Annals of Discrete Mathematics 28 (1985) 159 - 170
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PROBABILISTIC INEQUALITIES FROM EXTREMAL GRAPH RESULTS (A SURVEY)

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The aim of the paper is to survey the probabilistic inequalities proved by the method based on extremal combinatorial theorems.

1. Introduction

To illustrate the main idea of the field surveyed in the present paper, let us sketch the proof of the following theorem:

Theorem 1. [5] *If ξ and η are independent identically distributed random variables taking values from a Hilbert-space X, then*

$$P(\|\xi+\eta\|\geqslant x)\geqslant\tfrac{1}{2}P(\|\xi\|\geqslant x)^2$$

where $\|\ \|$ is the norm of X.

Proof.

1. We start with stating the following special case of the Turán theorem [17]: If a simple graph with n vertices contains no empty triangle (=for any 3 different vertices there is at least one edge) then the graph has at least $\left\lfloor\left(\frac{n-1}{2}\right)^2\right\rfloor$ edges.

2. We need the following simple statement from geometry:

If $a_1, a_2, a_3 \in X$ are of norm $\geqslant x$ ($\geqslant 0$) then $\|a_i+a_j\|\geqslant x$ holds for a pair $1\leqslant i<j\leqslant 3$.

The three vectors span a 3-dimensional Euclidean space. It is easy to see that the angle between a_i and a_j is $\leqslant 120°$ for some $1\leqslant i<j\leqslant 3$. Now it is easy to verify in the plane determined by them that $\|a_i+a_j\|\geqslant x$.

3. The following trivial inequality will be used:

$$P(\|\xi+\eta\|\geqslant x)\geqslant P(\|\xi+\eta\|\geqslant x,\ \|\xi\|\geqslant x,\ \|\eta\|\geqslant x). \tag{1}$$

Suppose for a while that ξ (and η) can have only m values, with equal probabilities: $P(\xi=a_i)=\frac{1}{m}$ $(1\leqslant i\leqslant m)$. Let a_i be ordered in the following way: $\|a_1\|\geqslant x, \ldots, \|a_n\|\geqslant x$, $\|a_{n+1}\|<x, \ldots, \|a_m\|<x$. Consider the following graph G. Let $a_1, \ldots, a_n$ be the vertices of G. Two vertices of G are connected with an edge iff the norm of their sum is $\geqslant x$. Then

$$\begin{aligned} &P(\|\xi+\eta\|\geqslant x,\ \|\xi\|\geqslant x,\ \|\eta\|\geqslant x)\\ &=m^{-2}\,(\text{the number of pairs } a_i, a_j\ (1\leqslant i,j\leqslant n) \text{ satisfying } \|a_i+a_j\|\geqslant x) \qquad (2)\\ &=n^{-2}\,(2\,(\text{the number of edges of } G)+n) \end{aligned}$$

holds since $\|a_i+a_i\|=2\|a_i\|\geqslant 2x\geqslant x$.

The graph G has no empty triangle by Section 2 of the proof. Applying the Turán theorem for G, we obtain a lower estimate for (2):

$$\frac{1}{m^2}\left(2\left[\left(\frac{n-1}{2}\right)^2\right]+n\right)\geqslant\frac{1}{2}\left(\frac{n}{m}\right)^2=\frac{1}{2}P(\|\xi\|\geqslant x)^2.$$

Theorem 1 is proved for this special case.

4. To prove the general case two approaches offer themselves:

a) Having an arbitrary distribution for ξ let us approximate it with the discrete distributions used in Section 3. This method was applied in [5] but the roughness of the elaboration led to unnecessary conditions for the distribution of ξ. Later Sidorenko [16] worked out this method properly. We do not treat it here in detail.

b) The other method can be found in [6] and [7]. Suppose that the distribution of ξ is arbitrary. Let the vertex-set X of G consist of the vectors satisfying $\|a\|\geqslant x$. Two vertices, a and b are connected if $\|a+b\|\geqslant x$. G is, in general, an infinite graph and it contains no empty triangle. The right-hand side of (1) is the measure, in a certain sense, of the set of edges of G. Namely, take the direct product of X with itself. Any edge (a, b) means two elements in the direct product X^2: the pairs (a, b) and (b, a). The set of edges is consequently a symmetric set in X^2. The measure P on X determines a product measure on X^2. The right-hand side of (1) is the measure of the above symmetric set according to this product measure. We have to give a good lower estimate of the measure of this set by terms of $P(\|\xi\|\geqslant x)$ (the measure of X) under the condition that G contains no empty triangle.

