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Polytopes Determined by Hypergraph Classes

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Let f_i denote the number of i-element members of a given family of subsets of a finite n-element set (=hypergraph). $(f_0, f_1, \ldots, f_n)$ is the profile of the hypergraph. The authors in two papers jointly written with Péter L. Erdös have determined the extreme points of the set of profiles for several hypergraph classes. This paper presents new short proofs.

1. Introduction

Let S be an n-element set and $\mathscr{F}$, a family of subsets of S, i.e., $\mathscr{F} \subset 2^S$. For $0 \leq i \leq n$, we denote by $\mathscr{F}_i$ the collection of i-element subsets in $\mathscr{F}$:

$$\mathscr{F}_i = \{F \in \mathscr{F} : |F| = i\}, \qquad f_i = |\mathscr{F}_i|.$$

The vector $\boldsymbol{f} = (f_0, f_1, \ldots, f_n)$ is called the *profile* of the family $\mathscr{F}$. We consider $\boldsymbol{f}$ as a point in $\mathbb{R}^{n+1}$, the $(n+1)$-dimensional Euclidean space. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n+1}$, $\boldsymbol{x}$ *dominates* $\boldsymbol{y}$ iff $x_i \geq y_i$ holds for $i = 0, \ldots, n$.

Definition 1.1. For a finite subset $X \subset \mathbb{R}^{n+1}$ we say that D is the *dominating* set of X if (a) $D \subset X$, (b) for all $\boldsymbol{x} \in X$ there exist $\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_j \in D$ and positive real numbers $\alpha_1, \ldots, \alpha_j$ so that $\sum \alpha_i = 1$ and $\sum \alpha_i \boldsymbol{d}_i$ dominates $\boldsymbol{x}$, (c) D is minimal with respect to these properties. The elements of D are called *dominating vertices*.

It is not hard to see that D consists of exactly those vertices of the convex hull of X which are not dominated by any other vertex.

Most extremal hypergraph problems can be formulated in the following way. Suppose we are given a weight function $w: \{0, \ldots, n\} \to \mathbb{Z}^+$, i.e. $w(i) \geq 0$ for all i. What is the maximum of $\sum_{i=0}^{n} w(i) f_i$ over all families $\mathscr{F} \subset 2^S$ satisfying certain properties (e.g. $F \cap F' \neq \varnothing$ holds for all $F, F' \in \mathscr{F}$)? That is, we have to find the maximum of a linear function with non-negative coefficients over the set X of all possible profile vectors. Clearly, this maximum equals the maximum over all dominating vertices. Thus, for most problems, it is sufficient to determine the dominating set of possible profile vectors.

This was done in [5] and [6] for some classes of hypergraphs. However, the proofs were lengthy and relied heavily on the duality theorem of linear programming. On the other hand [6] developed a unified treatment of these problems using no result of extremal set theory. Here we propose much shorter individual proofs using extremal set theory. Our theorems are generalizations of old results of extremal types. The connections and consequences can be found in [5] and [6].

2. Statement of the Results

Definition 2.1. Suppose k is a positive integer. The family $\mathscr{F}$ is called a k-Sperner family if there are no $k+1$ distinct members $F_0, F_1, \ldots, F_k$ forming a chain: $F_0 \subset F_1 \subset \cdots \subset F_k$.

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THEOREM 2.2. *The dominating set of the profile vectors of k-Sperner families consists exactly of the points*

$$\left(0, 0, \ldots, \binom{n}{i_1}, 0, \ldots, \binom{n}{i_2}, \ldots, 0, \binom{n}{i_k}, 0 \ldots 0\right),$$
$$0\ 1\ \ldots\qquad i_1\ i_1+1\qquad i_2\qquad i_k-1\ i_k\ i_k+1\qquad n.$$

that is, D is formed by the vectors having 0 or $\binom{n}{i}$ in the ith position and having exactly k non-zero entries.

A family $\mathcal{F}$ is called intersecting if $F \cap F' \neq \varnothing$ holds for all $F, F' \in \mathcal{F}$.

THEOREM 2.3. *The dominating set of profile vectors of intersecting families consists exactly of the points*

$$\mathbf{d}_i = \left(0, 0, \ldots, 0, \binom{n-1}{i-1}, \binom{n-1}{i}, \ldots, \binom{n-1}{n-i-1}, \binom{n}{n-i+1}, \ldots, \binom{n}{n}\right), \quad 1 \leqslant i \leqslant \frac{n+1}{2}.$$
$$0\ 1\ \ldots\qquad i\qquad i+1\ \ldots\qquad n-i\ \ldots\qquad n.$$

A family $\mathcal{F}$ is called *intersecting Sperner* if it is intersecting and 1-Sperner.

