EXTREME MORE-PART SPERNER FAMILIES

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__Abstract.__ Let $X_1 \cup \ldots \cup X_M$ be a partition of X. The $(i_1,\ldots,i_M)$th entry of the M-dimensional profile-matrix $P(\mathcal{F})$ of the family $\mathcal{F} \subseteq 2^X$ is the number of members containing exactly i_j elements from X_j. An M-part Sperner family contains no two members $F_1 \subset F_2$ being equal in exactly M-1 parts X_j. Consider the profile-matrices of all possible M-Sperner families. The paper determines the extreme ones of these profile-matrices and gives a survey of analogous statements and consequences.

__Zusammenfassung.__ Sei $X_1 \cup \ldots \cup X_M$ eine Partition von X. Das $(i_1,\ldots,i_M)$-te Element der M-dimensionalen Profilmatrix $P(\mathcal{F})$ der Familie $\mathcal{F} \subseteq 2^X$ ist die Anzahl der Mitglieder, die genau i_j Elemente aus X_j enthalten. Eine M-part-Spernerfamilie enthält keine zwei Mitglieder $F_1 \subset F_2$, die auf genau M-1 Teilen X_j gleich sind. Wir betrachten die Profilmatrizen aller möglichen M-part-Spernerfamilien. Diese Arbeit bestimmt die extremen dieser Profilmatrizen und gibt einen Überblick über analoge Resultate und Folgerungen.

__1. Introduction__

Let X be a finite set of n elements and $F_1,\ldots,F_m$ be distinct subsets of X such that $F_i \not\subseteq F_j$ $(i \neq j)$. The classical theorem of Sperner [16] states that

$$m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} . \tag{1}$$

Kleitman [14] and the author [12] independently discovered that considering any partition $X_1 \cup X_2 = X$ the weaker condition

$$F_i \subset F_j,\ F_j - F_i \subseteq X_1 \text{ or } X_2 \text{ implies } i = j$$

is sufficient to have (1). As it is shown in [12], this is not true for 3 parts.

In general, the family $\mathcal{F} \subseteq 2^X$ is called an M-__part Sperner family__ with respect to the partition $X = X_1 \cup \ldots \cup X_M$ iff

$$F_1, F_2 \in \mathcal{F},\ F_1 \subset F_2,\ F_2 - F_1 \subseteq X_i \ \underline{\text{for some}}\ i (1 \leq i \leq M) \quad (2)$$
$$\underline{\text{imply}}\ F_1 = F_2$$

Griggs [8] and Sali [15] found upper estimates for the size of an M-part Sperner family $\mathcal{F}$ and recently Griggs, Odlyzko, Shaerer [11] and Füredi [7] proved asymptotically good ones. [4] gives the exact maximum of $|\mathcal{F}|$ for the case $M = 3$, $|X_3| = 1$.

On the other hand, [13] and [10] give more complicated additional conditions which, completing (2), ensure the validity of (1).

After this brief history of the M-part Sperner families let us introduce another branch of the Sperner theory. If $\mathcal{F} \subseteq 2^X$ is a family then let p_i $(0 \leq i \leq n)$ denote the number of i-element members of $\mathcal{F}$. Take the vectors $(p_0, p_1, \ldots, p_n) \in R^{n+1}$ for all Sperner (= 1-Sperner) families. It is proved in [1] the extreme points of the convex hull of this set in R^{n+1} are $(0,\ldots,0)$ and $(0,\ldots,0,\binom{n}{i},0,\ldots,0)$ $(0 \leq i \leq n)$. In [1], [2] and [6] the analogous extreme points are determined for some other classes of families.

