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MINIMUM MATRIX REPRESENTATION OF CLOSURE OPERATIONS

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Let a be a column of the $m \times n$ matrix M and A a set of its columns. We say that A implies a iff M contains no two rows equal in A but different in a. It is easy to see that if $\mathscr{L}_M(A)$ denotes the columns implied by A, than $\mathscr{L}_M(A)$ is a closure operation. We say that M represents this closure operation. $s(\mathscr{L})$ is the minimum number of the rows of the matrices representing a given closure operation. $s(\mathscr{L})$ is determined for some particular closure operations.

1. Introduction

A simple model of a data base [4] is a matrix. A row contains the data of one individual. A column contains the data of the same sort (e.g. date of birth). Let X denote the set of columns. Choose a subset $A \subseteq X$ and suppose that the data of an individual are known in the columns belonging to A. The individual (or row) is not necessarily determined, there can be more individuals (rows) having these data in the columns belonging to A. However all these individuals (rows) might agree in a column $b \notin A$. We say that b belongs to the closure $\mathscr{L}(A)$ of A if this happens for b with any choice of data in the columns belonging to A. We will see that this function $\mathscr{L}$ mapping 2^X into 2^X satisfies (2)–(4). Such a function is called a closure operation. Conversely, if a closure operation $\mathscr{L}$ is given, one can find a matrix generating exactly this closure operation in the above defined way [1, 5, 6]. Let $s(\mathscr{L})$ denote the minimum cardinality of rows of such matrices.

The main aim of the paper is to investigate the function $s(\mathscr{L})$. There are three kinds of results. In Section 3 we more or less determine $s(\mathscr{L})$ for some very special closure operations: the closure of any set of cardinality $\geq k$ is X while the closure of smaller sets A is A. Section 4 determines $s(\mathscr{L}_1 \times \mathscr{L}_2)$ in terms of $s(\mathscr{L}_1)$ and $s(\mathscr{L}_2)$. Finally we quote a result producing an $\mathscr{L}$ with large $s(\mathscr{L})$.

2. Definitions

Let M be a matrix of m rows and n columns. The set of columns will be denoted by X. If $A \subseteq X$, $a \in X$ and M contains no two rows equal in A but different in a then we say that A *implies* a. The *closure* of A is

(1) $\mathscr{L}_M(A) = \{a\colon a \in X,\ A \text{ implies } a\}.$

It is easy to see that the following rules are valid for $\mathscr{L}_M = \mathscr{L}$:

(2) $A \subseteq \mathscr{L}(A),$

(3) $A \subseteq B \Rightarrow \mathscr{L}(A) \subseteq \mathscr{L}(B),$

(4) $\mathscr{L}(\mathscr{L}(A)) = \mathscr{L}(A).$

A function $\mathscr{L}\colon 2^X \to 2^X$ is called a *closure operation* if it satisfies (2)–(4).

Conversely, if $\mathscr{L}$ is an arbitrary closure operation on an n-element groundset X, then there is an $m \times n$ matrix M such that $\mathscr{L}_M = \mathscr{L}$ [1, 5, 6]. We say in this case that *M represents* $\mathscr{L}$. The definition of our main target is the following:

(5) $s(\mathscr{L}) = \min\{m\colon M \text{ is an } m \times n \text{ matrix, } \mathscr{L}_M = \mathscr{L}\}.$

A closure operation determines an important class of subsets, the class of keys. K is a *key* in $\mathscr{L}$ if $\mathscr{L}(K) = X$. $\mathscr{K} = \mathscr{K}(\mathscr{L})$ denotes the family of *minimal keys* (K is a minimal key if it is a key but no proper subset of K is a key). It is obvious that K_1, $K_2 \in \mathscr{K}$, $K_1 \neq K_2$ imply $K_1 \not\subset K_2$. Families of subsets satisfying this condition are called *Sperner-families.* Hence $\mathscr{K}$ is a Sperner-family. We say that a matrix M *represents* a given Sperner-family $\mathscr{K}$ if $\mathscr{K} = \mathscr{K}(\mathscr{L}_M)$ holds. The maximal non-keys are called *antikeys.* Their family is denoted by

$$\mathscr{K}^{-1} = \{A\colon \nexists B \in \mathscr{K},\ B \subseteq A,\ A \text{ is maximal for this property}\}.$$

Lemma 1. *M represents the Sperner-family* $\mathscr{K}$ *iff for any* $A \in \mathscr{K}^{-1}$ *M has two different rows having the same entries in the columns in A and any two rows equal in a* $K \in \mathscr{K}$ *are equal everywhere.*

Proof. If M represents $\mathscr{K}$, then $\mathscr{K} = \mathscr{K}(\mathscr{L}_M)$ holds.

$K \in \mathscr{K}$ implies $\mathscr{L}_M(K) = X$, the second condition obviously follows. Similarly, $A \in \mathscr{K}^{-1}$ implies $\mathscr{L}_M(A) \neq X$ and hence we obtain the first condition.

