset \cdots \subset F_{k+1}$ and $F_{k+1} - F_1 \subset X_j$ holds for some j $(1 \le j \le M)$. Analogously, a partial k-transversal I contains no $k + 1$ distinct members in $\{0, \ldots, n_1\} \times \cdots \times \{0, \ldots, n_M\}$ with $M - 1$ equal components.

Theorem 3.6.

$$\mu(M\text{-part } k\text{-Sperner families}) = \{S(I)\colon I \text{ is a partial } k\text{-transversal}\}.$$

We mention 3 other theorems determining $\max\{|\mathscr{F}|\colon \mathscr{F} \in \mathbb{A}\}$ for some classes $\mathbb{A}$ on 2 and 3 parts. 1) [14] gives a common generalization of the 2-part Sperner theorem and Paul Erdös's k-Sperner theorem. 2) [17] gives an additional condition to ensure the maximum to be $\binom{n}{\lfloor n/2 \rfloor}$. 3) [11] gives a nicer addition condition to ensure the same conclusion. In all these cases the extreme points can be easily determined by Theorem 3.2.

We developed Theorem 3.2 for classes of families on M parts, however, for our great surprise, it gives new results for 1 part, too. Let $\mathbb{F}_k$ denote the class of families $\mathscr{F} \subset 2^X$ $(|X| = n)$ such that $F_1 \subset F_2$, $F_1 \neq F_2$ imply $|F_2 - F_1| \ge k$. $\max\{|\mathscr{F}|\colon \mathscr{F} \in \mathbb{F}_k\}$ was determined for this class in [15]. Theorem 3.2 allows us to describe the extreme points, too.

Theorem 3.7. *The extreme points of the set $\mu(\mathbb{F}_k)$ of the profile-vectors are of the form $\left(0, \ldots, 0, \binom{n}{i_1}, 0, \ldots, 0, \binom{n}{i_2}, 0, \ldots, 0, \binom{n}{i_r}, 0, \ldots, 0\right)$ where $r \ge 0$ and the non-zero components are separate with at least $k - 1$ zeros.*

This theorem was independently discovered and proved by our-friend P. Frankl [6], too.

Kleitman [19] observed, that the result maximizing $|\mathscr{F}|$ in $\mathbb{F}_k$ can be generalized for other classes. This can be done here, again. If $\mathbb{A} \subset 2^{2^X}$ is a class of families (no partition of X) and $\mathbb{A}$ is described by conditions of type ($\nexists F_1, \ldots, F_m \in \mathscr{F}$ such that $F_1 \subset F_2 \subset \cdots \subset F_m$ and $|F_m - F_{m-1}|, \ldots, |F_2 - F_1|$ are in a certain set) then Theorem 3.2 determines the extreme points, therefore $\max|\mathscr{F}|$ can also be calculated.

4. Applications: Old Theorems as Consequences

In this section we will prove some old results using the previous theorems.

Actually, Theorem 2.1 is nothing else but the determination of the possible constructions maximizing any given linear function of the entries $p_{i_1,\ldots,i_M}$ of the profile matrix. However, the number of possible maximal constructions is too large for most of the functions. It is very hard to compare them to determine the really best one.

The most interesting case, of course, when the linear function is just the sum $\sum p_{i_1,\ldots,i_M}$, that is, when $|\mathscr{F}|$ has to be maximized. If $M = 2$ the above comparison can be done by the following lemma.