THEOREM 2.4. *The dominating set of profiles of intersecting Sperner families consists exactly of the points*

$$\mathbf{v}_j = \left(0, \ldots, 0, \binom{n}{j}, 0, \ldots, 0\right), j > \frac{n}{2}, \quad \mathbf{w}_{i,j} = \left(0, \ldots, 0, \binom{n-1}{i-1}, 0, \ldots, 0, \binom{n-1}{j}, 0, \ldots\right),$$
$$0\ \ldots\qquad j\ \ldots\qquad n\qquad\qquad 0\qquad i\qquad j$$

$$1 \leqslant i \leqslant \frac{n}{2}, \quad i+j > n.$$

To see that in all the above theorems the given points are actually possible profiles we give constructions of families with these profiles. For Theorem 2.2 take all i_1-element, $\ldots$, i_k-element sets. For Theorem 2.3 consider the family

$$\{F \subset S\colon s_0 \in F, i \leqslant |F| \leqslant n-i\} \cup \{F \subset S\colon |F| > n-i\}$$

for $1 \leqslant i \leqslant n/2$ where s_0 is a fixed element of the groundset. If n is odd and $i = (n+1)/2$, put

$$\{F \subset S\colon |F| > n/2\}.$$

Finally, the case of Theorem 2.4 is settled by the families

$$\{F \subset S\colon |F| = j\}, \qquad n/2 < j,$$
$$\{F \subset S\colon s_0 \in F, |F| = i\} \cup \{F \subset S\colon s_0 \notin F, |F| = j\}, \qquad 1 \leqslant i \leqslant n/2, i+j > n.$$

Consequently we have to prove only that any profile can be dominated by the convex linear combination of these points. In Section 7 we prove a similar statement for two families of sets.

3. PRELIMINARIES

THEOREM 3.1. [11] *Suppose $\mathcal{F}$ is a k-Sperner family, then*

$$\sum_{0 \leqslant i \leqslant n} f_i \Big/ \binom{n}{i} \leqslant k \tag{1}$$

holds.

THEOREM 3.2 (Erdös, Ko and Rado [4]). *Suppose $\mathcal{F}$ is an intersecting family consisting of only t-element subsets, $2t \leq n$. Then*

$$|\mathcal{F}| \leq \binom{n-1}{t-1} \tag{2}$$

holds.

For a family $\mathcal{F}$ and a positive integer l we define

$$\partial^l \mathcal{F} = \{G \subset S : |G| = l, \quad \exists F \in \mathcal{F} \text{ s.t. } G \supset F\}.$$

Suppose $S = \{s_1, \ldots, s_n\}$ and define the lexicographic ordering for subsets A, B of equal size by $A < B$ if and only if $\min_{s_i \in A-B} i < \min_{s_i \in B-A} i$. Given t, m denote by $\mathcal{L}(t, m)$ the lexicographically first (smallest) m t-element subsets of S.

THEOREM 3.3 (Kruskal [12], Katona [10]). *Suppose $0 \leq t < l$, $0 < m < \binom{n}{t}$ and $\mathcal{F}$ consists of m t-subsets of S. Then*

$$|\partial^l \mathcal{F}| \geq |\partial^l \mathcal{L}(t, m)| \tag{3}$$

holds.

Clements [1] and Daykin, Godfrey and Hilton [3] deduced the following.

COROLLARY 3.4. *Suppose f is the profile vector of a 1-Sperner family. Then there exists a 1-Sperner family $\mathcal{G}$ with the same profile such that $\partial^l \mathcal{G}$ consists of the lexicographically first few l-subsets of S, i.e. $\partial^l \mathcal{G} = \mathcal{L}(l, |\partial^l \mathcal{G}|)$.*

Another corollary to Theorem 3.3 is its following version due to Lovász [13], cf. [14] for a simple proof of both this and Theorem 3.3.

COROLLARY 3.5. *Suppose that $0 \leq t$, $\mathcal{F}$ consists of t-element subsets of S and $x \geq n-t$ is a real number such that*

$$0 < |\mathcal{F}| = \binom{x}{n-t} = \frac{x(x-1)\ldots(x-n+t+1)}{(n-t)!}$$

then

$$|\partial^{t+1} \mathcal{F}| \geq \binom{x}{n-t-1}.$$

4. PROOF OF THEOREM 2.2

Suppose $\boldsymbol{f} = (f_0, \ldots, f_n)$ is the profile of a k-Sperner family. Note that

$$0 \leq f_i \leq \binom{n}{i} \tag{4}$$

holds for all i.

In view of Theorem 3.1 the numbers f_i satisfy (1). We prove the following slightly stronger statement instead of Theorem 2.2.