Conversely, if both conditions are satisfied for M and $\mathscr{K}$, then (i) $\mathscr{L}_M(A) \neq X$ holds for any $A \in \mathscr{K}^{-1}$ and (ii) $\mathscr{L}_M(K) = X$ holds for any $K \in \mathscr{K}$.

(ii) and (3) imply that $\mathscr{L}_M(C) = X$ if $C \supseteq K$ for some $K \in \mathscr{K}$. Suppose now that C is not a superset of a member of $\mathscr{K}$. Then, by definition there is an $A \in \mathscr{K}^{-1}$ such that $C \subseteq A$. (i) and (3) imply $\mathscr{L}_M(C) \neq X$. That is, $\mathscr{L}_M(C) = X$ exactly for the supersets of members of $\mathscr{K}$: $\mathscr{K} = \mathscr{K}(\mathscr{L}_M)$. The proof is complete. □

The following definition is an analogue of (5):

(6) $$s(\mathcal{K})=\min\{m\colon M \text{ is an } m\times n \text{ matrix representing } \mathcal{K}\}$$

where $\mathcal{K}$ is a Sperner-family on an n-element set.

The *k-uniform closure operation* on an n-element groundset X is defined by

(7) $$\mathcal{L}_k^n(A)=\begin{cases} X, & \text{if } |A|\geq k,\\ A, & \text{if } |A|<k.\end{cases}$$

The family of all k-element subsets of X is denoted by $\binom{X}{k}$. In general, there exist more than one closure operation with the same system $\mathcal{K}$ of minimal keys ($\mathcal{K}=\mathcal{K}(\mathcal{L})$). The next Lemma states that if $\mathcal{K}$ is the family of all k-element subsets of X, then $\mathcal{L}$ is uniquely determined by $\mathcal{K}=\mathcal{K}(\mathcal{L})$.

Lemma 2. *Let any closure operation $\mathcal{L}$ be defined on an n-element set X. Then*

$$\mathcal{K}(\mathcal{L})=\binom{X}{k} \quad \textit{iff} \quad \mathcal{L}=\mathcal{L}_k^n.$$

Proof. $\mathcal{K}(\mathcal{L}_k^n)=\binom{X}{k}$ is trivial. We have to verify the converse statement only. Suppose that $a\in\mathcal{L}(A)-A$ for some $A\subseteq X$ such that $|A|<k$. Then one can find a set B satisfying $|B|=k$, $B\supseteq A\cup\{a\}$. (2) implies $\mathcal{L}(B-a)\supseteq B-a$; (3) implies $\mathcal{L}(B-a)\supseteq\mathcal{L}(A)\ni a$. Hence $\mathcal{L}(B-a)\supseteq B$ follows. We obtain $\mathcal{L}(B-a)=\mathcal{L}(\mathcal{L}(B-a))\supseteq\mathcal{L}(B)=X$ by (4) and (3). Consequently, there is a set $B-a$ of cardinality $<k$ with closure X. This contradiction shows that $a\in\mathcal{L}(A)-A$ cannot exist if $|A|<k$: $\mathcal{L}(A)=A$. $\mathcal{L}(A)=X$ for $|A|\geq k$ easily follows from $\mathcal{K}(\mathcal{L})=\binom{X}{k}$ and (3). $\mathcal{L}=\mathcal{L}_k^n$ holds, the lemma is proved. □

3. Minimum representation of uniform closure operations

First we repeat some results of [9]. We prove these statements for sake of completeness.

Lemma 3. *If an $m\times n$ matrix M represents $\mathcal{K}$, then*

$$\binom{m}{2}\geq|\mathcal{K}^{-1}|.$$

Proof. If $A\in\mathcal{K}^{-1}$, then by Lemma 1 there are different rows i,j such that they are equal in A. Take another member B of $\mathcal{K}^{-1}$. Let the corresponding pair of rows be i' and j'. If the unordered pairs $\{i,j\}$, $\{i',j'\}$ are equal, then these two different rows are equal in $A\cup B$. Consequently, $\mathcal{L}(A\cup B)\neq X$ and there is a $C\supseteq A\cup B$ with $C\in\mathcal{K}^{-1}$. By the definition of $\mathcal{K}^{-1}$ this is possible only when $C=A$ and $C=B$, contradicting our supposition $A \neq B$. Therefore to different members of $\mathscr{K}^{-1}$ we have different pairs of rows satisfying the above condition. The number of pairs of rows of M must be $\geq |\mathscr{K}^{-1}|$. The lemma is proved. □

Lemma 4.

$$\binom{s(\mathscr{L}_k^n)}{2} \geq \binom{n}{k-1}. \tag{8}$$

Proof. Let M be an $s(\mathscr{L}_k^n) \times n$ matrix representing $\mathscr{L}_k^n$. By Lemma 2, M also represents $\mathscr{K}(\mathscr{L}_k^n) = \binom{X}{k}$. It is easy to see that $\mathscr{K}^{-1}(\mathscr{L}_k^n) = \binom{X}{k-1}$. Then (8) follows by Lemma 3. □

We will see that (8) gives a fairly good lower estimate on $s(\mathscr{L}_k^n)$. It is sharp for $k = 1, 2, n-1$. It seems to be sharp for $k = 3$ and $n \geq 7$. On the other hand it is sharp up to a constant depending on k for any fixed k when $n \to \infty$. The case $k = n$ needs another lemma. If M is an $m \times n$ matrix let $G(M)$ denote the graph whose vertices are the rows of M, two vertices are connected with an edge iff the set A of columns where the two rows are equal is non-empty. The edge is *labelled* by A.