Lemma 4.1. *Suppose that $a_1 \geq \cdots \geq a_u \geq 1$ and $b_1 \geq \cdots \geq b_v \geq 1$ are integers. Choose entries from the matrix $(a_i \cdot b_j)_{1 \leq i \leq u, 1 \leq j \leq v}$, at most one from each row and each column. Their sum cannot exceed*

$$\sum_{i=1}^{\min\{u,v\}} a_i \cdot b_i.$$

Proof. The statement is proved by induction on $\min\{u,v\}$. Let $I \subset \{1, \ldots, u\} \times \{1, \ldots, v\}$ be a set containing at most one element in each row and each column. We shall show that there is an I' containing $(1, 1)$ and satisfying

$$\sum_{(i,j)\in I} a_i \cdot b_j \leq \sum_{(i,j)\in I'} a_i \cdot b_j. \tag{12}$$

Suppose first that I contains no element with 1 as a component. Then $I' = I \cup \{(1, 1)\}$ satisfies the requirements. If $(1, x) \in I$ but $(y, 1) \notin I$ for all y then choose $I' = (I - \{(1, x)\}) \cup \{(1, 1)\}$. Finally, if $(1, x) \in I$ $(y, 1) \in I$, then let $I' = (I - \{(1, x), (y, 1)\}) \cup \{(1, 1), (y, x)\}$.

To verify (12) take the difference of the two sides:

$$\sum_{(i,j)\in I'} a_i \cdot b_j - \sum_{(i,j)\in I} a_i \cdot b_j = a_1 b_1 + a_y b_x - a_1 b_x - a_y b_1 = (a_1 - a_y)(b_1 - b_x) \geq 0.$$

This proves that $(1, 1) \in I$ can be supposed. The induction hypothesis, applied with $a_2, \ldots, a_u$ and $b_2, \ldots, b_v$, proves the desired result. □

Apply this lemma with

$$\binom{n_1}{\left\lceil \frac{n_1}{2} \right\rceil} \geq \binom{n_1}{\left\lceil \frac{n_1}{2} \right\rceil - 1} \geq \binom{n_1}{\left\lceil \frac{n_1}{2} \right\rceil + 1} \geq \binom{n_1}{\left\lceil \frac{n_1}{2} \right\rceil - 2} \geq \ldots \quad \text{and}$$

$$\binom{n_2}{\left\lfloor \frac{n_2}{2} \right\rfloor} \geq \binom{n_2}{\left\lfloor \frac{n_2}{2} \right\rfloor + 1} \geq \binom{n_2}{\left\lfloor \frac{n_2}{2} \right\rfloor - 1} \geq \binom{n_2}{\left\lfloor \frac{n_2}{2} \right\rfloor + 2} \geq \ldots$$

and place of the a's and b's, respectively. (12) is equal to

$$\binom{n_1}{\lceil \frac{n_1}{2} \rceil}\binom{n_2}{\lfloor \frac{n_2}{2} \rfloor} + \binom{n_1}{\lceil \frac{n_1}{2} \rceil - 1}\binom{n_2}{\lfloor \frac{n_2}{2} \rfloor + 1} + \binom{n_1}{\lceil \frac{n_1}{2} \rceil + 1}\binom{n_2}{\lfloor \frac{n_2}{2} \rfloor - 1} + \cdots$$

$$= \sum_i \binom{n_1}{i}\binom{n_2}{\lceil \frac{n_1}{2} \rceil + \lfloor \frac{n_2}{2} \rfloor - i} = \binom{n_1 + n_2}{\lfloor \frac{n_1 + n_2}{2} \rfloor}$$

in this case.

Let $\mathscr{F} \in \mathbb{S}^2$ $(X = X_1 \cup X_2, |X_1| = n_1, |X_2| = n_2, n_1 + n_2 = n)$. By Theorem 2.1 (or [12]) we know that

$$|\mathscr{F}| \leq \max_I \sum_{(i,j) \in I} \binom{n_1}{i}\binom{n_2}{j}$$

where the maximum is over all partial transversals I. By Lemma 4.1 and the above calculations this sum is at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$:

Theorem 4.2. (Kleitman [18], Katona [13])

$$\max\{|\mathscr{F}|: \mathscr{F} \in \mathbb{S}^2\} = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Now, let X be again partitioned $X_1 \cup X_2$ and denote by $\mathbb{S}(l)$ the set of Sperner families $\mathscr{F} \subset 2^X$ $(F_1, F_2 \in \mathscr{F}, F_1 \neq F_2 \Rightarrow F_1 \not\subset F_2)$ satifying the additional condition

$$F \in \mathscr{F} \Rightarrow |F \cap X_1| \geq l.$$

Theorem 4.3. (Lih [20]) *If* $0 < n_1 \leq n_2$ *then*

$$\max\{|\mathscr{F}|: \mathscr{F} \in \mathbb{S}(1)\} = \binom{n}{\lceil \frac{n}{2} \rceil} - \binom{n_2}{\lceil \frac{n}{2} \rceil}.$$

Proof. If $\mathscr{F} \in \mathbb{S}(1)$ then its profile-matrix $P(\mathscr{F}) = (p_{ij})$ has only zeros in the 0th row. Hence $|\mathscr{F}| = \sum_{i=1}^{n_1} \sum_{j=0}^{n_2} p_{ij}$ follows. The latter sum is a linear combination of the entries of $P(\mathscr{F})$. It can be maximum only for the extreme points determined by Theorem 3.4. Therefore, we have to find

$$\max_I \sum_{(i,j) \in I} \binom{n_1}{i}\binom{n_2}{j}$$

where the maximum runs over all antichains $I \subset \{1, \ldots, n_1\} \times \{0, \ldots, n_2\}$. As an antichain is necessarily a partial transversal, we may use Lemma 4.1 with binomial coefficients, again. The only difference is that $\binom{n_1}{0}$ is missing now. If n_1 is odd, $\binom{n_1}{0}$

is the last one in the above ordering. Its coefficient is

$$\binom{n_2}{\left\lceil \frac{n_1}{2} \right\rceil + \left\lfloor \frac{n_2}{2} \right\rfloor}.$$

The sum is smaller by this amount. On the other hand, if n_1 is even then $\binom{n_1}{n_1}$ is the last number in the ordering, it leaps forward to replace $\binom{n_1}{0}$. Therefore the coefficient of $\binom{n_1}{n_1}$, that is,

$$\binom{n_2}{\left\lceil \frac{n_1}{2} \right\rceil + \left\lfloor \frac{n_1}{2} \right\rfloor - n_1}$$

will be deleted. In both cases the missing term is $\binom{n_2}{\left\lceil \frac{n}{2} \right\rceil}$. □

Remark 4.4. Actually, we proved a sharper theorem than Lih's one. We used only the 2-part Sperner condition rather than the usual Sperner condition. On the other hand, we did not determine here the optimal families as Griggs [10] did. It could be done with the present method, but it is rather inconvenient. (See the analogous result on all 2-part Sperner families in [5].)

In some further investigations the following inequality will be needed. We think it is interesting in its own right, too.

Lemma 4.5. *If* $I \subset \{0, \ldots, n_1\} \times \{0, \ldots, n_2\}$ *is an antichain then*

$$\sum_{(i,j) \in I} \frac{\binom{n_1}{i}\binom{n_2}{j}}{\binom{n_1 + n_2}{i + j}} \le 1 \tag{13}$$

holds.

Proof. Let I be an antichain. Suppose that $(i,j) \in I$ and that $(I - \{i,j\}) \cup \{(i+1,j)\}$ is also an antichain. An easy calculation shows that

$$\frac{\binom{n_1}{i}\binom{n_2}{j}}{\binom{n_1 + n_2}{i + j}} < \frac{\binom{n_1}{i+1}\binom{n_2}{j}}{\binom{n_1 + n_2}{i + 1 + j}}$$

holds iff $\dfrac{n_2}{n_1 + 1} < \dfrac{j}{i + 1}$. In this case the sum (13) is increased. We say briefly that the *change*

$$(i,j) \to (i+1,j) \text{ is } \textit{increasing} \text{ iff } \frac{n_2}{n_1+1} < \frac{j}{i+1}. \tag{14}$$