If X has finitely many, n elements, let the measure of each element be equal to 1. Then the Turán theorem says that the measure of the edge-set ($=2$ (the number of unoriented edges)$+n$) is $\leqslant \frac{1}{2}n^2$, that is, the half of the measure of X^2. We may expect the same statement for the general case. We will call the generalization of a discrete combinatorial statement for the product measures its "continuous version." Later we will precisely show that there is a transition (under very general condition) for continuous versions. Accepting the veracity of this statement, Theorem 1 follows easily by (1). □

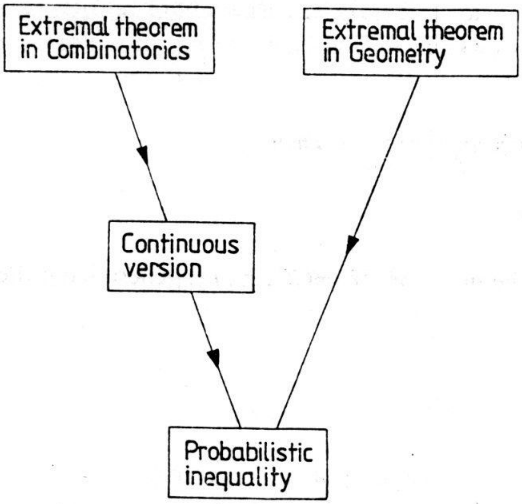

FIG. 1.

The above sketch of the proof can be illustrated with the diagram in Fig. 1. The aim of the present paper is to survey the results proved by this method.

2. Continuous versions of extremal theorems in combinatorics

The next lemma shows the connection between the continuous and finite graphs. Before stating it, let us give some definitions. Let $M=(X, \sigma, \mu)$ be a measure space, where σ is a σ-algebra on X and μ is a finite measure defined on σ. M^2 is the product of M with itself, that is, $M^2=(X^2, \sigma_2, \mu_2)$, where σ_2 is induced by the products of the members of σ and μ_2 is the product measure. If $E \subset X^2$ is measurable, that is, $E \in \sigma_2$ then $G=(X, E)$ is called a (directed) *graph*. Let Y be a subset of X, then G_Y denotes the graph *induced* by Y in G, that is, $G_Y=(Y, E_Y)$

where

$$E_Y=\{(a,b):a,b\in Y,\ (a,b)\in E\}.$$

M or μ is *atomless* if for any $A\in\sigma$, $\mu(A)>0$ there is a $B\subset A$, $B\in\sigma$ satisfying $0<\mu(B)<\mu(A)$.

Lemma 1. *Let $G=(X, E)$ be a graph on the atomless measure space $M=(X,\sigma,\mu)$. Suppose that*

$$|E_Y|\geqslant c|Y|^2$$

holds for any Y satisfying $|Y|>n_0$. Then

$$\mu_2(E)\geqslant c\mu(X)^2.$$

Proof. Introduce the notation $M^n=(X^n,\sigma_n,\mu_n)$ generalizing the case $n=2$. On the other hand, define

$$I(a,b;y_1,\dots,y_n)=\begin{cases}1 & \text{if } (y_a,y_b)\in E,\\ 0 & \text{otherwise},\end{cases}$$

$$\left(1\leqslant a,b\leqslant n,\quad \text{integers};\quad (y_1,\dots,y_n)\in X^n\right).$$

This function is obviously measurable since E is measurable. Take the integral

$$\begin{aligned}&\int_{X^n} I(a,b;y_1,\dots,y_n)\,d\mu_n\\ &=\int_{X^n} I(1,2;y_1,\dots,y_n)\,d\mu_n\\ &=\int_{E\times X^{n-2}} 1\,d\mu_n=\mu_2(E)\mu(X)^{n-2}\end{aligned}\tag{3}$$

when $a\neq b$. If $a=b$ we use

$$\begin{aligned}&\int_{X^n} I(a,a;y_1,\dots,y_n)\,d\mu_n\\ &=\int_{X^n} I(1,1;y_1,\dots,y_n)\,d\mu_n\\ &=\mu(\{y:(y,y)\in E\})\mu(X)^{n-1}.\end{aligned}\tag{4}$$