Lemma 5. *Let M be a matrix and let $A_1, \ldots, A_r$ be the labels along a circuit of $G(M)$. Then*

$$\left(\bigcap_{\substack{i=1\\ i \neq j}}^{r} A_i\right) - A_j = \emptyset \qquad (1 \leq j \leq r). \tag{9}$$

Proof. Suppose that, on the contrary, (9) is non-empty, that is, there is a column, say the uth one, which is an element of all A_i but A_j. Let the vertices of the circuit be $k_1, \ldots, k_r$ in such a way that the edge (k_i, k_{i+1}) is labelled by A_i $(1 \leq i < r)$ and (k_r, k_1) is labelled by A_r. From $u \in A_{j+1}$ it follows that the k_{j+1}st and k_{j+2}nd entries of the uth column are equal. The same holds for the k_{j+2}nd and k_{j+3}rd entries, etc. Consequently, the k_{j+1}st, k_{j+2}nd, $\ldots, k_r$th, k_1st, $\ldots, k_j$th entries in the uth column are all equal. This leads to $u \in A_j$ contradicting the assumption. The proof is complete. □

Theorem 1 [9].

$$s(\mathscr{L}_1^n) = 2, \qquad s(\mathscr{L}_2^n) = \lceil (1 + \sqrt{1 + 8n})/2 \rceil,$$
$$s(\mathscr{L}_{n-1}^n) = n, \qquad s(\mathscr{L}_n^n) = n + 1$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$.

Proof. By Lemma 2, $s(\mathscr{L}_k^n) = s(\mathscr{K}(\mathscr{L}_k^n)) = s(\binom{X}{k})$. We use the last form in the proof.

For $k=1$, (8) gives $s(\mathscr{L}_1^n)\geq 2$. The construction

$$\begin{matrix} 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \end{matrix}$$

proves the equality.

For $k=2$, (8) gives

$$\binom{s(\mathscr{L}_2^n)}{2}\geq n. \tag{10}$$

Suppose that $s(\mathscr{L}_2^n)$ satisfies (10); we construct an $s(\mathscr{L}_2^n)\times n$ matrix M representing $\binom{X}{2}$. Any column of M contains exactly two zeros. Different columns contain different pairs of zeros. The other entries of the ith row are i $(1\leq i\leq s(\mathscr{L}_2^n))$. It is easy to see, using Lemma 1, that M represents $\binom{X}{2}$. The least integer satisfying (10) can be expressed in the form given in the theorem.

For $k=n-1$, (8) gives $s(\mathscr{L}_{n-1}^n)\geq n$. The construction

$$\begin{matrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{matrix}$$

gives the equality.

Finally, suppose that the $m\times n$ matrix M realizes $\binom{X}{n}=\{X\}$. By Lemma 1 there is an edge in $G(M)$ labelled with A for any $(n-1)$-element subset of X. $G(M)$ has n different edges of this kind. These edges cannot form a circuit because the $(n-1)$-element subsets cannot satisfy (9), the lemma is applicable. $G(M)$ has at least $n+1$ vertices: $s(\mathscr{L}_n^n)\geq n+1$. The following construction gives equality

$$\begin{matrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1. \end{matrix}$$

The proof is complete. □

Substituting $k=3$ into (8) we obtain

$$s(\mathscr{L}_3^n)\geq n.$$

$s(\mathscr{L}_3^3)=4>3$ is proved and $s(\mathscr{L}_3^6)>6$ can be verified by checking all the cases. We conjecture that the above inequality is sharp for all other cases:

Conjecture 1. *$s(\mathscr{L}_3^n)=n$ for $n\geq 7$.*

We are able to reduce this conjecture in the case $n=3r+1$ for another conjecture concerning a certain kind of resolvable Steiner triple systems:

Conjecture 2. *There is a system of 3-element subsets of an $n(=3r+1)$-element set $\{1, 2, \dots, n\}$ satisfying the following conditions:*

(1) *Any pair of elements is contained in exactly two 3-sets.*

(2) *The family of 3-sets can be divided into n subfamilies where the ith subfamily is a partition of $\{1, 2, \dots, n\} - \{i\}$.*

(3) *Exactly one pair of members of two different subfamilies meet in 2 elements.*

We show the construction of an $n \times n$ matrix M representing $\mathscr{L}_3^n$ $(n=3r+1)$ if the family in Conjecture 2 exists. We write zeros in the main diagonal. The ith row jth entry will be l if i is an element of the lth triple in the jth sub-family. It follows by condition (3) that for any two columns of M there are two rows equal in these columns. The rows are, of course, different due to the zeros. The first condition of Lemma 1 is satisfied. Condition (1) implies that any two rows agree in exactly two entries. Hence there are no two rows equal in any given triple of columns. The second condition of Lemma 1 is also satisfied. M represents $\mathscr{L}_3^n$, indeed.