LEMMA 4.1. *The set D of Theorem 2.2 is the dominating set of all vectors $\boldsymbol{f}$ satisfying* (1) *and* (4).

PROOF. We have to prove that for any vector $\boldsymbol{f}$ satisfying (1) and (4) there are constants $\alpha_i \geqslant 0$, $\sum \alpha_i = 1$ such that

$$\boldsymbol{f} \leqslant \sum \alpha_i \boldsymbol{d}_i$$

where $\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots$ are the vectors given in Theorem 2.2. It is easy to see that it suffices to prove this statement for vectors $\boldsymbol{f}$ satisfying equality in (1). However, for such vectors $\boldsymbol{f}$ we can prove the existence of $\alpha_i \geqslant 0$, $\sum \alpha_i = 1$ such that

$$\boldsymbol{f} = \sum \alpha_i \boldsymbol{d}_i. \tag{5}$$

We apply induction on the number of components i for which $0 < f_i < \binom{n}{i}$ holds. If this number is zero, then $\boldsymbol{f} \in D$, the statement is trivial. In general let $I = \{i\colon 0 < f_i/\binom{n}{i} < 1\}$ and suppose that $|I| > 0$ (it implies $|I| \geqslant 2$) and the statement is proved for smaller values. Define now

$$\alpha^+ = \min_{i \in I} f_i \Big/ \binom{n}{i}, \qquad \alpha^- = \min_{i \in I} \left(1 - f_i \Big/ \binom{n}{i}\right), \qquad \alpha = \min\{\alpha^+, \alpha^-\}.$$

Suppose further that

$$f_{i_0} \Big/ \binom{n}{i_0} = \alpha^+ \quad \text{and} \quad 1 - f_{i_1} \Big/ \binom{n}{i_1} = \alpha^- \quad (i_0 \neq i_1).$$

Let $\boldsymbol{g} = (g_0, \ldots, g_n)$ be a new vector defined by

$$g_i = f_i \quad (i \neq i_0, i_1), \qquad g_{i_0} = f_{i_0} - \alpha \binom{n}{i_0} \quad \text{and} \quad g_{i_1} = f_{i_1} + \alpha \binom{n}{i_1}.$$

If $\boldsymbol{f}$ satisfies (1) and (4) with equality, then the same holds for $\boldsymbol{g}$. By the induction hypothesis, $\boldsymbol{g}$ satisfies (5):

$$\boldsymbol{g} = \sum \beta_i \boldsymbol{d}_i.$$

Clearly, either $g_{i_0} = 0$ or g_{i_1} equals $\binom{n}{i_1}$. Suppose first that $g_{i_0} = 0$. (5) implies that all $\boldsymbol{d}_i$ with positive β_i have a zero in the i_0th component.

Suppose that the vectors $\boldsymbol{d}_i$ are indexed in such a way that the vectors having $\binom{n}{i_1}$ in the i_1st component precede the ones having 0 here. Since the i_1st component of $\boldsymbol{g} = \sum \beta_i \boldsymbol{d}_i$ is $\geqslant \alpha\binom{n}{i_1}$ there is an index j such that the i_1st component of $\sum_{i=1}^{j-1} \beta_i \boldsymbol{d}_i + \gamma_j \boldsymbol{d}_j$ is equal to $\alpha\binom{n}{i_1}$, no vector with a zero in the i_1st components is used and $0 \leqslant \gamma_j \leqslant \beta_j$. Let $\boldsymbol{d}'_i$ denote the modification of $\boldsymbol{d}_i$: the i_0th and i_1st components are changed for $\binom{n}{i_0}$ and 0, respectively. The expansion

$$f = \sum_{i=1}^{j-1} \beta_i \boldsymbol{d}'_i + \gamma_j \boldsymbol{d}'_j + (\beta_j - \gamma_j)\boldsymbol{d}_j + \sum_{i=j+1} \beta_i \boldsymbol{d}_i$$

satisfies (5).

The case $g_{i_1} = \binom{n}{i_1}$ can be handled analogously, since this condition implies that all $\boldsymbol{d}_i$ with positive β_i have $\binom{n}{i_1}$ in the i_1th component. This proves the theorem.

5. PROOF OF THEOREM 2.3

Let $\boldsymbol{f}$ be the profile of an intersecting family $\mathcal{F}$.

$$0 \leqslant f_i, \qquad 0 \leqslant i \leqslant n, \tag{6}$$

is obvious.

