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# SUMS OF VECTORS AND TURÁN'S GRAPH PROBLEM

G.O.H. KATONA

*Mathematical Institute, Reáltanoda 13–15, 1053 Budapest, Hungary*

## 1. Introduction

Let $X$ be a Hilbert space and $0 \leqslant m \leqslant n$ integers. If $a_1, \ldots, a_n \in X$, then $N_g(a_1, \ldots, a_n)$ denotes the number of sets $A = \{i_1, \ldots, i_g\}$, $|A| = g$ such that

$$\left\| \sum_{j=1}^{g} a_{i_j} \right\| \geqslant 1.$$

Finally, let us consider the minimum

$$N_g(X, n, m) = \min N_g(a_1, \ldots, a_n),$$

where the minimum is taken over all the sequences $a_1, \ldots, a_n$, $\|a_i\| \geqslant 1$ $(1 \leqslant i \leqslant m)$, $\|a_i\| < 1$ $(m < i \leqslant n)$; $n$ and $m$ are fixed.

The aim of this paper is to investigate the values $N_2(X, n, m)$ and $N_3(X, n, m)$. It is, following the author's talk in Marseille, a version of [6], with less proofs, but giving a wider view of the subject.

## 2. Vectors and graphs

**Theorem 1** ([3]). *If $X$ is a Hilbert space of at least two dimensions then*

$$N_2(X, n, m) = \left\lfloor \left( \frac{m-1}{2} \right)^2 \right\rfloor .$$

**Proof.** It is easy to see that for any three vectors $a_1$, $a_2$, $a_3$ satisfying $\|a_1\|$, $\|a_2\|$, $\|a_3\| \geqslant 1$ there is a pair $i \neq j$ with $\|a_i + a_j\| \geqslant 1$. It is enough to verify this for 3 dimensions. Then it follows for any Hilbert space as the 3 vectors span a 3-dimensional subspace.

Suppose $a_1, \ldots, a_n$ satisfy the condition $\|a_i\| \geqslant 1$ $(1 \leqslant i \leqslant m)$. Define a graph $G = (X, E)$ on the vertex-set $X = \{a_1, \ldots, a_m\}$. The unoriented edge $\{a_i, a_j\}$ is in $E$ iff $\|a_i + a_j\| \geqslant 1$. By the above remark, any 3 vertices of $G$ span at least one edge, that is, $G$ contains no spanned empty triangle. So we can use the following special case of Turán's theorem [12]:

*If a graph on m vertices contains no spanned empty triangle then the number of edges is at least* $\lfloor((m-1)/2)^2\rfloor$.

Therefore,

$$N_2(a_1, \ldots, a_n) \geq N_2(a_1, \ldots, a_m) \geq \left\lfloor \left(\frac{m-1}{2}\right)^2 \right\rfloor,$$

that is,

$$N_2(X, n, m) \geq \left\lfloor \left(\frac{m-1}{2}\right)^2 \right\rfloor.$$

The following construction proves the equality. $X$ is not one-dimensional, consequently there are two orthogonal vectors $e$ and $f$ in $X$ with lengths $\|e\| = 0.99$, $\|f\| = 0.2$. Put $a_1 = \cdots = a_{\lfloor m/2 \rfloor} = e - f$, $a_{\lfloor m/2 \rfloor + 1} = \cdots = a_m = -e-f$ and $a_{m+1} = \cdots = a_n = f$. Here, $\|2f\| = \|(e-f)+(-e-f)\| = 0.4$, $\|(e-f)+f\| = \|(-e-f)+f\| = 0.99$ hold, hence $\|a_i + a_j\| \geq 1$ occurs iff either $1 \leq i, j \leq \lfloor m/2 \rfloor$ or $\lfloor m/2 \rfloor < i, j \leq m$. The number of these pairs is really $\lfloor (m-1)/2 \rfloor^2$. The proof is complete.

It is interesting that the one-dimensional $X$ needs somewhat more complex tools.

**Lemma 1** [4]. *If the graph* $G = (X_1 \cup X_2, E)(X_1 \cap X_2 = \emptyset, |X_1| = n_1, |X_2| = n_2)$ *contains no empty triangle spanned by at least two vertices from* $X_2$, *then*

$$|E| = \begin{cases} \left\lfloor \dfrac{(n_1+n_2)^2 - 2n_1^2 - 2n_2 + 1}{4} \right\rfloor & \text{if } n_1 \leq n_2, \\[2ex] \dbinom{n_2}{2} & \text{if } n_1 \geq n_2. \end{cases}$$

Unlike in the higher-dimensional case, here the sums $a_i + a_j$, $\|a_i\| \geq 1$, $\|a_j\| < 1$ should also be considered. In the optimal case they are not all smaller than 1. The above lemma makes us able to use these sums, too. The following theorem follows by a proof similar to that of Theorem 1.