Conjecture 2 follows for $n \equiv 1$ or 4 (mod 12) from the following result of Hanani [11, 12]. There exists a Steiner system $S(4, 2, n)$ for these n's. (I.e. we have a 4-uniform subsystem $\mathscr{S}$ on n-element set V such that for every two $v_1, v_2 \in V$ there exists exactly one member $S \in \mathscr{S}$ such that $\{v_1, v_2\} \subset S$.) Consider the 4-uniform set-system $\mathscr{S}$ over $\{1,2,\dots,n\}$ and replace every member $S \in \mathscr{S}$ with 4 3-element subsets. The obtained set-system $\mathscr{F}$ meets the condition of Conjecture 2, where the ith subfamily $\mathscr{F}_i = \{S - \{i\}: i \in S \in \mathscr{S}\}$.

We have proved the following.

Theorem 2.

$$s(\mathscr{L}_3^n) \geq n, \tag{11}$$

$$s(\mathscr{L}_3^n) = n \quad \textit{for } n = 12k+1 \textit{ and } n = 12k+4. \tag{12}$$

Corollary 1. $n \leq s(\mathscr{L}_3^n) \leq n+8$.

Proof. It follows from Theorem 2 and the inequality $s(\mathscr{L}_3^n) \leq s(\mathscr{L}_3^{n+1})$. □

Remark. In Theorem 2 we could have proved the following stronger statement: For $n \equiv 1$ or 4 (mod 12) we have a partition $G_1, G_2, \dots G_n$ of the edge-set of complete directed graph $\vec{K}_n$ ($\vec{K}_n = \{\langle u, v\rangle: 1 \leq u \neq v \leq n\}$, so it has $n(n-1)$ oriented edges) such that G_i consists of $(n-1)/3$ pairwise vertex-disjoint oriented triangles on the points $\{1,2,\dots n\} - \{i\}$ and $G_i \cup G_j$ $(i \neq j)$ contains exactly one pair of oppositely oriented edges.

Conjecture 2′. *The above-mentioned statement about the complete directed graph $\vec{K}_n$ holds for every $n \equiv 1$ (mod 3).*

This conjecture is much stronger than the usual statements about resolvable block-designs (cf. Hanani [12]).

Let us show the constructions for $n=4$ and 7:

0	0	0	1	0	2	2	2	1	1	1
0	0	1	0	1	0	2	1	2	2	1
0	1	0	0	1	1	0	2	1	2	2
1	0	0	0	1	2	1	0	2	1	2
				2	1	2	1	0	1	2
				2	1	1	2	2	0	1
				2	2	1	1	1	2	0

Theorem 3 [10].

$$\sqrt{2}\left(\frac{1}{k-1}\right)^{(k-1)/2} n^{(k-1)/2} < s(\mathscr{L}_k^n) < 2^{3k/2} n^{(k-1)/2} \quad (2\leq k<n). \tag{13}$$

Proof. The left-hand side of (13) is an easy consequence of (8). We now give a construction proving the right-hand side.

Let p be a prime number. We show that there is a set D of cardinality $2\lfloor\sqrt{p}\rfloor$ such that any integer satisfies

$$i \equiv d_1 - d_2 \pmod p \tag{14}$$

for some elements d_1, d_2 of D. The set D is defined by

$$D=\{0,1,2,\dots,a-1,\ 2a,\ 3a,\dots,(a-1)a\}$$

where $a=\lceil\sqrt{p}\rceil$. Suppose that i satisfies $0\leq i<p$ and $i=al+r$ $(0\leq r<a)$. If $1\leq l\leq a-2$ and $0<r<a$, then $d_1=(l+1)a$ and $d_2=a-r$ satisfy (14). If $i=al$ $(2\leq l\leq a-1)$, then $d_1=al$ and $d_2=0$ are suitable. $a=3a-2a$ and the rest can be expressed as a difference of zero and one of the numbers $1,2,\dots,a-1$. (We used $3\leq a-1$. Otherwise we have $p\leq 9$. These cases can be checked separately.)

As regards the cardinality of D we have

$$|D|=2a-2=2(\lceil\sqrt{p}\rceil-1)=2\lfloor\sqrt{p}\rfloor.$$

Let $\mathscr{P}$ be defined in the following way:

$$\mathscr{P}=\{c_{k-1}x^{k-1}+c_{k-2}x^{k-2}+\dots+c_1x+c_0 : c_0,\dots,c_{k-1}\in D, c_{k-1}=0 \text{ or } 1\}.$$

Note that

$$|\mathscr{P}|=2^k\lfloor\sqrt{p}\rfloor^{k-1}.$$

Let M be a $|\mathscr{P}|\times p$ matrix. Its rows are associated with members of $\mathscr{P}$. The jth entry of the row associated with $z(x)\in\mathscr{P}$ is $z(j) \pmod p$ $(0\leq j\leq p-1,\ 0\leq z(j)\leq p-1)$. We prove now that M represents $\mathscr{L}_k^p$. It is sufficient to prove, by Lemma 2, that M

represents $\binom{X}{k}$ (where $|X|=p$). Here we may use Lemma 1; we have to verify its conditions with $\mathscr{K}=\binom{X}{k}$, $\mathscr{K}^{-1}=\binom{X}{k-1}$ only.

