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CONVEX HULLS OF HYPERGRAPH CLASSES

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ABSTRACT

In this paper a survey of results concerning convex hulls of hypergraph classes is given. Some applications are discussed and some open problems are pointed out.

1. INTRODUCTION, SPERNER-HYPERGRAPHS

Let X be a finite set of n elements and let 2^X be its power set. We call the pair (X,H) *hypergraph* where $H \subset 2^X$. The elements of X and H are called *vertices* and *edges*, resp. (X,H) is *k-uniform* (or briefly *uniform*) if all of its edges are of size k. If no edge of (X,H) contains another one we say that (X,H) is a *Sperner-hypergraph*. It is easy to see that any uniform hypergraph is a Sperner-hypergraph.

Sperner theorem [16]. *A Sperner-hypergraph on* n *vertices has maximally*

(1) $$\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

edges.

The complete $\lfloor\frac{n}{2}\rfloor$ -uniform hypergraph shows that there is a Sperner-hypergraph with this many edges. The non-trivial part of the above theorem is that the number of edges is at most (1). For a short proof see [13].

In order to study the possible sizes of the edges a Sperner-hypergraph let us introduce the concept of the *profile* of a hypergraph:

$$p(H) = (p_0, p_1, \ldots, p_n)$$

where p_i is the number of edges of (X,H) of size i $(0 \le i \le n)$. Therefore the profile of a hypergraph is a point of the $(n+1)$-dimensional Euclidean space R^{n+1}. Let σ denote the set of profiles of all Sperner-hypergraphs. A good approximation of σ is its convex hull. The *convex hull* $\hat{\alpha}$ of a set $\alpha \subset R^{n+1}$ is

$$\hat{\alpha} = \left\{ \sum_{i=1}^{\ell} c_i A_i \; : \; A_i \in \alpha \; , \; c_i \ge 0 \quad (1 \le i \le \ell) \, , \quad \sum_{i=1}^{\ell} c_i = 1 \right\} \, ,$$

that is, the set of all *convex linear combinations* of the elements of α. $A \in \alpha$ is an *extreme point of* α iff A is not a convex linear combination of elements of α different from A. It is easy to see that 1) if α has finitely many extreme points then any element A of α can be expressed as a convex linear combination of the extreme points of α; 2) α and $\hat{\alpha}$ have the same extreme points; 3) the extreme points of α uniquely determine $\hat{\alpha}$.

After these preliminaries we can formulate our first (rather trivial)

Theorem 1. [5]. *The extreme points of* σ *(=set of profiles of Sperner-hypergraphs) are*

$$Z = (0,\dots,0)$$

and

$$V_i = (\underset{0}{0},\dots,0,\underset{i}{\tbinom{n}{i}},0,\dots,\underset{n}{0}) \qquad (0 \le i \le n).$$

Proof. Z is the profile of the hypergraph without any edge while V_i is the profile of the complete i-uniform hypergraph. These are Sperner-hypergraphs, so $Z, V_i \in \sigma$ $(0 \le i \le n)$. Let us show that they are extreme points of σ. Suppose that

$$(2) \qquad V_i = \sum_{j=1}^{\ell} c_j A_j$$

where $c_j > 0$, $A_j \in \sigma$ $(1 \le j \le \ell)$, $\sum_{j=1}^{\ell} c_j = 1$. The components of A_j are non-negative, therefore their k-th$(k \ne i)$ component must be 0 by (2). Their i-th components are $\le \binom{n}{i}$, so their i-th components must be equal to $\binom{n}{i}$, again by (2). Hence $V_i = A_j$ $(1 \le i \le n)$, that is, V_i is an extreme point. It can be shown in the same way that Z is also extreme point.

We will see that any element $A \in \sigma$ is a convex linear combination of Z and V_i $(0 \le i \le n)$. This obviously implies that these are all the extreme points of σ. Indeed, let $A = (p_0,\dots,p_n) \in \sigma$. We have to find coefficients c, $c_0,\dots,c_n \ge 0$ satisfying

(3) $$c + \sum_{i=0}^{n} c_i = 1$$

and

$$A = cZ + \sum_{i=0}^{n} c_i V_i .$$

This latter equation is equivalent to

$$p_i = c_i \binom{n}{i} \qquad (0 \le i \le n) .$$

That is, c_i are unambiguously determined. We can find a $c \ge 0$ satisfying (3) iff

$$\sum_{i=0}^{n} c_i =$$

(4) $$\sum_{i=0}^{n} \frac{p_i}{\binom{n}{i}} \le 1 .$$

However, (4) is well known and called the LYM inequality after Lubell [13], Yamamoto [17] and Meshalkin [14]. The proof is complete.