**Theorem 2** ([7]). *If X is the real line then*

$$N_2(X, n, m) = \begin{cases} \left\lfloor \dfrac{-n^2 - 2m^2 + 4nm - 2m + 1}{4} \right\rfloor & \text{if } n \leq 2m, \\[2ex] \dbinom{m}{2} & \text{if } n \geq 2m. \end{cases}$$

There is a more general question considered in the literature. The following notation is needed for posing it:

$$N_g(X, c, n, m) = \min N_g(a_1, \ldots, a_n),$$

where $0 \leq c < \infty$, $0 \leq m \leq n$ and the minimum is taken over all the sequences $a_1, \ldots, a_n$, $\|a_i\| \geq c$ $(1 \leq i \leq m)$, $\|a_i\| < c$ $(m < i \leq n)$. $N_2(X, c, n, m)$ is fully determined in [7] for Hilbert spaces of dimension 1, 2 or $\infty$. Lemmas like Lemma 1 are used in the proofs. Sidorenko [10] has considered linear normed spaces.

The problems concerning two-sums led to graph problems. Similarly, the three-sums lead to three-graphs. This makes these problems much harder since very little is known about three-graphs. Like before, we need some geometry in order to be able to use graph theory.

**Lemma 2.** *If $a_1, \ldots, a_4 \in X$ and $\|a_i\| \geq 1$ $(1 \leq i \leq 4)$ then, for some $i \neq j \neq k \neq i$,*

$$\|a_i + a_j + a_k\| \geq 1.$$

Let $T(m, 4, 3)$ denote the minimum number of edges of a 3-graph (no loops, no multiple edges) on $m$ vertices satisfying the condition that

$$\text{any 4-set of vertices contains at least one 3-edge.} \tag{1}$$

Using the idea of the proof of Theorem 1, one can prove [9]

$$N_3(X, n, m) \geq T(m, 4, 3). \tag{2}$$

However, this is not sharp enough. According to the famous conjecture of Turán

$$\lim T(m, 4, 3) \Big/ \binom{m}{3} = \tfrac{4}{9},$$

but only results like [8]

$$T(m, 4, 3) \Big/ \binom{m}{3} \geq \tfrac{5}{14} \quad (m \geq 8)$$

are proved. This gives

$$N_3(X, n, m) \geq \tfrac{5}{14}\binom{m}{3} \quad (m \geq 8). \tag{3}$$

First we show that even the order of magnitude of (3) is too small when $m$ is small in comparison to $n$. The reason is, again, that we did not use the sums with small components in (2). We can use them on the basis of the following.

**Lemma 3** ([6]). *Suppose $a_1$, $a_2$, $a_3$, $b_1$, $b_2 \in X$, $\|a_i\| \geq 1$ $(1 \leq i \leq 3)$. Then either there exist indices $1 \leq i < j \leq 3$ and $1 \leq k \leq 2$ such that*

$$\|a_i + a_j + b_k\| \geq 1,$$

*or,*

$$\|a_i + b_1 + b_2\| \geq 1$$

*holds for some* $i$ $(1 \leq i \leq 3)$.

The combinatorial tool which we use is a lemma about 3-graphs with two different classes of vertices. Its suppositions are clear from Lemma 3.

**Lemma 4** ([6]). *Let* $G = (V, E)$ *be a 3-graph (no loops, no multiple edges) where* $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, $|V_1| = n_1$, $|V_2| = n_2 \leq 2n_1 - 1$ *and* $E$ *contains an edge either of the form* $\{x_i, x_j, y_k\}$ $(1 \leq i < j \leq 3,\ 1 \leq k \leq 2)$ *or of the form* $\{x_i, y_1, y_2\}$ $(1 \leq i \leq 3)$ *for any choice of* $x_1, x_2, x_3 \in V_1$ *and* $y_1, y_2 \in V_2$. *Then,*

$$|E| \geq \tfrac{1}{6} n_1 \binom{n_2}{2} .$$

By Lemmas 3 and 4 we can easily prove the following.