Suppose that the rows associated with $z_1(x)$ and $z_2(x)$ have k equal entries:

$$z_1(t_i)\equiv z_2(t_i) \pmod p \quad (0\le t_1<\ldots<t_k<p).$$

Then the polynomial $z_1(x)-z_2(x)$ of degree $\le k-1$ has k different roots. This contradiction proves that z_1 and z_2 are the same, the 'two' rows are only one.

Choose now the integers $0\le t_1<\ldots<t_{k-1}<p$ arbitrarily. We have to find two different rows with equal entries in the t_1st, t_2nd, ..., t_{k-1}st places. Consider the polynomial

$$w(x)=(x-t_1)(x-t_2)\ldots(x-t_{k-1})=x^{k-1}+a_{k-2}x^{k-2}+\ldots+a_1x+a_0.$$

To a_i $(0\le i\le k-2)$ we can find two elements c_i and c_i' of D such that $a_i\equiv c_i-c_i' \pmod p$. Then $w(x)=z(x)-z'(x)$ holds where

$$z(x)=x^{k-1}+c_{k-2}x^{k-2}+\ldots+c_1x+c_0 \quad \text{and}$$

$$z'(x)=c'_{k-2}x^{k-2}+\ldots+c'_1x+c'_0.$$

$z(x)$ and $z'(x)$ are obviously different elements of $\mathscr{P}$. On the other hand, $z(t_i)\equiv z'(t_i) \pmod p$ holds, indeed. Both conditions of Lemma 1 are verified. M represents $\mathscr{L}_k^p$, indeed. This proves

$$s(\mathscr{L}_k^p)\le 2^k p^{(k-1)/2}.$$

For arbitrary n we choose a prime number p satisfying $n\le p\le 2n$. It exists by Chebyshev's theorem. Then we construct a matrix representing $\mathscr{L}_k^p$ and omit $p-n$ columns. The matrix represents $\mathscr{L}_k^n$. Hence

$$s(\mathscr{L}_k^n)\le 2^k p^{(k-1)/2}\le 2^k(2n)^{(k-1)/2}\le 2^{3k/2}n^{(k-1)/2}.$$

The theorem is proved. □

The method of Theorem 3 gives a good estimate only for small k. For instance, for $k=n/2$ a much better estimate is known. It is proved in [6] that

$$s(\mathscr{K})\le|\mathscr{K}^{-1}|+1 \tag{15}$$

holds for any Sperner-family. The following matrix proves it. Let the 0th row consist of zeros while the ith $(1\le i\le|\mathscr{K}^{-1}|)$ row contains zeros and i's: zeros in the column corresponding to the elements of the ith member of $\mathscr{K}^{-1}$. By (15),

$$s(\mathscr{L}_{n/2}^n)=s\left(\binom{X}{n/2}\right)\le\binom{n}{n/2}+1=2^{n+o(n)}$$

follows. Our feeling is that the truth is closer to the lower estimate given by (8):

Conjecture 3. $\log_2 s(\mathscr{L}_{n/2}^n)=n/2+o(n)$.

For comparison let us quote another related result [7, 8] stating

$$s(\mathscr{L}) \leq (1+o(1)) \binom{n}{n/2}$$

for any closure operation $\mathscr{L}$ on n elements.

4. Direct products

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be closure operations on the disjoint ground sets X_1 and X_2, resp. The *direct product* $\mathscr{L}_1 \times \mathscr{L}_2$ is defined by

$$(\mathscr{L}_1 \times \mathscr{L}_2)(A) = \mathscr{L}_1(A \cap X_1) \cup \mathscr{L}_2(A \cap X_2), \quad A \subseteq X_1 \cup X_2.$$

We prove the following, perhaps surprising

Theorem 4. $s(\mathscr{L}_1 \times \mathscr{L}_2) = s(\mathscr{L}_1) + s(\mathscr{L}_2) - 1.$

Proof. (1) Let us first prove the inequality

(16) $$s(\mathscr{L}_1 \times \mathscr{L}_2) \leq s(\mathscr{L}_1) + s(\mathscr{L}_2) - 1$$

with a construction. Let the $s(\mathscr{L}_1) \times n_1$ matrix M_1 and the $s(\mathscr{L}_2) \times n_2$ matrix M_2 represent $\mathscr{L}_1$ and $\mathscr{L}_2$, resp. We denote by α the last row of M_1 and by β the first row of M_2. The matrix M is constructed in Fig. 1.

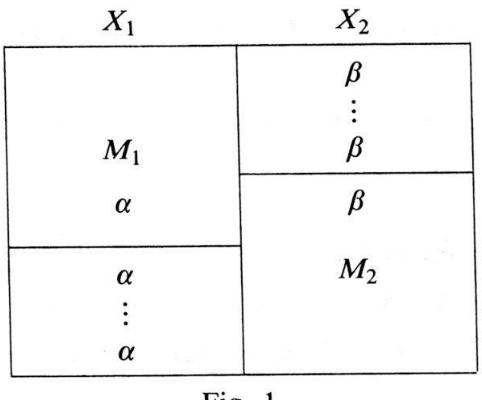

Fig. 1.