The convex hull of σ is bordered by the trivial hyperplanes

$$p_i \ge 0 \qquad (0 \le i \le n)$$

and by the hyperplane determined by (4). Therefore Theorem (1) is only a reformulation of the LYM inequality. However this is not so for other classes of hypergraphs. In general, more non-trivial inequalities are needed. In the next section we will list classes of hypergraphs whose extreme points are determined.

2. CONVEX HULLS OF SOME CLASSES OF HYPERGRAPHS

(X,H) is a *k-Sperner-hypergraph* if it contains no $k+1$ different edges $H_1,\dots,H_{k+1} \in H$ satisfying

$$H_1 \subset \dots \subset H_{k+1} .$$

A hypergraph is 1-Sperner iff it is a Sperner-hypergraph. The set of profiles of the k-Sperner-hypergraphs will be denoted by σ_k.

Theorem 2 [6]. *The extreme points of* σ_k *are the vectors whose i-th component is either zero or* $\binom{n}{i}$ *but the number of their non-zero components is at most* k.

Using this theorem, it is easy to determine the maximum number of edges of a k-Sperner-hypergraph (X,H). The number of edges of (X,H) is nothing else but the sum $\sum_{i=0}^{n} p_i$ of the profile $p(H) = (p_0,\dots,p_n)$. It is easy to see that

$$\max_{(X,H) \in \sigma_k} \sum_{i=0}^{n} p_i$$

can be attained only for extreme points of σ_k. Hence *the maximum number of edges in a k-Sperner-hypergraph is the sum*

$$\sum_{i=\lfloor\frac{n-k+1}{2}\rfloor}^{\lfloor\frac{n+k-1}{2}\rfloor} \binom{n}{i}$$

of the k *largest binomial coefficients.* This is a well known theorem of Erdös [3].

We say that (X,H) is an *intersecting* hypergraph if any two edges have non-empty intersection. The set of profiles of intersecting hypergraphs is denoted by ι.

Theorem 3 [6]. *The extreme points of* ι *are the following ones:*

$$(5)\qquad \Big(\underset{0}{0},\dots,\underset{k}{\tbinom{n-1}{k-1}},\underset{k+1}{\tbinom{n-1}{k}},\dots,\underset{n-k}{\tbinom{n-1}{n-k-1}},\underset{n-k+1}{\tbinom{n}{n-k+1}},\dots,\underset{n}{\tbinom{n}{n}}\Big)\qquad (1\le k<n/2),$$

$$(6)\qquad \Big(\underset{0}{0},\dots,\underset{n/2}{\tbinom{n-1}{n/2-1}},\underset{n/2+1}{\tbinom{n}{n/2+1}},\dots,\underset{n}{\tbinom{n}{n}}\Big)$$

$$(7)\qquad \left(0,\dots,0,\binom{n}{\frac{n+1}{2}},\dots,\binom{n}{n}\right)$$

and the vector obtained by replacing 1) any but the k*-th components of (5) by zero and 2) any components of (6) or (7) by zero.*

It is easy to construct hypergraphs with these profiles: take all the edges of size i with $k \le i \le n-k$ containing a fixed vertex x and all the possible edges of size $> n-k$. The rest of the proof is more complicated.

One can deduce from this theorem the maximum number of edges of an intersecting hypergraph: 2^{n-1}. However, even this deduction is longer than the original proof in [4]. A more interesting consequence of Theorem 3 is the Erdös-Ko-Rado theorem [4]: *The maximum number of edges of an intersecting k-uniform hypergraph is* $\binom{n-1}{k-1}$ *if* $k \le n/2$. Indeed, the i-th component $(k \le n/2)$ of any extreme point of ι is $\le \binom{n-1}{k-1}$. For a short direct proof of this theorem see [10].

The (up to now) deepest result of this theory is

Theorem 4 [5]. *The extreme points of* $\sigma \cap \iota$ *are*

$$Z, \qquad V_j \qquad (n/2 < j \le n)$$

$$W_i = (\underset{0}{0},\dots,\underset{i}{\tbinom{n-1}{i-1}},\dots,\underset{n}{0}) \qquad (1 \le i \le n/2)$$

$$W_{ij} = (\underset{0}{0},\dots,\underset{i}{\tbinom{n-1}{i-1}},\dots,\underset{j}{\tbinom{n-1}{j}},\dots,\underset{n}{0}) \qquad (1 \le i \le n/2,\ i+1>n)$$

The main part of the statement of this theorem is that the extreme points can have at most two non-zero components.