Note that Theorem 3.2 implies

$$f_i \leqslant \binom{n-1}{i-1}, \qquad 0 \leqslant i \leqslant n/2. \tag{7}$$

The fact that no set and its complement can be simultaneously in an intersecting family gives

$$f_i + f_{n-i} \leqslant \binom{n}{i}, \qquad (0 \leqslant i \leqslant n). \tag{8}$$

Any superset of a member of F of $\mathscr{F}$ can be added to $\mathscr{F}$ without violating the intersecting property. Suppose that $\mathscr{F}$ is maximal, then $F \subset G \subset S$ and $F \in \mathscr{F}$ imply $G \in \mathscr{F}$. In other words $\partial^l \mathscr{F} \subset \mathscr{F}$ holds for all l. Let $f_i = \binom{x}{n-i}$ ($x \geqslant n-i$, a real). By (7) and the monotonity of $\binom{x}{n-i}$ we obtain $x \leqslant n-1$ for $i \leqslant n/2$. Corollary 3.5 implies

$$\frac{f_{i+1}}{f_i} \geqslant \frac{|\partial^{i+1}\mathscr{F}_i|}{|\mathscr{F}_i|} \geqslant \frac{\binom{x}{n-i-1}}{\binom{x}{n-i}} \geqslant \frac{n-i}{i}$$

and hence

$$\frac{f_{i+1}}{\binom{n-1}{i}} \geqslant \frac{f_i}{\binom{n-1}{i-1}}, \qquad 0 \leqslant i < n/2 \tag{9}$$

follows. We prove the following slightly stronger statement instead of Theorem 2.3.

LEMMA 5.1. *The set D of Theorem* 2.3 *is the dominating set of the vectors f satisfying conditions* (6)–(9).

PROOF. We have to prove that for any vector $\boldsymbol{f}$ satisfying (6)–(9) there are constants $\alpha_i \geqslant 0$, $\sum_{1 \leqslant i \leqslant (n+1)/2} \alpha_i = 1$ such that $\boldsymbol{f} \leqslant \sum_{1 \leqslant i \leqslant (n+1)/2} \alpha_i \boldsymbol{d}_i$ where $\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots$ are the vectors given in Theorem 2.3. Increasing f_i ($i > n/2$) until equality is obtained in (8) shows that it is sufficient to prove our statement for vectors $\boldsymbol{f}$ satisfying (8) with equality. We will show in this case the existence of $\alpha_i \geqslant 0$, $\sum \alpha_i = 1$ such that

$$\boldsymbol{f} = \sum_{1 \leqslant i \leqslant (n+1)/2} \alpha_i \boldsymbol{d}_i. \tag{10}$$

Adding up (8) (with equality) for i we infer $2\sum_{i=0}^n f_i = 2^n$ i.e. $\sum_{i=0}^n f_i = 2^{n-1}$. Let us set

$$\alpha_1 = f_1$$

$$\alpha_i = \frac{f_i}{\binom{n-1}{i-1}} - \frac{f_{i-1}}{\binom{n-1}{i-2}}, \qquad 2 \leqslant i \leqslant n/2,$$

$$\alpha_{(n+1)/2} = 1 - \frac{f_{(n-1)/2}}{\binom{n-1}{(n-3)/2}}, \qquad n \text{ is odd.}$$

(9) and (7) imply $\alpha_i \geqslant 0$ ($1 \leqslant i \leqslant (n+1)/2$). If n is odd $\sum_{1 \leqslant i \leqslant (n+1)/2} \alpha_i = 1$ is trivial. If n is even, this sum is equal to $f_{n/2}/\binom{n-1}{n/2-1}$. This is really 1 because of the equality in (8) with $i = n/2$. To prove (10) we have to check it componentwise. The jth component on

the left hand side for $j \leqslant n/2$ is

$$\binom{n-1}{j-1} \sum_{1 \leqslant i \leqslant j} \alpha_i = \binom{n-1}{j-1} \frac{f_j}{\binom{n-1}{j-1}} = f_j.$$

To prove the same for $j > n/2$, let us note that $f_j + f_{n-j} = \binom{n}{j}$ $(0 \leqslant j \leqslant n)$ holds by the equality in (8) and similarly, the sum of the jth and $(n-j)$th components of any $\boldsymbol{d}_i$ is $\binom{n}{j}$, as well. The proof is complete.

6. Proof of Theorem 2.4

Suppose $\mathscr{F}$ is an intersecting Sperner family with profile $\boldsymbol{f}$. Let $\mathscr{G}$ be the Sperner family with the same profile and satisfying $\partial^l \mathscr{G} = \mathscr{L}(l, |\partial^l \mathscr{G}|)$ for all l. The existence of such a $\mathscr{G}$ is guaranteed by Corollary 3.4.