For the combination of the above two branches of the theory we have to define the <u>profile-matrix</u> $P(\mathcal{F})$ of the family $\mathcal{F}$ given on $X = X_1 \cup \ldots \cup X_M$ $(X_i \cap X_j = 0,\ i \neq j; |X_i| = n_i)$. It is an M-dimensional matrix with the entries

$$p_{i_1, i_2, \ldots, i_M}(\mathcal{F}) = |\{F : F \in \mathcal{F}, |F \cap X_j| = i_j (1 \leq j \leq M)\}|.$$

$P(\mathcal{F})$ can be considered as a point of $R^{(n_1+1)\ldots(n_M+1)}$. Let S^M denote the class of all M-part Sperner families (with respect to $X_1 \cup \ldots \cup X_M$). In general, if A_X is a class of some families $\mathcal{F}$ of subsets of X $(A \subseteq 2^{2^X})$ then $\mu(A)$ denotes the set of points in $R^{(n_1+1)\ldots(n_M+1)}$ corresponding to all profile-matrices $P(\mathcal{F})$ $(\mathcal{F} \in A)$. $\langle \mu(A) \rangle$ is its <u>convex hull</u> and $\mathcal{E}(A)$ denotes the set of extreme points of $\langle \mu(A) \rangle$.

The aim of the present talk is to survey the results of [3], [4] and [5] on this subject.

2. The extreme profile-matrices of the class of M-part Sperner families

A set $I \subseteq \{0,\dots,n_1\} \times \dots \times \{0,\dots,n_M\}$ is called a partial transversal if $(i_1,\dots,i_j,\dots,i_M) \in I$, $(i_1,\dots,i'_j,\dots,i_M) \in I$ imply $i_j = i'_j$, that is, if I contains no two elements in the same "row". On the other hand, I is called an antichain if $(i_1,\dots,i_M)$, $(j_1,\dots,j_M) \in I$ and $i_1 \leq j_1,\dots,i_M \leq j_M$ imply $i_1 = j_1,\dots,i_M = j_M$. The M-dimensional matrix A(I) is defined by the entries

$$a_{i_1,\dots,i_M}(I) = \begin{cases} \binom{n_1}{i_1}\dots\binom{n_M}{i_M} & \text{if } (i_1,\dots,i_M) \in I \\ 0 & \text{otherwise.} \end{cases}$$

After these definitions we are able to present our main theorems.

Theorem 1 [3]. *$\mathcal{E}(S^M)$ consists of the matrices A(I) where I is a partial transversal.*

Theorem 2 [3]. *$\mathcal{E}(S)$ consists of the matrices A(I) where I is an antichain.*

3. Consequences

The most important question here is to determine max $|\mathcal{F}|$ over the M-part Sperner families. As

$$|\mathcal{F}| = \sum_{i_1=0}^{n_1} \dots \sum_{i_M=0}^{n_M} p_{i_1,\dots,i_M} \quad \text{is a}$$

linear function of the entries of $P(\mathcal{F})$, $|\mathcal{F}|$ attains its maximum at one of the extreme points of $\langle \mu(S^M) \rangle$.
Thus Theorem 1 implies the next

Theorem 3 [11]. *There is an $\mathcal{F} \in S^M$ such that*

$$|\mathcal{F}| = \max\{|\mathcal{F}'| : \mathcal{F}' \in S^M\}$$

and $F \in \mathcal{F}$ implies that all sets $G \subseteq X$ satisfying $|F \cap X_j| = |G \cap X_j|$ for all j $(1 \leq j \leq M)$ belong to $\mathcal{F}$.

At the first glance it might seem that this solves the problem of finding max $|\mathcal{F}|$. However, this is not true because it is hard to compare the sum of entries of the possible A(I)'s. The asymptotic results [11] and [7] are based on different methods. But the case $M = 2$ is easy enough. Then a simple transformation shows that $A(I)$ is maximum if the large $\binom{n_1}{i_1}$ are paired with large $\binom{n_2}{i_2}$ and the small ones with each other:

$$\sum_i \binom{n_1}{i} \binom{n_2}{\left\lceil \frac{n_1}{2} \right\rceil + \left\lfloor \frac{n_2}{2} \right\rfloor - i} = \binom{n_1 + n_2}{\left\lfloor \frac{n_1 + n_2}{2} \right\rfloor} = \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor}.$$

This is an old result ([14],[12]). However, all 2-part Sperner families maximizing $|\mathcal{F}|$ were determined only recently in [5]. To obtain them it is necessary to determine first all the extreme points (given in Theorem 1, $M = 2$) maximizing the sum of the entries. The profile-matrices of all other optimal $\mathcal{F}$ can be convex linear combinations of the above extreme points. It is shown that the latter ones do not exist.