Summing up (3) and (4) for all pairs $1 \leqslant a, b \leqslant n$, we obtain

$$\int_{X^n} \Big(\sum_{1 \leqslant a, b \leqslant n} I(a, b; y_1, \dots, y_n) \Big) d\mu_n$$

$$= n(n-1)\mu_2(E)\mu(X)^{n-2} + n\mu(\{y : (y, y) \in E\})\mu(X)^{n-1}. \tag{5}$$

Observe that $\sum_{1 \leqslant a, b \leqslant n} I(a, b; y_1, \dots, y_n)$ is nothing else but $|E_{\{y_1, \dots, y_n\}}|$, that is, the number of edges of the subgraph induced by $\{y_1, \dots, y_n\}$ if $y_1, \dots, y_n$ are all distinct. This latter condition holds with the exception of a set of measure 0. This is heuristically obvious and can be rigorously proved (see e.g. [7]). Using the assumption $|E_{\{y_1, \dots, y_n\}}| \geqslant cn^2$, $(n > n_0)$,

$$cn^2 \mu(X)^n \leqslant \int_{X^n} \Big(\sum_{1 \leqslant a, b \leqslant n} I(a, b; y_1, \dots, y_n) \Big) d\mu_n \tag{6}$$

(5) and (6) imply

$$c \leqslant \frac{n-1}{n} \frac{\mu_2(E)}{\mu(X)^2} + \frac{1}{n} \frac{\mu(\{y : (y, y) \in E\})}{\mu(X)}.$$

If $n \to \infty$ this leads to $\mu_2(E) \geqslant c\mu(X)^2$. □

Let $\mathscr{G}$ be an arbitrary class of graphs $G = (X, E)$ determined on the measure space $M = (X, \sigma, \mu)$. $\mathscr{G}$ is called hereditary if $G \in \mathscr{G}$ implies $G_Y \in \mathscr{G}$ for any measurable $Y \subset X$. Then

$$H(M, \mathscr{G}) = \inf_{E \in \mathscr{G}} \frac{\mu_2(E)}{\mu(X)^2}$$

can be considered as the continuous analogue of the "minimum number" of edges in $\mathscr{G}$. Analogously, let us define

$$H(n, \mathscr{G}) = \min \frac{|E|}{n^2}, \tag{7}$$

where the minimum runs over all members of $\mathscr{G}$ having exactly n vertices. It is proved in [7] that (7) has a limit if $n \to \infty$. The inequality

$$H(M, \mathscr{G}) \geqslant \lim H(n, \mathscr{G})$$

is an easy consequence of Lemma 1 if M is atomless. However, this inequality holds for measures with atoms supposing that $\mathcal{G}$ has certain properties.

Let $G=(\{x, x_1, ...\}, E)$ be a graph, and define $G^x=(\{x', x'', x_1, ...\}, E^x)$, where E^x consists of the pairs obtained by substituting x either by x' or x'' in any way in any pair which is in E. In other words, we form two copies of x in all edges of G. $\mathcal{G}$ is called *doublable* iff $G \in \mathcal{G}$ implies $G^x \in \mathcal{G}$ for any vertex x of G.

Theorem 2. [7] *Suppose that $\mathcal{G}$ is a hereditary class of graphs on the measure space M,*

$$H(M, \mathcal{G}) \geqslant \lim H(n, \mathcal{G}) \tag{8}$$

if M is atomless or $\mathcal{G}$ is doublable.

For our applications we need this direction of the inequality. One may guess, however, that equality holds in (8) under some reasonable conditions. Indeed, if $\mathcal{G}$ is doublable then (8) holds with equality (see [7]).