Lemma 6.1. *For all* $0 \leqslant t \leqslant n$ *we have*

$$|\partial^t \mathscr{F}| + |\partial^{n-t} \mathscr{F}| \leqslant \binom{n}{t}.$$

Proof. In the opposite case we can find $F \in \partial^t \mathscr{F}$, $F' \in \partial^{n-t} \mathscr{F}$ such that $F = S - F'$. By definition there exist $E, E' \in \mathscr{F}$ with $E \subset F$, $E' \subset F'$ and consequently $E \cap E' = \varnothing$, a contradiction.

By Theorem 3.3 we have $|\partial^t \mathscr{G}| \leqslant |\partial^t \mathscr{F}|$ for all t. Hence Lemma 6.1 implies

$$|\partial^t \mathscr{G}| + |\partial^{n-t} \mathscr{G}| \leqslant \binom{n}{t}. \tag{11}$$

Also, Theorem 3.2 gives $|\partial^t \mathscr{F}| \leqslant \binom{n-1}{t-1}$ for $t \leqslant n/2$. Thus

$$|\partial^t \mathscr{G}| \leqslant \binom{n-1}{t-1} \tag{12}$$

holds for $t \leqslant n/2$. By the lexicographic ordering we infer $s_1 \in G$ for all $G \in \mathscr{G}$, $|G| \leqslant n/2$.

Let us define $\mathscr{G}^1 = \{G - \{s_1\}: s_1 \in G \in \mathscr{G}\}$, $\mathscr{G}^0 = \{G \in \mathscr{G}: s_1 \notin G\}$. The members of $\mathscr{G}^0$ are of size $> n/2$ by the above remark. Clearly, both $\mathscr{G}^1$ and $\mathscr{G}^0$ are Sperner families on $S - \{s_1\}$. Denote their profiles by $\boldsymbol{g}^1$, $\boldsymbol{g}^0$, resp. By Theorem 3.1 we have

$$\sum g_i^1 \Big/ \binom{n-1}{i} \leqslant 1$$

and

$$\sum g_i^0 \Big/ \binom{n-1}{i} \leqslant 1 \qquad (g_i^0 = 0 \quad \text{for} \quad 0 \leqslant i \leqslant n/2).$$

Let us set

$$\alpha_i = g_{i-1}^1 \Big/ \binom{n-1}{i-1} \quad \text{and} \quad \beta_i = g_i^0 \Big/ \binom{n-1}{i}.$$

The above inequalities can be rewritten as

$$\sum_{1 \leqslant j \leqslant n} \alpha_j \leqslant 1, \qquad \sum_{n/2 < j \leqslant n-1} \beta_j \leqslant 1, \quad \beta_j = 0, j \leqslant n/2. \tag{13}$$

We define $\beta_n = 1 - \sum_{n/2 < j \leqslant n-1} \beta_j \geqslant 0$.

LEMMA 6.2.

$$\sum_{1\le i\le t} \alpha_i \le \frac{|\partial^{t-1}\mathcal{G}^1|}{\binom{n-1}{t-1}} \tag{14}$$

where ∂^{t-1} is understood to act on $S-\{s_1\}$.

PROOF. We use induction on t. If $t=1$ the statement is trivial. Suppose $t>1$.

$$\partial^t\mathcal{G}^1 = \partial^t(\partial^{t-1}(\mathcal{G}^1)) \cup \mathcal{G}_t^1 \quad \text{and} \quad \partial^t(\partial^{t-1}(\mathcal{G}^1)) \cap \mathcal{G}_t^1 = \varnothing$$

are obvious since $\mathcal{G}^1$ is a Sperner family. Hence we have

$$|\partial^t\mathcal{G}^1| = |\partial^t(\partial^{t-1}(\mathcal{G}^1))| + g_t^1. \tag{15}$$

Counting the number of pairs A, B $A\in\partial^{t-1}(\mathcal{G}^1)$, $B\in\partial^t(\partial^{t-1}(\mathcal{G}^1))$, $A\subset B$ in two different ways the inequality

$$|\partial^{t-1}(\mathcal{G}^1)|\cdot(n-t) \le |\partial^t(\partial^{t-1}(\mathcal{G}^1))|t$$

is obtained. This implies

$$|\partial^t(\partial^{t-1}(\mathcal{G}^1))| \ge \frac{|\partial^{t-1}(\mathcal{G}^1)|\binom{n-1}{t}}{\binom{n-1}{t-1}}. \tag{16}$$

(15) and (16) result in

$$\frac{|\partial^t\mathcal{G}^1|}{\binom{n-1}{t}} \ge \alpha_{t+1} + \frac{|\partial^{t-1}(\mathcal{G}^1)|}{\binom{n-1}{t-1}}.$$

The application of the induction hypothesis gives (14) for $t+1$. The lemma is proved.