The Erdös-Ko-Rado theorem can be easily deduced, again. Indeed, the k-th component of any extreme point is $\le \binom{n-1}{k-1}$ if $k \le n/2$.

Let us consider the problem, what is the maximum number of edges of an intersecting Sperner hypergraph. It is sufficient to maximize the sum of the components of the extreme points in Theorem 4. Examine first the extreme points W_{ij}. $j \ge n-i+1 > n/2$ follows from the conditions $1 \le i \le n/2$, $i+j > n$. Therefore

$$\binom{n-1}{i-1}+\binom{n-1}{j} \le \binom{n-1}{j-1}+\binom{n-1}{n-i+1} = \binom{n-1}{i-1}+\binom{n-1}{i-2} = \binom{n}{i-1} \le \binom{n}{\lfloor \frac{n-2}{2} \rfloor}$$

gives an upper estimate for the sum of the components in W_{ij}. The only non-zero components of W_i and V_j are $\binom{n-1}{i-1} \le \binom{n-1}{\lfloor \frac{n-2}{2} \rfloor}$ and $\binom{n}{j} \le \binom{n}{\lceil \frac{n+1}{2} \rceil}$, resp. Consequently, *the complete* $\lfloor \frac{n+1}{2} \rfloor$*-uniform hypergraph has the maximum number of edges among all intersecting Sperner-hypergraphs*. This is a special case of a theorem of Milner [15].

It is worth-while to determine the nontrivial hyperplanes bordering the convex hull of $\sigma \cap \iota$. The following class of inequalities contains the inequalities corresponding to these hyperplanes:

$$(8) \qquad \sum_{1 \le i \le n/2} (1-y_{n-i+1}) \frac{p_i}{\binom{n-1}{i-1}} + \sum_{n/2 \le j \le n-1} y_j \frac{p_j}{\binom{n-1}{j}} \le 1$$

for any $(p_0, \dots, p_n) \in \sigma \cap \iota$ and for any sequence $y_{\lfloor n/2 \rfloor+1} \ge \dots \ge y_n \ge 0$ satisfying

$$(9) \qquad y_j \le 1 - \frac{j}{n} \qquad (n/2 < j \le n).$$

It is interesting to mention that some authors tried to find inequalities well characterizing the elements of $\sigma \cap \iota$. Bollobás [1] proved

$$\sum_{1 \le i \le n/2} \frac{p_i}{\binom{n-1}{i-1}} \le 1$$

which can be obtained from (8) by substituting $y_{\lfloor n/2 \rfloor+1} = \dots = y_n = 0$, while the inequality

$$\sum_{1\le i\le n/2} \frac{p_i}{\binom{n}{i-1}} + \sum_{n/2<j\le n} \frac{p_j}{\binom{n}{j}} \le 1$$

of Greene, Katona and Kleitman [8] follows from (8) by choosing $y_j = 1-\frac{j}{n}$ $(n/2<j\le n)$. Now it is clear that these inequalities were too weak to characterize the elements of $\sigma \cap \iota$, alone. Many of them are needed.

Let us investigate a problem of somewhat different character. Suppose that the hypergraphs $(X, H_1), \ldots, (X, H_t)$ satisfy the following condition:

(10) $$G, H \subset H, \quad G \ne H, \quad G \in H_i, \quad H \in H_j, \quad i \ne j$$
implies
$$G \not\subset H$$

That is, two different hypergraphs cannot contain different edges, one containing the other one. (But $H_i \cap H_j$ is not necessarily empty.) The *profile* of the sequence of the hypergraphs $(X, H_1), \ldots, (X, H_t)$ is

$$p(H_1, \ldots, H_t) = \sum_{i=1}^{t} p(H_i) .$$

Let $\sigma(t)$ denote the set of the profiles of hypergraphs satisfying (10).

Theorem 5 [6]. *The extreme points of* $\sigma(t)$ *are*

$$Z, \quad tV_i$$

if $t \ge n+1$. *Otherwise there are some additional extreme points* *with at least* $t+1$ *non-zero components* $p_i = \binom{n}{i}$.