However, there is another class of $\mathcal{G}$'s, for which the equality in (8) is proved. $\mathcal{G}$ is called *strongly hereditary* if (i) $\mathcal{G}$ is hereditary, (ii) adding a new edge to a member of $\mathcal{G}$, the new graph is also in $\mathcal{G}$, (iii) adding a new vertex to a member of $\mathcal{G}$ (until a certain fixed cardinality) with all the possible edges containing x, the new graph is also in $\mathcal{G}$. It is proved in [11] that to any strongly hereditary class $\mathcal{G}$ there is another class $\mathcal{G}_0$ of graphs that a graph H has all its induced subgraphs from $\mathcal{G}$ iff the complement $\bar{H}$ contains no subgraph from $\mathcal{G}_0$. $\mathcal{G}_0$ is called in the literature the class of forbidden graphs. The equality in (8) for strongly hereditary graphs is an easy consequence of a theorem of Brown, Erdös and Simonovits [3]. (The conditions of this theorem and of Theorem 2 are stated incorrectly in [7].)

The above results are formulated for directed graphs but, in fact, we need them for undirected graphs. The connection is obvious: each edge (a, b) $(a \neq b)$ of an undirected graph is replaced by two oppositely directed edges (a, b), (b, a). Let us remark that Bollobás [2] independently proved (8) with equality for strongly hereditary classes of undirected graphs on an atomless measure space. His proof is easier for this special case.

Let us see how we can obtain the "continuous version" of the Turán theorem by Theorem 2. Let $\mathcal{G}$ be the class of all graphs $G=(X, E)$ such that (i) $(a, b) \in E$ iff $(b, a) \in E$, (ii) $(a, a) \in E$ for all $a \in X$, (iii) if a, b, c are different vertices $(\in X)$ then at least one of (a, b), (b, c), (c, a) is in E. By the usual Turán theorem, the graph $G=(X, E)$, $|X|=n$, $G \in \mathcal{G}$ must contain at least $\left\lceil \frac{(n-1)^2}{4} \right\rceil$ pairs of edges

$(a, b), (b, a)$ $(a \neq b)$. Hence, by property (ii), the number of edges is

$$|E| \geqslant 2\left\lceil \frac{(n-1)^2}{4} \right\rceil + n \geqslant \frac{1}{2} n^2 .$$

This implies lim $H(n, \mathcal{G}) \geqslant \frac{1}{2}$ and (8) implies $H(M, \mathcal{G}) \geqslant \frac{1}{2}$.

The proof of Theorem 1 can be completed if this inequality is used for measure space induced by the set $\{\|\xi\| \geqslant x\}$ in the probability measure P, and for the set

$$E = \{(\xi, \eta) \colon \|\xi + \eta\| \geqslant x\}$$

$$\frac{P(\|\xi+\eta\| \geqslant x)}{P(\|\xi\| \geqslant x)^2} \geqslant \frac{1}{2}$$

is a consequence of $H(M, \mathcal{G}) \geqslant \frac{1}{2}$.

Let us remark that [7] states the results on the "continuous versions" for g-graphs, however the equality in (8) is known for strongly hereditary graphs only when $g=2$. [7] also contains some results for the case when M has atoms and $\mathcal{G}$ is not doublable. Finally, [8] gives the "continuous versions" of a completely different class of combinatorial extremal problems: a transformation T of g-graphs to h-graphs is given; the number of vertices and g-edges is fixed, the number of edges of the transformed graph has to be minimized.

3. Two random variables

One can see with an easy construction that Theorem 1 is sharp in the following sense. For any p $(0 \leqslant p \leqslant 1)$ and any $x \geqslant 0$ there is a distribution of ξ (and η) in a more than two-dimensional space such that $P(\|\xi+\eta\| \geqslant x) = \frac{1}{2}p^2$ and $p = P(\|\xi\| \geqslant x)$. In other words, $P(\|\xi+\eta\| \geqslant x)$ has no better lower estimate in terms of $P(\|\xi\| \geqslant x)$ (If the dimension is at least 2. In one-dimension there is a better estimate.) The next theorem investigates the same problem if $P(\|\xi\| \geqslant cx)$ is used in place of $P(\|\xi\| \geqslant x)$.