LEMMA 6.3.

$$\sum_{1\le i\le t} \alpha_i \le \sum_{1\le i\le t} \beta_{n+1-i}, \qquad 1\le t\le n.$$

PROOF. As $\mathcal{G}$ is an intersecting Sperner family, $\partial^{t-1}\mathcal{G}^1 \cup \{S-(G\cup\{s_1\})\colon G\in\mathcal{G}^0, |G|\le n-t\}$ is a Sperner family where ∂^{t-1} acts still on $S-\{s_1\}$. (1) implies

$$\frac{|\partial^{t-1}\mathcal{G}^1|}{\binom{n-1}{t-1}} + \beta_{\lceil (n+1)/2\rceil} + \cdots + \beta_{n-t} \le 1, \qquad 1\le t\le n.$$

The application of Lemma 6.2 and $\sum_{n/2<i\le n-t} \beta_i = 1 - \sum_{t\le i\le n} \beta_{n+1-i}$ completes the proof of the lemma.

Let us turn back to the proof of the theorem. Define the numbers γ_{ij} ($1\le i\le j\le n$, $i+j>n$) recursively. Start with $\gamma_{1n}=\alpha_1$. Suppose $\gamma_{i'j'}$ is defined for all $i'<i$ and $i'=i, j'>j$.

Set

$$\gamma_{ij} = \min\left\{\alpha_i - \sum_{j'>j} \gamma_{ij'},\ \beta_j - \sum_{n-j<i'<i} \gamma_{i'j}\right\} \tag{17}$$

where the void sum means zero.

LEMMA 6.4.

$$\gamma_{ij}\geqslant 0,\qquad 1\leqslant i\leqslant j\leqslant n,\ i+j>n, \tag{18}$$

$$\sum_{n-j<i\leqslant j}\gamma_{ij}\leqslant\beta_j,\qquad n/2<j\leqslant n, \tag{19}$$

$$\sum_{\max\{i,n-i+1\}\leqslant j\leqslant n}\gamma_{ij}=\alpha_i,\qquad 1\leqslant i\leqslant n. \tag{20}$$

PROOF. Let $n/2<j\leqslant n$. Then (17) yields

$$\gamma_{jj}\leqslant\beta_j-\sum_{n-j<i<j}\gamma_{ij}.$$

This is nothing else but (19).

To prove (18) and (20) we use induction following the recursive definition. For $i=1$ $0\leqslant\gamma_{1n}=\alpha_1$ follows from the definition and Lemma 6.3. Let $i>1$ and suppose that (18) and (20) are proved for all (i',j) with $i'<i$. By (17) we have

$$\gamma_{in}=\min\left\{\alpha_i,\ \beta_n-\sum_{1\leqslant i'<i}\gamma_{i'n}\right\}.$$

Using (19) we infer $0\leqslant\gamma_{in}\leqslant\alpha_i$. By induction on j, using (17), we can prove that $0\leqslant\gamma_{ij}$ and $\sum_{j\leqslant j'\leqslant n}\gamma_{ij'}\leqslant\alpha_i$ hold for any j such that $\max\{i,\ n-i+1\}\leqslant j\leqslant n$. Especially,

$$\sum_{\max\{i,n-i+1\}\leqslant j\leqslant n}\gamma_{ij}\leqslant\alpha_i. \tag{21}$$

Suppose that $\gamma_{ij}=\alpha_i'-\sum_{j'>j}\gamma_{ij'}$ holds for some j. Then $\gamma_{i,j-1}=\gamma_{i,j-2}=\cdots=0$ and (20) hold. If $\gamma_{ij}<\alpha_i-\sum_{j'>j}\gamma_{ij'}$ then we have $\gamma_{ij}=\beta_j-\sum_{n-j<i'<i}\gamma_{i'j}$ by (17). Hence, if $i<n/2$,

$$\sum_{\max\{i,n-i+1\}\leqslant j\leqslant n}\gamma_{ij}=\sum_{\max\{i,n-i+1\}\leqslant j\leqslant n}\beta_j-\sum_{\max\{i,n-i+1\}\leqslant j\leqslant n}\ \sum_{n-j<i'<i}\gamma_{i'j}$$

$$=\sum_{\max\{i,n-i+1\}\leqslant j\leqslant n}\beta_j-\sum_{1\leqslant l<i}\ \sum_{\max\{l,n-l+1\}\leqslant j\leqslant n}\gamma_{lj}$$

follows. In the last terms we may use the induction hypothesis for (20):

$$\sum_{\max\{i,n-i+1\}\leqslant j\leqslant n}\gamma_{ij}=\sum_{\max\{i,n-i+1\}\leqslant j\leqslant n}\beta_j-\sum_{1\leqslant l<i}\alpha_l.$$

The right hand side is $\geqslant\alpha_i$ by Lemma 6.3. This inequality and (21) prove (20) for i. In the other case when $n/2<i$ the proof is similar, but (19) should also be used. The lemma is proved.