Although Theorem 3 does not help to determine $\max |\mathcal{F}|$ in general, it leads to an exact solution in the very special case $M = 3$, $|X_3| = 1$ [4].

Let us show now an application of Theorem 2. Let $X_1 \cup X_2$ be a partition of X and suppose that $\mathcal{F}$ is a Sperner family and its members meet X_1 in at least l elements. Griggs [9] proved the inequality

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n-l}{|F|-l}} \frac{\binom{|F \cap X_1|}{l}}{\binom{|X_1|}{l}} \le 1 \qquad (3)$$

for such families $\mathcal{F}$. The left hand side is, in fact a linear combination of the quantities p_{ij} $(l \le i \le n_1,\ 0 \le j \le n_2)$. Therefore it is enough to prove (3) for the extreme points of Theorem 2 ($M = 2$):

$$\sum_{(i,j) \in I} \binom{n_1}{i}\binom{n_2}{j} \frac{1}{\binom{n-l}{i+j-l}} \frac{\binom{i}{l}}{\binom{n_1}{l}} \le 1$$

for any antichain $I \subset \{l,\dots,n_1\} \times \{0,\dots,n_2\}$. Observe that this inequality involves no families. It speaks about binomial coefficients and antichains, only. It equivalent and more pleasent form is

$$\sum_{(i,j) \in I'} \frac{\binom{m_1}{i}\binom{m_2}{j}}{\binom{m_1 + m_2}{i + j}} \le 1$$

(I' is an antichain in $\{0,\ldots,m_1\} \times \{0,\ldots,m_2\}$ $m_1 = n_1 - 1$, $m_2 = n_2$). Its prove can be found in [3].

As a final remark, let us observe that Theorems 1 and 2 can be formulated in the following way. The extreme points of a class of families satisfying a certain condition in $X_1 \cup \ldots \cup X_M$ are the matrices A(I) where I satisfies "the same" condition in $\{0,\ldots,n_1\} \times \ldots \times \{0,\ldots,n_M\}$. This law is valid in a more general context (see [3]).

References

[1] Erdős, P.L., Frankl, P. and Katona, G.O.H.: Intersecting Sperner families and their convex hulls, Combinatorica 4 (1984) 21-34.

[2] Erdős, P.L., Frankl, P. and Katona, G.O.H.: Extremal hypergraph problems and convex hulls (to appear in Combinatorica).

[3] Erdős, P.L. and Katona, G.O.H.: Convex hulls of more-part Sperner families (submitted to Graphs and Combinatorics).

[4] Erdős, P.L. and Katona, G.O.H.: An exact 3-part Sperner theorem (submitted to Studia Sci. Math. Hungar.).

[5] Erdős, P.L. and Katona, G.O.H.: The extremal 2-part Sperner families (in preparation).

[6] Frankl, P. and Katona G.O.H.: Polytopes determined by hypergraph classes (to appear in European J. of Combinatorics).

[7] Füredi, Z.: A Ramsey-Sperner theorem, (to appear in Graphs and Combinatorics 1 (1985)).

[8] Griggs, J.R.: The Littlewood-Offord problem: tightest packing and an M-part Sperner theorem, European J. Combinatorics 1 (1980) 225-234.

[9] Griggs, J.R.: Collection of subsets with Sperner property, Trans. Amer. Math. Soc. 269 (1982) 575-591.

[10] Griggs, J.R. and Kleitman, D.J.: A three-part Sperner theorem, Discrete Math. 17 (1977) 281-289.

[11] Griggs, J.R., Odlyzko, A.M. and Shearer, J.B.: K-color Sperner theorems, preprint.

[12] Katona, G.O.H.: On a conjecture of Erdös and a stronger form of Sperner's theorem, Studia Sci. Math. Hungar. 1 (1966) 59-63.

[13] Katona, G.O.H.: A three-part Sperner theorem, Studia Sci. Math. Hungar. 8 (1973) 379-390.

[14] Kleitman, D.J.: On a lemma of Littlewood and Offord and the distribution of certain sums, Math. Z. 90 (1965) 251-259.

[15] Sali, A.: Stronger form of an M-part Sperner theorem, European J. of Combinatorics 4 (1983) 179-183.

[16] Sperner, E.: Ein Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1928) 544-548.