M is an $(s(\mathscr{L}_1) + s(\mathscr{L}_2) - 1) \times (n_1 + n_2)$ matrix. We have to show that it represents $\mathscr{L}_1 \times \mathscr{L}_2$, that is,

$$a \in \mathscr{L}_M(A) \Leftrightarrow a \in \mathscr{L}_1(A \cap X_1) \cup \mathscr{L}_2(A \cap X_2).$$

We may assume $a \in X_1$ because of the symmetry. Then the above condition can be divided into two implications:

(17) $$a \in \mathscr{L}_1(A \cap X_1) \Rightarrow \text{any two rows of } M \text{ equal in } A \text{ are equal in } a,$$

(18) $$a \notin \mathscr{L}_1(A \cap X_1) \Rightarrow M \text{ has two rows equal in } A \text{ but different in } a.$$

To prove (17) suppose that $a \in \mathcal{L}_1(A \cap X_1)$ choose two rows of M with equal entries in A. If both of them start with α, they are equal in a. If one of them does not start with α then the first parts of these rows are two different rows of M_1. By the definition of M_1, if they are equal in $A \cap X_1$ then they are equal in a.

To prove (18) suppose that $a \notin \mathcal{L}_1(A \cap X_1)$. M_1 contains two rows equal in $A \cap X_1$ but different in a. The extensions of these rows in M satisfy the right hand side of (18).

M represents $\mathcal{L}_1 \times \mathcal{L}_2$, consequently (16) is proved.

(2) With the help of two lemmas we prove now the inequality

$$s(\mathcal{L}_1 \times \mathcal{L}_2) \geq s(\mathcal{L}_1) + s(\mathcal{L}_2) - 1. \tag{19}$$

Let M be a matrix representing $\mathcal{L}_1 \times \mathcal{L}_2$ and suppose that the first n_1 columns correspond to the groundset X_1 of $\mathcal{L}_1$ and the remaining n_2 columns correspond to the groundset X_2 of $\mathcal{L}_2$. We want to prove that the number of rows of M is at least $s(\mathcal{L}_1) + s(\mathcal{L}_2) - 1$. The submatrix determined by the first n_1 columns in M is denoted by M_1. The rest is denoted by M_2.

Lemma 6. $\mathcal{L}_{M_2} = \mathcal{L}_2$.

Proof. Suppose that $A \subseteq X_2$, $a \in X_2$ and $a \in \mathcal{L}_2(A)$. Hence $a \in (\mathcal{L}_1 \times \mathcal{L}_2)(A)$ follows. If two rows of M are equal in A then, by the definition of M, they are also equal in a. Of course, this remains true if we consider the submatrix M_2 only. That is, we have proved $a \in \mathcal{L}_{M_2}(A)$. Conversely, suppose now $A \subseteq X_2$, $a \in X_2$ and $a \notin \mathcal{L}_2(A)$. Since $a \notin (\mathcal{L}_1 \times \mathcal{L}_2)(A)$ follows, M has two rows equal in A and different in a. These two rows in M_2 prove $a \notin \mathcal{L}_{M_2}(A)$. The proof is complete. □

$\mathcal{L}_{M_1} = \mathcal{L}_1$ follows analogously. However, we need a somewhat stronger statement for M_1:

Lemma 7. *Let N be a matrix. Suppose that the set of rows of N can be partitioned into k classes such that whenever $a \notin \mathcal{L}_N(A)$ holds, then there are two rows in one class which are equal in A and different in a. Then*

$$(\textit{number of rows of } N) \geq s(\mathcal{L}_N) + k - 1. \tag{20}$$

Proof. We use induction on k. For $k=1$, (20) follows by the definitions of $s(\mathcal{L}_N)$. Suppose now that N is partitioned into $k \geq 2$ classes satisfying the conditions of the lemma and that the statement is proved for smaller values.

$\mathcal{L}_N$ depends only on the relationship of the entries in N: which ones are equal, which ones are different. Therefore we may suppose that N contains only positive integers.

If N has one or more columns with the same entry everywhere, then delete these columns and denote the new matrix by N_1. The partition of rows of N is a 'good' partition in N_1, too. On the other hand

(21) $$s(\mathscr{L}_{N_1})=s(\mathscr{L}_N)$$

obviously holds. Moreover $\mathscr{L}_{N_1}(\emptyset)=\emptyset$.

Let $p_1,\ldots,p_k$ be different prime numbers greater than any entry of N_1. Multiply all entries in the ith class of rows by p_i $(1\le i\le k)$. The new matrix is denoted by N_2. It is easy to see that

(22) $$\mathscr{L}_{N_2}=\mathscr{L}_{N_1}$$

(since $\mathscr{L}N_1(\emptyset)=\emptyset$) and N_2 contains no equal entries in different classes of rows.