Extend the definition of the vectors $\boldsymbol{w}_{ij}$ of Theorem 2.4 for all $i<j$, $i+j>n$, in a natural way. Let $\boldsymbol{w}_{ii}=\boldsymbol{v}_i$. We prove now that

$$\sum_{\substack{i\leqslant j\\ i+j>n}}\gamma_{ij}\boldsymbol{w}_{ij}+\sum_{n/2<j\leqslant n}\gamma_j\boldsymbol{v}_j\geqslant\boldsymbol{f} \tag{22}$$

where $\gamma_j=\beta_j-\sum_{n-j<i\leqslant j}\gamma_{ij}$. This will be checked componentwise. For the 0th component (22) is trivial because $f_0=0$ by the intersection property. Let $1\leqslant i\leqslant n/2$. The ith component of the left hand side is

$$\sum_{n-i+1\leqslant j\leqslant n}\gamma_{ij}\binom{n-1}{i-1}=\alpha_i\binom{n-1}{i-1}=g^1_{i-1}$$

and this is equal to the ith component $f_i=g^1_{i-1}+g^0_i$ of the right hand side since $g^0_i=0$ for $1\leqslant i\leqslant n/2$. Let $n/2<j\leqslant n$. The jth component of the left hand side of (22) is

$$\geqslant\sum_{n-j<i<j}\gamma_{ij}\binom{n-1}{j}+\gamma_{jj}\binom{n}{j}+\sum_{j<l\leqslant n}\gamma_{jl}\binom{n-1}{j-1}+\left(\beta_j-\sum_{n-j<i\leqslant j}\gamma_{ij}\right)\binom{n-1}{j}$$

$$=\alpha_j\binom{n-1}{j-1}+\beta_j\binom{n-1}{j}.$$

This is equal to the ith component $g_{j-1}^1+g_j^0$ of $\boldsymbol{f}$ if $j<n$. For $j=n$ this latter component is g_{n-1}^1. Thus (22) is proved.

The sum of the coefficients in (22) is

$$\sum_{\substack{i\leq j\\ i+j>n}} \gamma_{ij}+\sum_{n/2<j\leq n} \delta_j=\sum_{n/2<j\leq n} \beta_j=1.$$

Replace each w_{ij} with $n/2<i<j\leq n$ by

$$\frac{\binom{n-1}{i-1}}{\binom{n}{i}} \boldsymbol{v}_i+\frac{\binom{n-1}{j}}{\binom{n}{j}} \boldsymbol{v}_j$$

on the left hand side of (22).

This modification does not change the left hand side and the validity of (22). However, the sum of the coefficients is decreased since

$$\binom{n-1}{i-1}\bigg/\binom{n}{i}+\binom{n-1}{j}\bigg/\binom{n}{j}=\frac{i+n-j}{n}<1.$$

This proves that $\boldsymbol{f}$ is dominated by a linear combination of the vectors given in Theorem 2.4 and the sum of the coefficients is ≤ 1.

7. An Extremal Problem for Two Families

Suppose $\mathscr{F}$ and $\mathscr{G}$ are families of subsets of S, such that $\mathscr{F}$ is a Sperner family and if $F\in\mathscr{F}$, $G\in\mathscr{G}$ then $F\not\subset G$ and $G\not\subset F$ hold (in particular $\mathscr{F}\cap\mathscr{G}=\varnothing$). We consider the profile vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ of these families as one vector $(f_0, f_1, \ldots, f_n, g_0, \ldots, g_n)\in\mathbb{R}^{2n+2}$, and call it the double profile.

Theorem 7.1. *The dominating set of all possible double profiles of $\mathscr{F}$, $\mathscr{G}$ with the above properties consists of the vectors*

$$\boldsymbol{v}_i=\Big(\underset{0}{0},\ldots,0,\underset{i}{\binom{n}{i}},0,\ldots,\underset{n}{0},\underset{n+1}{0},\ldots,\underset{2n+1}{0}\Big),\qquad 0\leq i\leq n$$

and

$$\boldsymbol{w}=\Big(\underset{0}{0},0,\ldots,\underset{n}{0},\underset{n+1}{\binom{n}{0}},\ldots,\underset{n+i+1}{\binom{n}{i}},\ldots,\underset{2n+1}{\binom{n}{n}}\Big).$$

Before proving Theorem 7.1 we state a theorem proved by Daykin *et al.* [2] which can be easily derived from it.