Theorem 3. ([16] and [9] independently) *Let X be an infinite-dimensional Hilbert-space, ξ and η be X-valued independent, identically distributed random variables, then the best possible functions f in the inequality $P(\|\xi+\eta\| \geqslant x) \geqslant f(P(\|\xi\| \geqslant cx))$ are the following ones:*

$$f(p) = \begin{cases} \text{[illegible]} & \text{if } p \geqslant \frac{1}{3}, \\ 2p - \frac{3}{2}p^2 & \text{otherwise}, \end{cases} \qquad \text{when } \tfrac{5}{2} \leqslant c < \infty$$

$$f(p)=\begin{cases}\frac{1}{2} & \text{if } p\geqslant\frac{1}{2},\\ 2p(1-p) & \text{otherwise},\end{cases}\qquad \text{when } \tfrac{3}{2}\leqslant c<\tfrac{5}{2}$$

$$f(p)=\begin{cases}-\frac{1}{2}+2p-p^2 & \text{if } p\geqslant\frac{1}{2},\\ p^2 & \text{otherwise},\end{cases}\qquad \text{when } \tfrac{\sqrt{5}}{2}\leqslant c<\tfrac{3}{2}$$

$$f(p)=\tfrac{1}{2}p^2,\qquad \text{when } 1\leqslant c<\tfrac{\sqrt{5}}{2}$$

$$f(p)=\frac{1}{k-1}p^2,\ \text{when } \sqrt{\frac{k-1}{2(k-2)}}\leqslant c<\sqrt{\frac{k-2}{2(k-3)}}\qquad (4\leqslant k<\infty)$$

$$f(p)=0,\qquad \text{when } 0<c\leqslant\frac{1}{\sqrt{2}}.$$

Each row of the theorem can be proved following the proof of Theorem 1, that is, the scheme given in Fig. 1. We show a new phenomenon of the proof in the case $\sqrt{5}/2\leqslant c<3/2$.

We start with a very brief sketch of the proof. Fix the real number $x>0$ and put $X_1=\{a : a\in X,\ \|a\|<\frac{\sqrt{5}}{2}x\}$, $X_2=X-X_1$. The graph $G=(X, E)$ is defined by $E=\{(a, b) : \|a+b\|\geqslant x\}$. The following simple geometric statement is true. If a_1, a_2, a_3 are vectors in a Hilbert-space and $\|a_1\|\geqslant\frac{\sqrt{5}}{2}$, $\|a_2\|\geqslant\frac{\sqrt{5}}{2}$ then there is a pair $i\neq j$ satisfying $\|a_i+a_j\|\geqslant 1$. Hence the graph G has no empty triangle with at least two vertices in X_2. If G is finite and $|X_1|=n_1$, $|X_2|=n_2$ then according to Lemma 2 of [6] the number of edges is at least

$$n_1\left\lfloor\frac{n_2-n_1}{2}\right\rfloor+\binom{\lfloor(n_2-n_1)/2\rfloor}{2}+\binom{\lceil(n_2-n_1)/2\rceil}{2}$$

if $n_2\geqslant n_1$ and $\binom{n_2}{2}$ otherwise. The "continuous version" of this lemma proves the statement for $\frac{\sqrt{5}}{2}\leqslant c<\frac{3}{2}$.

The first novelty here is that we cannot disregard the small ($\|\ \|<\frac{\sqrt{5}}{2}x$) vectors, like in the case of Theorem 1. This causes the trouble with two classes, so new types of extremal graph results are needed. Finally, we need a generalization of Theorem 2 for two (or more) classes. This generalization is straightforward and can be found in [7] (Theorem 3).

The proof of this row and any other row is valid for any (lower-dimensional) Hilbert-space. But the estimates are not the best, in general. The constructions do not work.

In the d-dimensional space the necessary geometric problems are unsolved (even for $d=3$). Namely, the quantities

$$\delta(2, k, X) = \inf \max_{1 \leqslant i < j \leqslant k} \{\|a_i + a_j\|\},$$

where the infimum is taken over all vectors $a_1, \ldots, a_k \in X$ satisfying $\|a_i\| \geqslant 1$ ($1 \leqslant i \leqslant k$), are unknown in general. Thus the "sharp" estimates analogous to Theorem 3 for d-dimension involve the constants $\delta(2, k, X)$. Sidorenko [16] stated Theorem 3 in such a generalized form. He has sharp estimates for any linear normed space X, supposing that $\delta(2, k, X)$ are known. However, the method can be extended for additive groups X having an invariant metric (see Theorem 5 of [6]).