Let $\gamma=(\gamma_1,\ldots,\gamma_u)$ and $\delta=(\delta_1,\ldots,\delta_u)$ be one of the rows of the first and second classes in N_2, resp. We now delete γ and change any γ_i for δ_i in the ith column (for all i, $1\le i\le u$). The new matrix is denoted by N_3. The number of its rows is equal to the number of rows of N_2 minus 1. Let us prove that

(23) $$\mathscr{L}_{N_3}=\mathscr{L}_{N_2}.$$

Suppose first that $a\in\mathscr{L}_{N_2}(A)$ and choose two rows, μ_3 and ν_3 of N_3 equal in A. The corresponding rows in N_2 are denoted by μ_2 and ν_2, resp. If μ_2 and ν_2 are in the same class but not in the first one in N_2, then $\mu_2=\mu_3$, $\nu_2=\nu_3$. Therefore μ_2 and ν_2 are equal in A; $a\in\mathscr{L}_{N_2}(A)$ implies that they are equal in a. The same holds for μ_3 and ν_3. $a\in\mathscr{L}_{N_3}(A)$ is proved. If μ_2 and ν_2 are both in the first class, then μ_2 and μ_3 differ only in the sense that γ_i is changed for δ_i everywhere. The same holds for ν_2 and ν_3. It follows that μ_2 and ν_2 are equal in A, consequently in a. We obtain that μ_3 and ν_3 are also equal in $a: a\in\mathscr{L}_{N_3}(A)$. The last case is when μ_2 and ν_2 are in different classes. The supposition that μ_3 and ν_3 are equal in A implies either $A=\emptyset$ or that μ_2 and ν_2 are in the first and second classes, resp. $A=\emptyset$ is excluded by $\mathscr{L}_{N_2}(\emptyset)=\emptyset$. We may conclude that μ_2 is in the first class, ν_2 in the second one and they are different from γ and δ. $\nu_2=\nu_3$ is obvious. Since μ_3 and ν_3 are equal in A they both must contain δ_i in the ith place if $i\in A$. Then $\nu_2=\nu_3$ and δ are equal in A, consequently they are also equal in a. Their common entry here is δ_a. If μ_3 contains δ_i in the ith place when $i\in A$, then μ_2 contains γ_i there. Consequently, μ_2 and γ are equal in A and hence they are equal in a. Their common entry here is γ_a. We obtain that μ_3 contains δ_a in this column. Hence μ_3 and ν_3 are equal in $a: a\in\mathscr{L}_{N_3}(A)$.

Suppose now that $a\notin\mathscr{L}_{N_2}(A)$. N_2 contains two rows equal in A and different in a. If $A\neq\emptyset$, then the two rows are in the same class, consequently the corresponding rows in N_3 are also equal in A and different in a. $a\in\mathscr{L}_{N_3}(A)$ follows. If $A=\emptyset$, $a\in\mathscr{L}_{N_3}(\emptyset)$ would mean that there is a column with equal entries. This is impossible for $k\ge 3$. It is possible for $k=2$ only when N_2 contains merely γ_a and δ_a in the column corresponding to a. However in this case we are not able to find two rows in one class satisfying the conditions of the lemma for $a\notin\mathscr{L}_{N_2}(\emptyset)$. This contradiction proves $a\notin\mathscr{L}_{N_3}(A)$ and (23).

Moreover the conditions of the lemma are satisfied with at least $k-1$ classes for

N_3. Therefore we may use the induction hypothesis:

$$(\text{number of rows of } N_3) \geq s(\mathscr{L}_{N_3}) + k - 2.$$

Hence we obtain

$$(\text{number of rows of } N) \geq s(\mathscr{L}_N) + k - 1$$

by (21), (22) and (23). The lemma is proved. □

Let us turn back to the proof of the theorem, that is, more exactly, of (19). Form a partition of the rows of M_1 putting two rows in one class if their extension in M_2 is equal. Our aim is to apply Lemma 7 for M_1. We know that $\mathscr{L}_{M_1} = \mathscr{L}_1$ by Lemma 6. Choose a and A so that $a \notin \mathscr{L}_{M_1}(A) = \mathscr{L}_1(A)$. Then $a \notin \mathscr{L}_1(A) \cup X_2 = (\mathscr{L}_1 \times \mathscr{L}_2)(A \cup X_2)$ holds and M contains two rows equal in $A \cup X_2$ but different in a. That is, there are two rows of M_1 equal in A, different in a and being in the same class of the partition. We may apply Lemma 7 for M_1:

$$(24) \qquad (\text{number of rows of } M_1) \geq s(\mathscr{L}_1) + (\text{number of different rows of } M_2) - 1.$$

Using Lemma 6, again, we obtain

$$(25) \qquad (\text{number of different rows of } M_2) \geq s(\mathscr{L}_2).$$

(24) and (25) result in

$$(\text{number of rows of } M) \geq s(\mathscr{L}_1) + s(\mathscr{L}_2) - 1.$$

This proves (19) and the theorem. □

The analogous question for Sperner-families as minimal keys is not really answered. If $\mathscr{K}_1$ and $\mathscr{K}_2$ are Sperner-families on the disjoint sets X_1 and X_2, resp., then $\mathscr{K}_1 \times \mathscr{K}_2$ is defined as the family $\{A \cup B : A \in \mathscr{K}_2\}$ of subsets of $X_1 \cup X_2$. The proof of (16) works also here:

Theorem 5. $s(\mathscr{K}_1 \times \mathscr{K}_2) \leq s(\mathscr{K}_1) + s(\mathscr{K}_2) - 1$.