Theorem 7.2. *Suppose $\mathscr{F}^1,\ldots,\mathscr{F}^t$ are families of subsets of S so that for $1\leq i\neq j\leq t$ and $F_i\in\mathscr{F}^i$, $F_j\in\mathscr{F}^j$, $F_i\neq F_j$ we have $F_i\not\subset F_j$. Then* (*a*), (*b*) *and* (*c*) *hold*:

(a) $$\sum_{1\leq i\leq t}|\mathscr{F}^i|\leq\max\left\{t\binom{n}{\lfloor n/2\rfloor}, 2^n\right\},$$

(b) $$\sum_{1\leq i\leq t}\sum_{0\leq j\leq n}\frac{f_j^i}{\binom{n}{j}}\leq\max\{t, n+1\},$$

(c) $$\sum_{1\le i\le t} |\mathcal{F}^i| \le rt + \left(1 - \frac{r}{\binom{n}{\lfloor n/2\rfloor}}\right)2^n,$$

where r denotes the number of subsets of S which occur in at least two of the $\mathcal{F}^i$.

PROOF OF THEOREM 7.1. The conditions imply that for each i, $1\le i\le n$, $\mathcal{F}\cup\mathcal{G}_i$ is a Sperner family (recall that $\mathcal{G}_i=\{G\in\mathcal{G}: |G|=i\}$). By Theorem 3.2 we have

$$\sum_{0\le j\le n} f_j\Big/\binom{n}{j} + g_i\Big/\binom{n}{i} \le 1. \tag{23}$$

Define $\alpha_j = f_j/\binom{n}{j}$ and $\beta=\max_{0\le i\le n} g_i/\binom{n}{i}$. Clearly, $(f_0, f_1, \dots, f_n, g_0, \dots, g_n) \le \sum_j \alpha_j \boldsymbol{v}_j + \beta\boldsymbol{w}$ holds, while (23) implies $\sum \alpha_j+\beta\le 1$. The proof is complete.

THE PROOF OF THEOREM 7.2. Let us define $\mathcal{F}$ as the family of those subsets of S which are contained in at least two of the $\mathcal{F}^i$, and set $\mathcal{G}=\bigcup_{1\le i\le t}\mathcal{F}^i-\mathcal{F}$. Clearly, we have

$$\sum_{i=1}^{t} |\mathcal{F}^i| \le t|\mathcal{F}|+|\mathcal{G}|. \tag{24}$$

However, $\mathcal{F}$ and $\mathcal{G}$ fulfill the assumptions of Theorem 7.1. Therefore, the maximum of $t|\mathcal{F}|+\mathcal{G}=\sum_i tf_i+\sum_i g_i$ is attained in some point of the dominating set of profiles. For $\boldsymbol{v}_j$ we obtain $t\binom{n}{j}$, for $\boldsymbol{w}$ 2^n, yielding (a). For (b) note first

$$\sum_{1\le i\le t}\sum_{0\le j\le n} f_j^i\Big/\binom{n}{j} \le \sum_{0\le j\le n}\frac{t}{\binom{n}{j}} f_j + \sum_{0\le j\le n}\frac{1}{\binom{n}{j}} g_j.$$

Here the right hand side is again maximized for some point of the dominating set. For $\boldsymbol{v}_i$ we obtain t, for $\boldsymbol{w}$ $n+1$, proving (b).

To prove (c) we rearrange (23):

$$g_i\Big/\binom{n}{i} \le 1 - \sum_{0\le j\le n} f_j\Big/\binom{n}{j} \le 1-\frac{r}{\binom{n}{\lfloor n/2\rfloor}}.$$

Multiplying by $\binom{n}{i}$ and summing up over all i we infer

$$|\mathcal{G}| = \sum_{0\le i\le n} g_i \le \left(1-\frac{r}{\binom{n}{\lfloor n/2\rfloor}}\right)2^n, \text{ yielding (c).}$$

8. OPEN PROBLEMS

It would be interesting to determine the dominating set of profiles of other classes of families, e.g. t-intersecting families ($|F\cap F'|\ge t$), intersecting-union families ($F\cap F'\ne\emptyset$, $F\cup F'\ne S$). However, these cases seem to be much more difficult. A more hopeful case might be that of k-wise intersecting Sperner families ($F_1\cap\cdots\cap F_k\ne\emptyset$, $\mathcal{F}$ is Sperner). We conjecture that all dominating vertices have one or two non-zero coordinates. For $k=3$, this would completely settle the problem of the maximum size of a 3-wise intersecting Sperner family. This is known to be $\binom{n-1}{\lceil (n-1)/2\rceil}+\varepsilon$, where $\varepsilon=1$ for n even and $\varepsilon=0$ for n odd, for all but 12 values of n (cf. Frankl [7], Gronau [8], [9]).

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