There is another generalization of Theorem 1 in [16]. The best lower estimates of

$$P(\|a\xi + b\eta\| \geqslant x)$$

in terms of $P(\|\xi\| \geqslant cx)$ are determined, where a and b are fixed reals, ξ and η are independent, identically distributed real random variables. Another theorem (Theorem 22 of [16]) deals with the case $X = l_2$.

We think that our method is helpful in a more general context. Let f_1 and f_2 be a one-variable and a two-variable function, respectively. A lower estimate is needed for $P(f_2(\xi, \eta) \geqslant x)$ in terms $P(f_1(\xi) \geqslant cx)$. Examples are $f_1(\xi) = |\xi|$, $f_2 = \xi\eta$, or $f_1(\xi)$ is the vector of the coordinates of ξ.

4. More random variables

Probabilists claim that the real task of probability theory is to say something about a large set of random variables. Thus, they would need a generalization of Theorem 3 for $\|\xi_1 + \xi_2 + \ldots + \xi_l\|$ rather than for $\|\xi_1 + \xi_2\|$. Let us try the case $l=3$.

If we copy the proof of Theorem 1, the geometry works: $\|a_1\| \geqslant 1$, $\|a_2\| \geqslant 1$, $\|a_3\| \geqslant 1$, $\|a_4\| \geqslant 1$ implies that there are 3 distinct ones of them so that $\|a_i + a_j + a_k\| \geqslant 1$. However, there is a little trouble with the combinatorics. We need the minimum number $T(n, 4, 3)$ of 3-element subset of an n-element set under the condition that any 4-element subset contains one of them. It is conjectured that $T(n, 4, 3)/\binom{n}{3} \to \frac{4}{9}$. The proof of this conjecture would imply

$$P(\| \;_1 + \xi_2 + \xi_3\| \geqslant x) \geqslant \tfrac{4}{9} P(\|\xi_1\| \geqslant x)^3 \tag{9}$$

for any 3 independent, identically distributed random variables in a Hilbert-space (see [12]). The first problem is that even the order of magnitude of this estimate is not correct. It is proved in [10] that

$$P(\|\xi_1+\xi_2+\xi_3\|\geqslant x)\geqslant\tfrac{1}{2}P^2(\|\xi_1\|\geqslant x)(1-P(\|\xi_1\|\geqslant x)) \tag{10}$$

holds if $P(\|\xi_1\|\geqslant x)\leqslant\frac{1}{3}$. (10) is much stronger for small values of $P(\|\xi_1\|\geqslant x)$ than (9). However, the constant $\frac{1}{2}$ is not the best possible.

The reason why the situation here is different from the case $l=2$ is that the small vectors also play role. This problem is circumvented if we consider $P(\|\xi_1+\xi_2+\xi_3\|, \|\xi_1\|, \|\xi_2\|, \|\xi_3\|\geqslant x)$. Indeed,

$$P(\|\xi_1+\xi_2+\xi_3\|, \|\xi_1\|, \|\xi_2\|, \|\xi_3\|\geqslant x)\geqslant\tfrac{4}{9}P(\|\xi_1\|\geqslant x)^3 \tag{11}$$

is proved for the independent, identically distributed two-dimensional random variables. In fact (11) is proved in [10] only for one-dimensional variables and with 0.44444. Since then Bereznai and Varecza [1] proved that (37) of [10] tends to $\frac{4}{9}$ and Ha Le Anh [4] proved Lemmas 2.2 and 2.4 for two-dimensions. Ha Le Anh also gave counterexamples for those lemmas if the dimension is higher. So the method of [10] does not work for higher dimensions. However, we still conjecture

$$P(\|\xi_1+\xi_2+\xi_3\|, \|\xi_1\|, \|\xi_2\|, \|\xi_3\|\geqslant x)\geqslant\tfrac{5}{9}P(\|\xi_1\|\geqslant x)^3 \tag{12}$$

for any Hilbert-space.