A theorem of Daykin, Frankl, Greene and Hilton [2] easily follows:

$$\max_{\substack{(X,H_1),\dots,(X,H_t)\\ \text{satisfy (10)}}} \sum_{i=1}^{t} |H_i| = \max\left(t\binom{n}{\lfloor \frac{n}{2} \rfloor},\ 2^n \right).$$

3. APPLICATIONS

Let c_i $(0 \le i \le n)$ be reals and suppose that $\sum_{i=0}^{n} c_i p_i$ has to be maximized for a certain class of hypergraphs. Let α be the set of profiles of these hypergraphs. If the extreme points of α are determined, our situation is very easy. We have to maximize $\sum_{i=0}^{n} c_i p_i$ only for these extreme points.

In the previous section we applied this idea only for the cases when (i) $c_0 = c_1 = \dots = c_n = 1$ and (ii) $c_i = 1$, $c_j = 0$ $(j \ne i)$. However, more complicated functions can be arised. For instance, one could ask for the maximum of the sums of the sizes of the edges in a hypergraph. That is, $c_i = i$ $(0 \le i \le n)$. [11] solves this problem for Sperner-hypergraphs:

$$\text{(11)} \qquad \max_{(p_0,\dots,p_n) \in \sigma} \sum_{i=0}^{n} i p_i = \binom{n}{\lceil \frac{n}{2} \rceil} \left\lceil \frac{n}{2} \right\rceil .$$

This is an easy consequence of Theorem 1 (that is, of the LYM inequality). Indeed, the extreme point V_i gives $\binom{n}{i} i$. It is easy to verify that $\max_i \binom{n}{i} i$ is equal to the right hand side of (11).

Another application can be found in [5] where

$$(12) \qquad \sum_{i=0}^{n} (n-i)!\, p_i$$

is to be maximized for intersecting Sperner-hypergraphs. We have to use Theorem 4. (12) gives more for W_{ij} than for W_i, therefore we have to check only V_j $(n/2<j\leq n)$ and W_{ij} $(1\leq i\leq n/2, i+j>n)$ from the extreme point. If $V_j = (p_0,\ldots,p_n)$ then we have trivial inequality for (12)

$$(n-j)!\binom{n}{j} \leq (n-1)!+1 \quad .$$

On the other hand, if $W_{ij} = (p_0,\ldots,p_n)$ then the following sequence of inequalities gives the same estimate:

$$(n-i)!\binom{n-1}{i-1}+(n-j)!\binom{n}{j} = \frac{(n-j)!}{(i-1)!} + \frac{(n-1)!(n-j)}{j!} \leq$$

$$\leq \frac{(n-1)!}{(i-1)!} + \frac{(n-1)!(i-1)}{(n-i+1)!} \begin{cases} = (n-1)! & \text{if } i=1 \\ = (n-1)!+1 & \text{if } i=2 \\ \leq \frac{(n-1)!}{2} + \frac{(n-1)!}{2} & \text{if } 3\leq i\leq n/2 \\ (n\leq 4 \text{ should be checked separately}). & \end{cases}$$

Summarizing, $(n-1)!+1$ is the maximum of (12). The hypergraph (X,H), where

$$H = \{\{x,y\}: y \in X-x\} \cup \{X-x\} \qquad (x \in X \quad \text{fixed}) ,$$

gives the equality.

4. OPEN PROBLEMS

1. Any problem of the extremal hypergraphs (see e.g. [12]) can be extended in the present way. However, some of these extended questions are blocked by longstanding open problems. See e.g. the following condition for (X,H)

$$(13) \qquad H_1, H_2 \in H \qquad \text{implies} \qquad |H_1 \cap H_2| \geq k .$$

Let us denote by $\iota(k)$ the set of profiles of the hypergraphs satisfying (13). The extreme points of $\iota(1) = \iota$ are determined in Theorem 3. However knowing the extreme points of $\iota(k)$ would imply the determination of $\max p$ for $(p_0,\dots,p_n) \in \iota(k)$, too. This is known to be $\binom{n-k}{\ell-k}$ only for $n > (\ell+1)(k+1)$ $(k \geq 15)$ (see [4], [7]). A nice open problem of this kind from [4]: Is it true that the optimal construction of the above problem for $k = 2$, $n = 4m$ $\ell = 2m$ is the hypergraph (X,H) with

$$H = \{A: |A \cap X_1| > m\}$$

where

$$X_1 \subset X , \qquad |X_1| = 2m \ ?$$

On the other hand, one extreme point of $\iota(k)$ maximizing $\sum_{i=0}^{n} p_i$ is known [17].

2. We determined the extreme points for several cases. However it is non-trivial to make a detailed description of these convex hulls. Determine e.g. the graph of the edges (1-dimensional faces) of the convex hulls.

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