One can see in the same way that

$$(i,j) \to (i-1,j) \text{ is increasing iff } \frac{n_2}{n_1+1} > \frac{j}{i}, \tag{15}$$

$$(i,j) \to (i,j+1) \text{ is increasing iff } \frac{n_1}{n_2+1} < \frac{i}{j+1}, \tag{16}$$

and

$$(i,j) \to (i,j-1) \text{ is increasing iff } \frac{n_1}{n_2+1} > \frac{i}{j}. \tag{17}$$

Suppose that I is an antichain maximizing the left hand side of (13). We will show that I consists of all pairs with a constant $i+j$.

Order the elements of I according to their first components. It is easy to see that the second components are then ordered backwards. Let (i,j) and $(i+k, j-l)$, $k \geq 1$, $l \geq 1$, be two neighbouring elements of I in this order. Suppose first that $k \geq 2$. The change $(i,j) \to (i+1,j)$ cannot be increasing by the optimality of I. Therefore $\frac{n_2}{n_1+1} \geq \frac{j}{i+1}$ must hold by (14). Hence $\frac{n_2}{n_1+1} > \frac{j-l}{i+k}$ follows, consequently $(i+k, j-l)$ $(i+k-1, j-l)$ is an increasing change by (15). This contradiction proves $k=1$.

Suppose now $l \geq 2$. The change $(i,j) \to (i,j-1)$ cannot be increasing by the optimality of I. Therefore $\frac{n_1}{n_2+1} \leq \frac{i}{j}$ must hold by (17). Hence $\frac{n_1}{n_2+1} < \frac{i+1}{j-l+1}$ follows, consequently $(i+1, j-l) \to (i+1, j-l+1)$ is an increasing change by (16). This contradiction proves $l=1$.

We have proved that the neighbouring elements of I are of the form (i,j) and $(i+1, j-1)$. Consequently, there is a constant c such that the optimal I consists of all elements (i,j) satisfying $i+j=c$. For such an I (13) holds with equality, whence it holds for any antichain. □

It will be easy to prove the next theorem after this lemma.

Theorem 4.6. (Griggs [10]) *$\mathscr{F} \in \mathbb{S}(l)$ implies the inequality*

$$\sum_{F \in \mathscr{F}} \frac{1}{\binom{n-l}{|F|-l}} \frac{\binom{|F \cap X_1|}{l}}{\binom{|X_1|}{l}} \leq 1. \tag{18}$$

Proof. If $\mathscr{F} \in \mathbb{S}(l)$ then its profile-matrix $P(\mathscr{F}) = (p_{ij})$ has only zeros in the 0th, 1st, $\ldots, (l-1)$st rows. (18) can be rewritten into the form

$$\sum_{i=l}^{n_1} \sum_{j=0}^{n_2} p_{ij} \frac{1}{\binom{n-l}{i+j-l}} \frac{\binom{i}{l}}{\binom{n_1}{l}} \leq 1. \tag{19}$$

The left hand side of (19) is a linear combination of the entries of $P(\mathscr{F})$. It attains its maximum for one of the extreme points determined by Theorem 3.4. Therefore, we have to prove (19) only for these extreme points:

$$\sum_{(i,j)\in I} \binom{n_1}{i}\binom{n_2}{j} \frac{1}{\binom{n-l}{i+j-l}} \frac{\binom{i}{l}}{\binom{n_1}{l}} \le 1$$

should be proved for any antichain (not containing elements with a first component $< l$). It is easy to check that this is equivalent to

$$\sum_{(i,j)\in I} \frac{\binom{n_1-l}{i-l}\binom{n_2}{j}}{\binom{n_1+n_2-l}{i+j-l}} \le 1.$$

This latter inequality is a consequence of (13). □

Of course, the original proof in [10] is easier and shorter than ours, which however shows some new connections.

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