Sidorenko [16, 15] found similar results for the case when the lower estimate uses $P(\|\xi_1\|\geqslant cx)$.

Finally, let us mention another result of Sidorenko [16]. He gives lower estimates of

$$P(\max_{1\leqslant i_1<\ldots<i_q\leqslant l}\|\xi_{i_1}+\ldots+\xi_{i_q}\|\geqslant x)$$

and

$$P(\min_{1\leqslant i_1<\ldots<i_q\leqslant l}\|\xi_{i_1}+\ldots+\xi_{i_q}\|\geqslant x)$$

in terms of $P(\|\xi_1\|\geqslant x)$, where q also runs in the min and max.

5. Open problems

Although the papers in this field contain many open questions (actually, they contain more open questions than results) we would like to emphasize some of them.

1. Do we always have equality in (8)? It is known that if $\mathcal{G}$ is doublable and if $g=2$, $\mathcal{G}$ is strongly hereditary and M is atomless. So, it is unknown for some cases, even if $g=2$, and very little is known for $g>2$.

2. It is easy to see that if $|E_Y|\geqslant c|Y|^2$ is replaced by $|E_Y|\leqslant c|Y|^2$ in Lemma 1 then $\mu_2(E)\leqslant c\mu(X)^2$ can be concluded. Suppose now $|E_Y|\leqslant c|Y|$ (or $c|Y|^\alpha$ $(0<\alpha<2)$), only. We conjecture that (under some measurability condition) the Hausdorff dimension of E is at most 1 (α) and its 1-dimensional (α-dimensional) outer Hausdorff-measure is at most c.

(A *square* is a set $S\subset X^2$ of form $S=A\times B$, where $A, B\in\sigma$ and $\mu(A)=\mu(B)$. $\mu(A)=\mu(B)$ is the size $\rho(S)$ of S. If $E\subset X^2$ then $\lambda(E, \alpha, \rho)$ is defined by

$$\lambda(E, \alpha, \rho)=\inf \sum_{i=1}^{\infty} \rho(S_i)^\alpha,$$

where $E\subset \bigcup_{i=1}^{\infty} S_i$ and $\rho(S_i)\leqslant\rho$. $\lambda(E, \alpha, \rho)$ is a non-decreasing function of ρ, therefore the limit

$$\lambda(E, \alpha)=\lim_{\rho\to 0} \lambda(E, \alpha, \rho)$$

exists. This is called the α-dimensional *outer Hausdorff-measure* of E. It is easy to see that there is an α_0 such that $\lambda(E, \alpha)=\infty$ if $\alpha<\alpha_0$ and $\lambda(E, \alpha)=0$ if $\alpha>\alpha_0$. This α_0 is the *Hausdorff-dimension* of E.)

3. Our estimates determine the "best" function f of the distribution function of $||\xi||$ at a given place (cx). Another problem is to find the best operator f, where $P(||\xi||\geqslant x)$ is considered to be a function of x.

A modest result in this direction can be found in [6]:

$$P(||\xi+\eta||\geqslant x)\geqslant\begin{cases}(p_1+p_2)^2/2 & \text{if } p_1\leqslant p_2,\\ 2p_1 p_2 & \text{if } p_1\geqslant p_2,\end{cases}$$

where $p_1=P(1/\sqrt{2}\leqslant||\xi||<(1+\sqrt{3})/2)$ and $p_2=P(||\xi||\geqslant(1+\sqrt{3})/2)$.

4. Determine the values $\delta(l, k, X)=\inf \max_{1\leqslant i_1<\ldots<i_l\leqslant k} \{||a_{i_1}+\ldots+a_{i_l}||\}$, where the infimum is taken over all vectors $a_1, \ldots, a_k\in X$ satisfying $||a_i||\geqslant 1$ $(1\leqslant i\leqslant k)$.

A survey of some results can be found in [16, 15] and for the 3-dimensional X in [6].

5. Prove (12).

A final remark. Most of the work in this field is done by Sidorenko and by the author. I know the results of the latter one better so this survey is based on them. Consequently, the interested reader should carefully study the quoted and forthcoming papers of Sidorenko.

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