**Theorem 3**

$$N_3(X, n, m) \geq (n - m) \binom{m}{2} + \binom{m}{3} .$$

For $n = m$ the order of magnitude of (3) is, of course, correct. However, the constant is too small. We conjecture [6]

$$N_3(X, n, n) = \binom{n}{3} - \left\lceil \frac{2n}{3} \right\rceil \binom{\lfloor n/3 \rfloor}{2} . \tag{4}$$

The right-hand side of (4) can be realized with $a_i = e$ $(1 \leq i \leq \lceil 2n/3 \rceil)$, $a_i = -2e$ $(\lceil 2n/3 \rceil < i \leq n)$ where $e$ is an arbitrarily chosen vector of length 1. Eq. (4) is about $\frac{5}{9}\binom{n}{3}$. Eq. (3) gives $\frac{5}{14}\binom{n}{3}$ and the Turán conjecture would give $\frac{4}{9}\binom{n}{3}$. That is, condition (1) is not strong enough. In [6] it is proved for the one-dimensional $X$ that there are no 5-vectors $a_i \in X$, $|a_i| \geq 1$ $(1 \leq i \leq 5)$ such that $|a_1 + a_2 + a_3| \geq 1$, $|a_1 + a_2 + a_4| \geq 1$ and $|a_3 + a_4 + a_5| \geq 1$, but all the other 7 combinations give a sum with length $< 1$. (This statement is probably true for any $X$.) Hence we have the following condition for our 3-graphs (defined by $|a_i + a_j + a_k| \geq 1$):

> There are no 5 vertices $x_1, x_2, \ldots, x_5$ spanning the graph $(\{x_1, \ldots, x_5\}, \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_3, x_4, x_5\}\})$. (5)

If we denote by $h(n)$ the minimum number of 3-edges of a 3-graph on $n$ vertices satisfying (1) and (5) then a combination of [6] and [1] proves

$$h(n) \geq (4/9) \binom{n}{3} \quad (n \geq 8).$$

Consequently, the same is true for $N_3(X, n, n)$. However, we conjecture [6]

$$\lim \frac{h(n)}{\binom{n}{3}} = 4 - 2\sqrt{3} = 0.5358984\ldots .$$

In [6] a further condition is given which excludes another spanned subgraph on 5 vertices and which can be imposed on the basis of some easy geometrical lemma. We conjecture that under this condition + (1) + (5), the number of edges cannot be smaller than the right-hand side of (4).

## 3. Connections with probability theory

Let $\xi$ and $\eta$ be independent identically distributed random vectors taking on the values $a_1, \ldots, a_n \in X$ with equal probabilities where $\|a_i\| \geq x$ $(1 \leq i \leq m)$, $\|a_i\| < x$ $(m < i \leq n)$. Then at least $N_2(X, n, m)$ pairs $i \neq j$ satisfy $\|a_i + a_j\| \geq x$ and $\|a_i + a_i\| > x$ holds for $1 \leq i \leq m$. Therefore

$$P(\|\xi + \eta\| \geq x) \geq \frac{2N_2(X, n, m) + m}{n^2}. \tag{6}$$

The inequality

$$\frac{2N_2(X, n, m) + m}{n^2} \geq \frac{\frac{1}{2}(m^2 - 2m) + m}{n^2} = \frac{1}{2}\left(\frac{m}{n}\right)^2 \tag{7}$$

follows from Theorem 1. However, $m/n = P(\|\xi\| \geq x)$. By (6) and (7) we proved [3]

$$P(\|\xi + \eta\| \geq x) \geq \tfrac{1}{2}P^2(\|\xi\| \geq x). \tag{8}$$

However, if $\xi$ and $\eta$ are (independent and identically distributed) arbitrary random vectors then we need a 'continuous' variant of Theorem 1, that is, of Turán's theorem, where 'number of' (vertices, edges) is substituted by 'measure of'. This is worked out in [5] and independently in [2]. Consequently (8) can be proved for such general random vectors too [4].

Theorem 2 similarly implies

$$P(|\xi + \eta| \geq x) \geq \begin{cases} \frac{1}{2} - (1 - P(|\xi| \geq x))^2 & \text{if } P(|\xi| \geq x) \geq \frac{1}{2}, \\ P^2(|\xi| \geq x) & \text{otherwise.} \end{cases}$$

It is interesting to mention that these inequalities are improvable. Simple constructions give equality in them. Lower estimations of $P(\|\xi + \eta\| \geq x)$ by use of $P(\|\xi\| \geq cx)$ are worked out in [7] and [10].

The probabilistic version of Theorem 3 is

$$P(\|\xi_1+\xi_2+\xi_3\| \geq x) \geq \tfrac{1}{2}P^2(\|\xi_1\| \geq x)(1-P(\|\xi_1\| \geq x),$$

while our efforts concerning $N_3(X, n, n)$ can be formulated in the form

$$P(\|\xi_1+\xi_2+\xi_3\| \geq x, \|\xi_1\| \geq x, \|\xi_2\| \geq x, \|\xi_3\| \geq x) \geq c\,P^3(\|\xi_1\| \geq x),$$

where $c = 5/14$ is proved for any $X$, $c = 4/9$ is proved for the real $X$, and $c = 5/9$ is conjectured.

Interested readers can find more about the geometrical tools in [11]. I also would like to draw attention to the forthcoming paper (or papers) of Sidorenko [10].

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