We found equality in many particular cases but it is not true in general, as the following example shows: let $X_1 = \{1,2,3,4,5\}$, $X_2 = \{6,7,8,9,10\}$, $\mathscr{K}_1 = \{\{1,2\}, \{3,4\}, \{1,5\}, \{2,5\}, \{3,5\}, \{4,5\}\}$, $\mathscr{K}_2 = \{\{6,7\}, \{8,9\}, \{6,10\}, \{7,10\}, \{8,10\}, \{9,10\}\}$. We show first $s(\mathscr{K}_1) = s(\mathscr{K}_2) \geq 5$. It is easy to see that

$$\mathscr{K}_1^{-1} = \{\{5\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}.$$

Lemma 3 implies $s(\mathscr{K}_1) \geq 4$. Suppose that $s(\mathscr{K}_1) = 4$ and the matrix M realizes it. $G(M)$ (see Lemma 5) has 4 vertices and its 5 edges are labelled with the members of $\mathscr{K}_1^{-1}$. Distinguishing several cases one can see that Lemma 5 implies that the sixth edge is labelled with $\{1,2\}$, $\{1,2,5\}$, $\{3,4\}$ or $\{3,4,5\}$. This contradicts the supposition that the key-set of M is $\mathscr{K}_1$. This proves $s(\mathscr{K}_1) \geq 5$. On the other hand, the following matrix shows $s(\mathscr{K}_1 \times \mathscr{K}_2) \leq 8$:

$$\begin{array}{ccccc|ccccc}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&1&1&1&0\\
0&0&0&0&0&0&1&0&1&1\\
1&1&1&1&0&0&0&0&0&0\\
1&1&1&1&0&1&1&1&1&0\\
1&1&1&1&0&1&0&0&1&2\\
0&1&0&1&1&0&0&0&0&0\\
1&0&0&1&2&1&1&1&1&0
\end{array}$$

(non-trivial!).

5. Closure relations with large $s(\mathscr{L})$

In [7] and [8] it is proved that there is an $\mathscr{L}$ satisfying

$$s(\mathscr{L}) \geq s(\mathscr{K}(\mathscr{L})) \geq \frac{1}{n^2}\binom{n}{\lfloor n/2 \rfloor}.$$

However the proof is non-constructive; we are not able to find one with $s(\mathscr{L})$ more than $\sqrt{2}\,\binom{n}{\lfloor n/2 \rfloor}^{1/2}$.

We pose here the analogous question for $\mathscr{K} \subseteq \binom{X}{k}$. Let

$$f_k(n) = \max\{s(\mathscr{K}) : \mathscr{K} \subseteq \tbinom{X}{k},\ |X| = n\}.$$

Theorem 6.

$$f_k(n) \geq \sqrt{2}\,\binom{2k-2}{k-1}^{\lfloor n/(2k-2) \rfloor /2}$$

Proof. Take a partition $X = X_1 \cup \ldots \cup X_q \cup Y$, where $q = \lfloor n/(2k-2) \rfloor$ and $|X_i| = 2k-2$ $(1 \leq i \leq q)$. $\mathscr{K}$ is defined by $\mathscr{K} = \{K : |K| = k,\ K \subseteq X_i \text{ for some } i\}$. It is easy to see that

$$\mathscr{K}^{-1} = \{A : |A \cap X_i| = k-1 \text{ for all } i, |A \cap Y| = |Y|\}.$$

Hence

$$|\mathscr{K}^{-1}| = \binom{2k-2}{k-1}^{\lfloor n/(2k-2) \rfloor}$$

follows and the theorem can be obtained by Lemma 3. □

It is easy to see that $f_1(n) = 2$. Theorem 6 gives $f_2(n) \geq \sqrt{2}\, 2^{\lfloor n/2 \rfloor /2} > 2^{n/4}$. It is surprising that such a 'simple' construction can have a big $s(\mathscr{K})$. However, we do not know the correct order of magnitude of $f_2(n)$.

6. Open problems

Besides Conjectures 1–3 we would like to pose some other related problems:

Problem 1. Give sufficient conditions for equality in Theorem 5.

Problem 2. Give methods for lower estimates of $s(\mathscr{L})$ and $s(\mathscr{K})$ deeper than Lemmas 3,4 and 5.

Problem 3. Determine $\max\{|\mathscr{K}^{-1}| : \mathscr{K} \subseteq \binom{X}{k},\ |X| = n\}$.

If $k=2$, then $\mathscr{K}$ is a graph on n vertices. $\mathscr{K}^{-1}$ is the family of all maximal vertex-sets containing no edge of this graph. The problem asks for what graph is this family the largest. Moon and Moser [13] solved this graph-theoretical question. Our problem is its analogue for hypergraphs.

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